Parameter Least-Squares Estimate for Time-Inhomogeneous Ornstein-Uhlenbeck Process

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Position of Research: Part of the dissertation

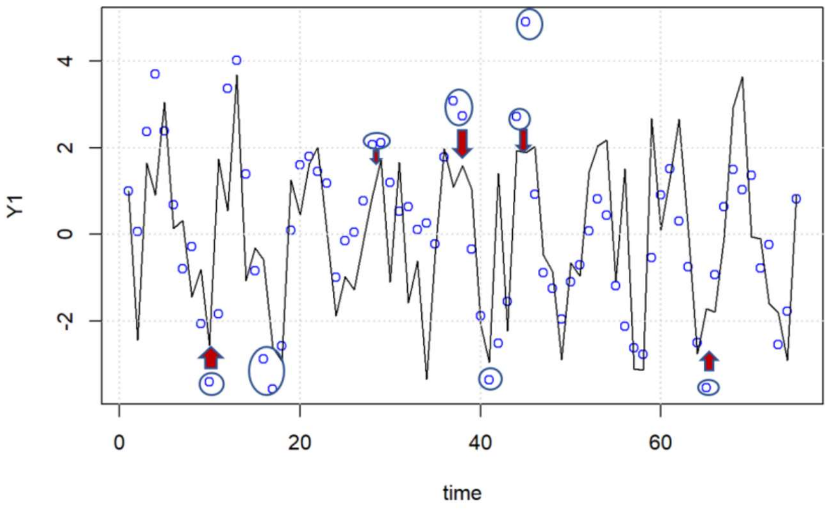
Motivation of the study: Statistical inference of time-inhomogeneous Ornstein-Uhlenbeck Process

Abstract

We address the least-squares estimation of the drift coefficient parameter θ of a time-inhomogeneous Ornstein-Uhlenbeck process, that is observed at high frequency, in which the discretized step size h satisfies $h \to 0$. In this paper, under the conditions $nh \to \infty$ and $nh^2 \to 0$, we prove the consistency and the asymptotic normality of the estimators. We obtain the convergence of the parameters at rate $\sqrt{nh}$, except for $\boldsymbol{\omega}$ at $\sqrt{n^3h^3}$. Finally, the proposed model is applied to real data for the measurement of energy use of light fixtures in one household in Belgium.

Keywords: Least-squares; estimation; Ornstein-Uhlenbeck

Motivations and Model



Background

- ❑ Consistency assumption;
- ❑ Asymptotic numerical proven;
- ❑ High frequency of a time-inhomogeneous OU process

High frequency

$(Y_t)_{t \in R_+}$ is observed at

$0 \equiv t_0 < \cdots < t_n = T_n$;

$t_j = jh$;

h nonrandom sampling, $h \to 0$ s.t

- $T_n := nh \to \infty$
- $nh^2 \to 0$

Model Setup

Let Y_t be a real-valued stationary process satisfying

$$Y_t = y_0 + \int_0^t \left(-\lambda Y_s + \sum_{k=1}^{K} [A_k \cos(\omega_k s) + B_k \sin(\omega_k s)] \right) ds + \sigma w_t$$

- ➢ y_0 is a deterministic initial condition; $K \in \mathrm{N}$ is fixed; $\sigma > 0$;
- ➢ $\boldsymbol{\theta} = (\lambda, A, B, \omega)$ unknown parameter;
- ➢ $A_k, B_k \in \mathbb{R} \backslash \{0\}$ is the amplitude of the kth sinusoidal signal;
- ➢ w is the standard Wiener process.

Goal:

- ❑ The Least-Squares Estimation (LSE) of $\boldsymbol{\theta}$
- ❑ Application propose model to appliance energy in Belgium

1. The consistency

$$Sup_{\theta}|f_n(\theta) - f_0(\theta)| \xrightarrow{p} 0 \text{ as } T_n \to \infty, h \to 0,$$

- $f_0(\theta) = (\lambda - \lambda_0)^2 C + \frac{1}{2}\sum_{k=1}^{K}\left[(A_k - A_{0,k})^2 + (B_k - B_{0,k})^2\right];$
- $\frac{1}{n}\sum_{j=1}^{n} Y_{t_{j-1}}^2 \xrightarrow{p} C, 0 < C < \infty;$
- $\underset{\theta \in \overline{\Theta}}{\text{argmin}}\, f_0(\theta) = \{\theta_0\}$

2. The asymptotic normality

$$\mathbf{D}_n(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}_{(3K+1)}\left(\mathbf{0}, \boldsymbol{\sigma}^2 \mathbf{M}_0(\theta_0)^{-1}\right)$$

- $\mathbf{D}_n = diag\left(\sqrt{nh}, \sqrt{nh}\mathbb{I}_K, \sqrt{nh}\mathbb{I}_K, \sqrt{n^3h^3}\mathbb{I}_K\right)$ - $(3K+1)$x $(3K+1)$ diagonal matrix,
- $\mathbf{M}(\boldsymbol{\theta}_0)$ is $(3K+1)$x $(3K+1)$ - matrix

$$\mathbf{M}_0(\theta_0) := \begin{cases} \begin{pmatrix} C & 0 & 0 & 0 \\ & \frac{1}{2} & 0 & \frac{B_{0,K}}{4} \\ & & \frac{1}{2} & -\frac{A_{0,K}}{4} \\ Sym. & & & \frac{1}{6}\left(A_{0,K}^2 + B_{0,K}^2\right) \end{pmatrix}, & K = 1 \\ \begin{pmatrix} \mathbf{M}_{(3K-2)\times(3K-2)} & \mathbf{0}_{(3K-2)\times 3} \\ \mathbf{0}_{3\times(3K-2)}^{\top} & \mathbf{M}_K \end{pmatrix}, & K \geq 2. \end{cases}$$

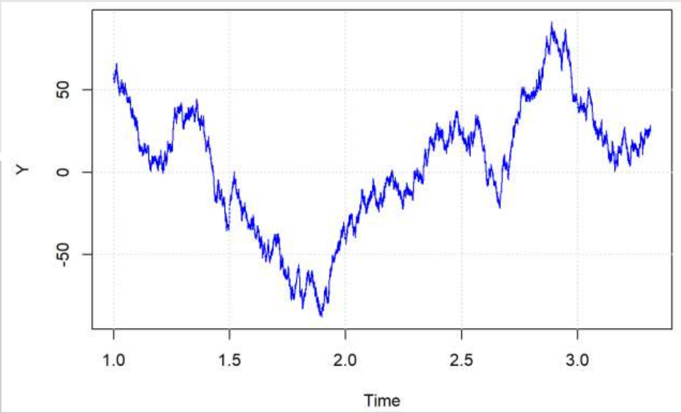
$$\mathbf{M}_K := \begin{pmatrix} \frac{1}{2} & 0 & \frac{B_{0,K}}{4} \\ & \frac{1}{2} & -\frac{A_{0,K}}{4} \\ Sym. & & \frac{1}{6}\left(A_{0,K}^2 + B_{0,K}^2\right) \end{pmatrix}$$

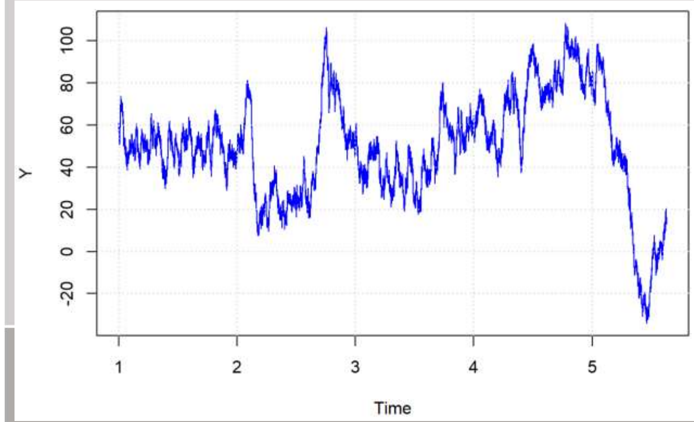
Simulation

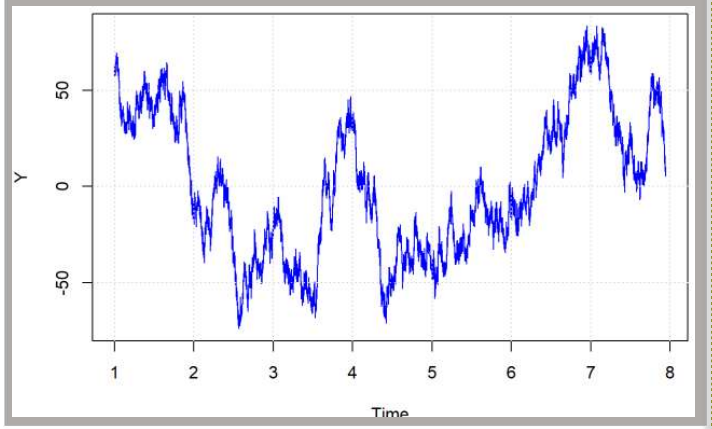
The model examined was applied to real data for the energy use of light fixtures of the house in Belgium. The data that support the findings of this study are available in UCI Machine Learning Repository at [https://archive.ics.uci.edu/ml/datasets/], and the relevant paper can be found at Candanedo *et al* (2017). All calculations have been performed in R program. We consider $K = 3$ paths over 1500 replication and h =0.0023. A pseudorandom number generator of θ_0 : $\lambda_0 = 0.910, A_{0,1} = 2.23, A_{0,2} = -1.206, A_{0,3} = 5.927, B_{0,1} = -2.516, B_{0,2} = -1.038, B_{0,3} = 4.163, \omega_{0,1} = 2.436, \omega_{0,2} = 3.012, \omega_{0,3} = 2.301.$

	$h = 0.00023$		
$\hat{\theta}_n$	$n = 10.000$	$n = 20.000$	$n = 30.000$
λ	0.162 (0.217)	0.654 (0.331)	0.857 (0.372)
A_1	2.341 (0.284)	2.643 (0.680)	2.295 (0.168)
A_2	-0.445 (1.293)	-0.947 (0.910)	-1.111 (0.107)
A_3	5.740 (0.264)	5.998 (0.361)	5.846 (0.146)
B_1	-1.516 (1.112)	-2.626 (0.339)	-2.489 (0.383)
B_2	-1.296 (0.471)	-1.108 (0.650)	-0.661 (0.433)
B_3	4.182 (0.632)	4.378 (0.477)	4.093 (0.269)
ω_1	2.420 (0.030)	2.411 (0.055)	2.770 (0.114)
ω_2	3.032 (0.063)	3.041 (0.117)	2.492 (0.125)
ω_3	2.324 (0.020)	2.394 (0.018)	2.712 (0.044)

- ✓ The performance of $\hat{\theta}_n$ tends to be better for larger n;
- ✓ The convergence performance of $\hat{\omega}_n$ prone to be slower than $(\hat{\lambda}_n, \hat{A}_n, \hat{B}_n)$.







Conclusion

- The LSE, $\hat{\theta}_n$ is a consistent estimator and tend to a $(3K + 1)$ -variate Normal distribution;
- The rate of convergence of (λ, A, B) at $\sqrt{nh}$; ω at $\sqrt{n^3h^3}$.

Remarks

- The explicit term of $\left(\omega_k - \omega_{0,k}\right)_k$ does not appear at f_0. The consistency of ω can be shown by a function that minimizes the least-squares function, which is this function would be given by the location of K highest peaks of the signals. It is mean it should be defined maximization function at the minimization least-squares function, see, for example: Stoica and Moses (1997)

Future Works

- The least-squares approach;
- Model selection for the number components K;
- The driving noise process such as Levy process;
- Non-stationary and ergodic diffusion process.

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