Least squares estimators based on the Adams method for discretely sampled SDEs with small Lévy noise

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Abstract

We consider stochastic differential equations (SDEs) driven by small Lévy noise with some unknown parameters, and propose a new type of least squares estimators (LSEs) based on discrete samples from the SDEs. To approximate the increments of a process from the SDEs, we shall use not the usual Euler method, but the Adams method, that is, a well-known numerical approximation of the solution to the ordinary differential equation appearing in the limit of the SDE. We show the asymptotic distribution of the proposed estimators in a suitable observation scheme. We also show that our estimators can be better than the usual LSE in the finite sample performance.

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Model

Stochastic differential equation with small Lévy noise:

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon, \theta_0)\,dt + \varepsilon\, dL_t \quad (0 < t \le 1), \\ X_0^\varepsilon = x_0 \in \mathbb{R}^d, \end{cases}$$

where

- Unknown parameter: $\theta_0 \in \mathbb{R}^p$
- Given: $b : \mathbb{R}^d \times \Theta \to \mathbb{R}^d$, L_t : Lévy process
- Observed data: $X_{t_0}^\varepsilon, \ldots, X_{t_n}^\varepsilon \quad (t_0 = 0,\ t_n = 1,\ t_i - t_{i-1} = 1/n)$

ODE in the limit $(\varepsilon \to 0)$:

$$\frac{dx_t}{dt} = b(x_t, \theta_0) \quad (0 \le t \le 1).$$

Aim

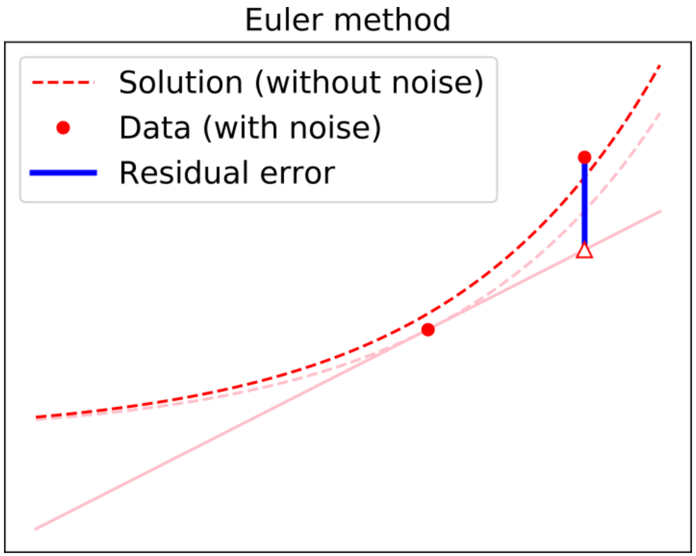
Establish new LSEs for θ_0, and compare their finite sample performance.

Background

A usual LSE is given by

$$\Psi_{n,\varepsilon}(\theta) = \sum_{k=1}^{n} \frac{\left| X^{\varepsilon}_{t_k} - X^{\varepsilon}_{t_{k-1}} - b(X^{\varepsilon}_{t_{k-1}}, \theta)\, \Delta t_{k-1} \right|^2}{\varepsilon^2 \Delta t_{k-1}},$$

$$\hat{\theta}_{n,\varepsilon} := \arg\min_{\theta \in \Theta} \Psi_{n,\varepsilon}(\theta).$$



There is a known result for the usual LSE (see Long, Shimizu and Sun (2013)):

$$\varepsilon^{-1}\left(\hat{\theta}_{n,\varepsilon} - \theta_0\right) \xrightarrow{P_{\theta_0}} I(\theta_0)^{-1} S(\theta_0)$$

as $n \to \infty$, $\varepsilon \to 0$, and $n\varepsilon \to \infty$, where

$$I_{ij}(\theta) := \int_0^1 \partial_{\theta_i} b(x_t, \theta) \cdot \partial_{\theta_j} b(x_t, \theta)\, dt, \;\; S_i(\theta) := \int_0^1 \partial_{\theta_i} b(x_t, \theta) \cdot dL_t.$$

The same convergence is desired for our new LSEs.

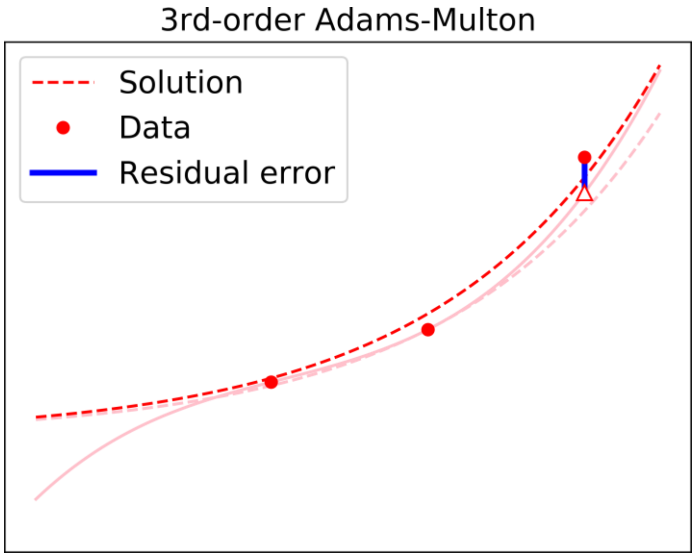
New LSEs

LSEs based on the Adams method $(\ell = 1, 2, \dots)$:

$$\Psi_{n,\varepsilon,\ell}(\theta) := \sum_{k=\ell\vee 1}^{n} \frac{\left| X^\varepsilon_{t_k} - X^\varepsilon_{t_{k-1}} - A_\ell b(\boldsymbol{X}^\varepsilon_{t_k:t_{k-\ell}}, \theta)\, \Delta t_{k-1} \right|^2}{\varepsilon^2 \Delta t_{k-1}},$$

where $\boldsymbol{X}^\varepsilon_{t_k:t_{k-\ell}} := (X^\varepsilon_{t_k}, \dots, X^\varepsilon_{t_{k-\ell}})$ and

$$A_\ell b(\boldsymbol{X}^\varepsilon_{t_k:t_{k-\ell}}, \theta) = \sum_{\nu=0}^{\ell} \beta_{\ell\nu} b(X_{t_{k-\nu}}, \theta), \qquad \beta_{\ell\nu} := \frac{(-1)^\nu}{\nu!(\ell-\nu)!} \int_0^1 \prod_{\substack{j=0 \\ j\neq\nu}}^{\ell} (u + j - 1)\, du.$$



Theoretical Result

$$\varepsilon^{-1}\left(\hat{\theta}_{n,\varepsilon,\ell} - \theta_0\right) \xrightarrow{P_{\theta_0}} I(\theta_0)^{-1} S(\theta_0)$$

as $n \to \infty$, $\varepsilon \to 0$, $\ell 2^{4\ell}/n \to 0$, $2^\ell \varepsilon \to 0$ and $\ell 2^{2\ell}/n\varepsilon \to 0$.

Numerical Result

OU-process with $(\theta_0, x_0) = (1.0, 1.0)$:

$$dX_t = -\theta_0 X_t dt + \varepsilon dB_t, \qquad X_0 = x_0,$$

where B is the standard Brownian motion.

$$\hat{\theta}_{n,\varepsilon} = -\frac{\sum_{k=1}^{n} (X^{\varepsilon}_{t_k} - X^{\varepsilon}_{t_{k-1}}) X^{\varepsilon}_{t_{k-1}}}{\frac{1}{n}\sum_{k=1}^{n} |X^{\varepsilon}_{t_{k-1}}|^2},$$

$$\hat{\theta}_{n,\varepsilon,\ell} = -\frac{\sum_{k=\ell}^{n} (X^{\varepsilon}_{t_k} - X^{\varepsilon}_{t_{k-1}}) A_\ell b(\boldsymbol{X}^{\varepsilon}_{t_k:t_k-\ell})}{\frac{1}{n}\sum_{k=\ell}^{n} |A_\ell b(\boldsymbol{X}^{\varepsilon}_{t_k:t_k-\ell})|^2} \qquad (\ell = 1, \ldots, 6).$$