ON THURSTON'S CONSTRUCTION OF A SURJECTIVE HOMOMORPHISM $H_{2n+1}(B\Gamma_n, \mathbb{Z}) \to \mathbb{R}$

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Translator's remarks

This article is an English translation of notes by T. Mizutani on a theorem of Thurston [3]. The notes include a construction which seems not quite well-known, of a family of foliations of which the Godbillon-Vey class varies continuously. The contents are kept as it was. Some apparent errors are corrected, while historical comments are left original.

1. Introduction

Thurston constructed codimension-one foliations of S^3 which are non-cobordant and showed that there exists a surjective homomorphism from $H_3(B\Gamma_1, \mathbb{Z})$ to $\mathbb{R}$ in [2]. The homomorphism is given by the integration of the Godbillon-Vey form of foliations over manifolds. The Godbillon-Vey forms are also defined for foliations of codimension greater than one, and it has been conjectured that an analogue also holds. A simple adaptation of constructions in codimension-one case does not work in higher codimensional case, however, there still exists a surjective homomorphism from $H_{2n+1}(B\Gamma_n, \mathbb{Z})$ to $\mathbb{R}$. Indeed, Thurston showed the following

Theorem. *For any $r \in \mathbb{R}$, there exist a closed manifold W^{2n+1} of dimension $(2n+1)$ and a foliation $\mathcal{F}$ of W of codimension n such that*

$$\mathrm{gv}(W, \mathcal{F}) = r.$$

We give an outline of the proof after Thurston, omitting detailed calculations[†3]. We remark that Heitsch recently extends Thurston's theorem to show the existence

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[†3]We slightly add some calculations for conveniences.

of surjective homomorphisms from $H_{2n+1}(B\Gamma_n, \mathbb{Z})$ to $\mathbb{R}^s$, where $s \geq 1$ is a certain integer, by using the Godbillon-Vey class as well as other exotic characteristic classes [7].

Finally we remark that this article is partly based on notes of Thurston's lectures taken by S. Morita[†4] of Osaka City University.

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2. Godbillon-Vey form

Let $(W^{n+p}, \mathcal{F})$ be a foliation of a smooth manifold W^{n+p} of codimension n. We assume that $\mathcal{F}$ is transversely orientable. If $\mathcal{F}$ is locally defined by a system of 1-forms $\{\omega_1, \ldots, \omega_n\}$ with the equation $\omega_1 = \cdots = \omega_n = 0$, then there exists a global n-form Ω such that $\Omega = k\omega_1 \wedge \cdots \wedge \omega_n$ locally holds, where k is a positive function (it can be shown by partition of unity arguments). By the Frobenius theorem there exists a 1-form α such that

$$d\Omega = \alpha \wedge \Omega.$$

Note that the integrability of the distribution defined by $\omega_1 = \cdots = \omega_n = 0$ is equivalent to the existence of such a 1-form α as above also by the Frobenius theorem.

Definition 1. The differential form $\gamma = \alpha \wedge (d\alpha)^n$ is called the Godbillon-Vey form. The cohomology class represented by γ is called the Godbillon-Vey class.

It is indeed known that γ is a closed $(2n+1)$-form and that the cohomology class represented by γ depends only on $\mathcal{F}$ but not on the choice of Ω and α [1]. Therefore, if W is a closed manifold of dimension $(2n+1)$, then the integration of γ over W determines a real number, which we denote by $\mathrm{gv}(W, \mathcal{F})$ and call the Godbillon-Vey characteristic.

3. A formula for foliated M-products

Let N and M be closed manifolds of dimension $(n+1)$ and n, respectively. Suppose that W is a fiber bundle over N with fibers M. A foliation $\mathcal{F}$ of W of codimension n which is transverse to fibers is called a foliated bundle. In particular if W is a trivial bundle, then we call $(W, \mathcal{F})$ a foliated M-product.

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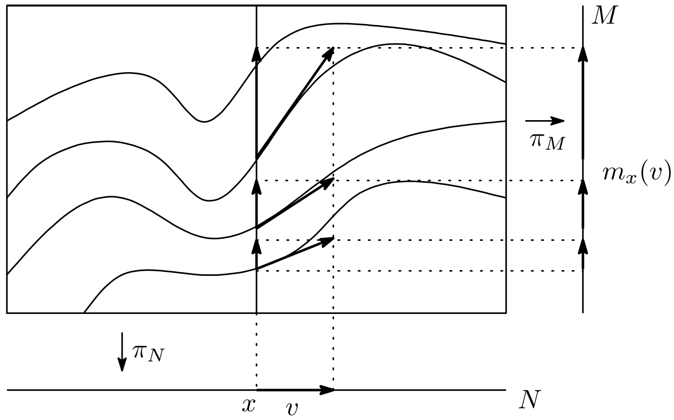

FIGURE 1. the map m_x

Let $(W, \mathcal{F})$ be a foliated M-product. We denote by $\mathcal{L}(M)$ the Lie algebra of smooth (of class C^∞) vector fields on M. For $x \in N$, we will define a linear map $m_x \colon T_xN \to \mathcal{L}(M)$ as follows. Let $\pi_N \colon W = N \times M \to N$ and $\pi_M \colon W = N \times M \to M$ be the projections. Given $v \in T_xN$ and $y \in \pi_N^{-1}(x)(\cong M)$, let $\widetilde{v}_y$ be the unique element of $T_y\mathcal{F}$ such that $\pi_{N*}(\widetilde{v}_y) = v$. We set then $m_x(v)(y) = \pi_{M*}(\widetilde{v}_y)$. It is easy to see that $m_x(v)$ is smooth if $\mathcal{F}$ is smooth. Next we introduce a Gel'fand-Fuchs cocycle which we denote by β. We fix a Riemannian metric on M and let ω be the volume form. Let $X \in \mathcal{L}(M)$ and denote by L_X the Lie derivative with respect to X. Then the function $\operatorname{div} X$ is defined by the equality

$$L_X\omega = (\operatorname{div} X)\,\omega.$$

We define β by the formula

$$\beta(X_1, X_2, \ldots, X_{n+1}) = \int_M (\operatorname{div} X_1)\, d(\operatorname{div} X_2) \wedge \cdots \wedge d(\operatorname{div} X_{n+1}).$$

The cocycle β, homomorphism m_x and the Godbillon-Vey characteristic are related as follows.

Lemma 2 (Thurston, cf. [4], [5], [6], [8])**.** *Let $(N^{n+1} \times M^n, \mathcal{F})$ be a foliated M-product. Then, we have*

$$\begin{aligned}\mathrm{gv}(N \times M, \mathcal{F}) &= \int_N \beta\left(m_x\left(\frac{\partial}{\partial x^1}\right), \ldots, m_x\left(\frac{\partial}{\partial x^{n+1}}\right)\right) dx^1 \wedge \cdots \wedge dx^{n+1} \\ &= \int_N (m_x^*\beta)\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n+1}}\right) dx^1 \wedge \cdots \wedge dx^{n+1},\end{aligned}$$

where $x = (x^1, \ldots, x^{n+1})$ is a system of local coordinates on N.

4. Proof of Theorem and Construction of foliations

We will show the following theorem of Thurston.

Theorem 3 (Thurston)**.** *For any $r \in \mathbb{R}$, there exist a closed manifold W^{2n+1} of dimension $(2n+1)$ and a foliation $\mathcal{F}$ of W of codimension n such that*

$$\mathrm{gv}(W, \mathcal{F}) = r.$$

Corollary 4. *There exists a surjective homomorphism from $H_{2n+1}(B\Gamma_n, \mathbb{Z})$ to $\mathbb{R}$.*

Thurston's proof in the case where $n = 1$ appeared in [2]. We will explain an outline of the proof in the case where $n > 1$ after Thurston. In the arguments, W will be an S^n-bundle over $\Sigma \times T^{n-1}$, where Σ is a closed hyperbolic surface and $(W, \mathcal{F})$ will be a foliated bundle. The strategy is as follows: we will construct enough number of representations from $\mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ to $\mathrm{Diff}(S^n)$, namely, actions of $\mathrm{SL}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ on S^n. Then construct $\mathcal{F}$ on $\Gamma \times \mathbb{Z}^{n-1} \backslash (\mathbb{H} \times \mathbb{R}^{n-1} \times S^n)$, where $\mathbb{H} = \{z = x + \sqrt{-1}y \,|\, x, y \in \mathbb{R},\ y > 0\}$ is the Poincaré upper half plane and Γ is a cocompact lattice of $\mathrm{SL}(2;\mathbb{R})/\mathrm{SO}(2)$ such that $\Sigma = \Gamma\backslash\mathbb{H}$. Let $\mathfrak{sl}(2;\mathbb{R})$ be the Lie algebra of $\mathrm{SL}(2;\mathbb{R})$. We consider an action of $\mathrm{SL}(2;\mathbb{R})$ on $\mathbb{R}^{n+1} = \mathbb{R}^2 \times \mathbb{R}^{n-1}$ such that the action on the $\mathbb{R}^2$ is the linear one and the one on $\mathbb{R}^{n-1}$ is trivial. Then, there is a homomorphism of Lie algebras

$$\lambda_{n+1} \colon \mathfrak{sl}(2;\mathbb{R}) \to \mathcal{L}(\mathbb{R}^{n+1}).$$

Let (x^1, x^2) be the standard coordinates on $\mathbb{R}^2$ and e_2 the Euler vector field. If we introduce the polar coordinates (r, θ) on $\mathbb{R}^2 \setminus \{o\}$, then $e_2 = r\dfrac{\partial}{\partial r}$. We trivialize $T(\mathbb{R}^2 \setminus \{o\})$ by $\left\{r\dfrac{\partial}{\partial r}, \dfrac{\partial}{\partial \theta}\right\}$. We will extend $r\dfrac{\partial}{\partial r}$ and $\dfrac{\partial}{\partial \theta}$ to the whole $\mathbb{R}^2$ by the formulas $e_2 = r\dfrac{\partial}{\partial r} = x^1\dfrac{\partial}{\partial x^1} + x^2\dfrac{\partial}{\partial x^2}$ and $\dfrac{\partial}{\partial \theta} = -x^2\dfrac{\partial}{\partial x^1} + x^1\dfrac{\partial}{\partial x^2}$. Let $a = \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix} \in \mathfrak{sl}(2;\mathbb{R})$. If we set $b = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} a \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, then we can represent

$$\begin{aligned} \lambda_2(a) &= (a^1_1 x^1 + a^1_2 x^2)\frac{\partial}{\partial x^1} + (a^2_1 x^1 + a^2_2 x^2)\frac{\partial}{\partial x^2} \\ &= b^1 r\frac{\partial}{\partial r} + b^2\frac{\partial}{\partial \theta} \\ &= k(\theta)e_2 + \rho_2(a) \end{aligned}$$

on $\mathbb{R}^2 \setminus \{o\}$. Note that $\rho_2(a)$ is the projectivization of λ_2. Indeed, by regarding S^1 as the set of oriented lines in $\mathbb{R}^2$ which pass through the origin, we obtain ρ_2 from λ_2. Note also that $\rho_2(a)$ is parallel to $\dfrac{\partial}{\partial \theta}$ and depends only on θ. We consider the standard metric on $\mathbb{R}^2$. Then, $\operatorname{div}\lambda_2(a) = 0$ because $a \in \mathfrak{sl}(2;\mathbb{R})$, and we have $k(\theta) = -\dfrac{1}{2}\operatorname{div}\rho_2(a)$. Therefore

$$\lambda_2(a) = -\frac{1}{2}\operatorname{div}\rho_2(a)e_2 + \rho_2(a).$$

Assume that $n \geq 2$ and introduce the polar coordinates on the first factor of $(\mathbb{R}^2 \setminus \{o\}) \times \mathbb{R}^{n-1}$. Let $(r, \theta, x^3, \cdots, x^{n+1})$ be the natural coordinates and $e_{n+1} =$

$r\dfrac{\partial}{\partial r}$. We trivialize $T((\mathbb{R}^2 \setminus \{o\}) \times \mathbb{R}^{n-1})$ by $\left\{e_{n+1}, \dfrac{\partial}{\partial\theta}, \dfrac{\partial}{\partial x^3}, \dots, \dfrac{\partial}{\partial x^{n+1}}\right\}$. Then we can represent $\lambda_{n+1}(a)$ as

$$\lambda_{n+1}(a) = k(\theta)e_{n+1} + \widetilde{\rho}_2(a),$$

where $\widetilde{\rho}_2(a)$ is parallel to $\dfrac{\partial}{\partial\theta}$ and depends only on θ. By the same reason as above, $k(\theta) = -\dfrac{1}{2}\operatorname{div}\widetilde{\rho}_2(a)$. Therefore,

$$\lambda_{n+1}(a) = -\frac{1}{2}\operatorname{div}\widetilde{\rho}_2(a)\, e_{n+1} + \widetilde{\rho}_2(a)$$

on $(\mathbb{R}^2 \setminus \{o\}) \times \mathbb{R}^{n-1}$. Note that

$$\begin{cases} 1) \quad \widetilde{\rho}_2(a) \text{ is parallel to } \dfrac{\partial}{\partial\theta} \text{ and depends only on } \theta. \\ 2) \quad \operatorname{div}\widetilde{\rho}_2(a) = \operatorname{div}\rho_2(a) \text{ and it depends only on } \theta. \end{cases} \tag{5}$$

We remark for later use that $\operatorname{div}\rho_2(Y) = -2\sin\theta\cos\theta$ and $\operatorname{div}\rho_2(Z) = -\cos^2\theta + \sin^2\theta$, where $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Z = \dfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We denote by D^l_t the round open ball of radius t in $\mathbb{R}^l$. Let $\epsilon \in (0, 1/2)$ and regard[†5] $S^n = (D^2_{1+\epsilon} \times S^{n-2}) \cup (S^1 \times D^{n-1}_{1+\epsilon})$, where $(r, \theta, p) \in D^2_{1+\epsilon} \times S^{n-2}$ is identified with $(\theta, p/r) \in S^1 \times D^{n-1}_{1+\epsilon}$ if $|r - 1| < \epsilon$. Let $f^i \colon S^{n-2} \to \mathbb{R}$ be any C^∞-functions, where $3 \leq i \leq n+1$, and let g be a function on $\mathbb{R}$ such that $g(r) = 0$ if $r > 1 - \epsilon$ and $g(r) = 1$ if $r < \epsilon$. We will define $\sigma_{n+1} \colon \mathfrak{sl}(2;\mathbb{R}) \times \mathbb{R}^{n-1} \to \mathcal{L}(S^n)$ as follows. First let

$$\begin{aligned} U_0 &= D^2_{\epsilon/2} \times S^{n-2}, \\ U_1 &= \{(r, \theta, p) \in D^2_{1+\epsilon} \times S^{n-2} \mid r > \epsilon/3\}. \end{aligned}$$

We then define $\sigma_{n+1} \colon \mathfrak{sl}(2;\mathbb{R}) \times \mathbb{R}^{n-1} \to \mathcal{L}(D^2_{1+\epsilon} \times S^{n-2})$ by

$$\begin{aligned} \sigma_{n+1}(a) &= \begin{cases} \lambda_{n+1}(a), & \text{on } U_0, \\ -\dfrac{1}{2}(\operatorname{div}\rho_2(a))\, g \cdot r\dfrac{\partial}{\partial r} + \widetilde{\rho}_2(a), & \text{on } U_1, \end{cases} \quad a \in \mathfrak{sl}(2;\mathbb{R}), \\ \sigma_{n+1}(t_i) &= f^i g \cdot r\frac{\partial}{\partial r}, \quad 3 \leq i \leq n+1, \end{aligned}$$

where $\mathbb{R}^{n-1}$ is regarded as the Lie algebra of $\mathbb{R}^{n-1}$ and $\{t_3, \dots, t_{n+1}\}$ is the standard basis for $\mathbb{R}^{n-1}$, and the natural images of elements of $\mathfrak{sl}(2;\mathbb{R})$ and $\mathbb{R}^{n-1}$ in $\mathfrak{sl}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ are denoted by the same symbols by abuse of notation. Note that $\sigma_{n+1}(a)$ and $\sigma_{n+1}(t_i)$ are indeed tangent to $D^2_{1+\epsilon} \times S^{n-2}$. Since $\sigma_{n+1}(a)$ depends only on θ and parallel to $\dfrac{\partial}{\partial\theta}$ on a neighborhood of $\partial(D^2_{1+\epsilon} \times S^{n-2})$, and since $\sigma_{n+1}(t_i)$ is independent of θ and vanishes outside $D^2_1 \times S^{n-2}$, these vector fields naturally extends to S^n. By abuse of notations, we denote thus obtained mapping from $\mathfrak{sl}(2;\mathbb{R}) \times \mathbb{R}^{n-1}$ to $\mathcal{L}(S^n)$ again by σ_{n+1}. Then, by the property (5), σ_{n+1} is indeed

[†5] The original construction makes use of joins instead of decomposing S^n. We modified the construction for clarity.

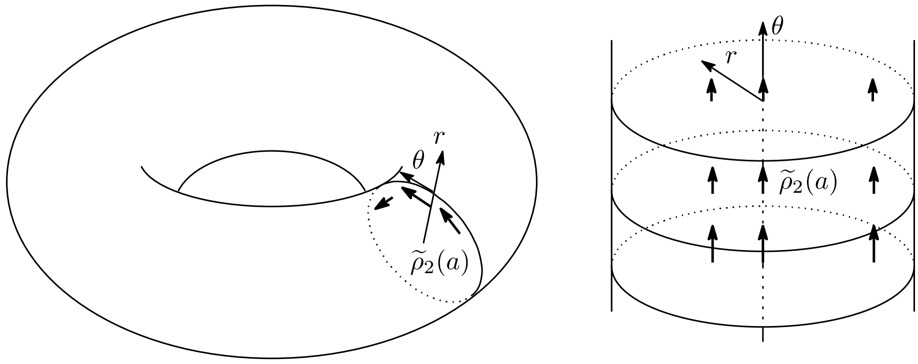

FIGURE 2. extension of $\sigma_{n+1}(a)$

a morphism of Lie algebras. Moreover, if $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $\sigma_{n+1}(a) = \widetilde{\rho}_2(a) = -\dfrac{\partial}{\partial\theta}$. Therefore, the $\mathbb{R}$-action generated by a is periodic and σ_{n+1} induces a group action of $\mathrm{SL}(2;\mathbb{R})\times\mathbb{R}^{n-1}$ on S^n which we denote by $\widetilde{\sigma}_{n+1}$. We will equip the trivial bundle $\mathrm{SL}(2;\mathbb{R})\times\mathbb{R}^{n-1}\times S^n \to \mathrm{SL}(2;\mathbb{R})\times\mathbb{R}^{n-1}$ with a foliation[†6] such that the leaf $\widetilde{L}_{(g,u,w)}$ which passes $(g,u,w)\in \mathrm{SL}(2;\mathbb{R})\times\mathbb{R}^{n-1}\times S^n$ is given by

$$\widetilde{L}_{(g,u,w)} = \{(gh, u+v, \widetilde{\sigma}_{n+1}(h,v)^{-1}w) \mid (h,v)\in \mathrm{SL}(2;\mathbb{R})\times\mathbb{R}^{n-1}\}.$$

Note that $\mathrm{SL}(2;\mathbb{R})\times\mathbb{R}^{n-1}$ acts on $\mathrm{SL}(2;\mathbb{R})\times\mathbb{R}^{n-1}\times S^n$ on the right by $(g,u,w)(h,v) = (gh, u+v, \widetilde{\sigma}_{n+1}(h,v)^{-1}w)$ and on the left by $(h,v)(g,u,w) = (hg, v+u, w)$, respectively. The foliation $\{\widetilde{L}_{(g,u,w)}\}$ is invariant under the both actions. Therefore, by first taking the quotient by $\mathrm{SO}(2)$ on the right, we obtain a foliated S^n-bundle over $\mathbb{H}\times\mathbb{R}^{n-1}$ which is in fact a foliated product as we will explain below. Now let Γ be a cocompact lattice of $\mathrm{SL}(2;\mathbb{R})/\mathrm{SO}(2)$, and take the quotient of $(\mathrm{SL}(2;\mathbb{R})\times\mathbb{R}^{n-1}) \underset{\mathrm{SO}(2)}{\times} S^n \cong \mathbb{H}\times\mathbb{R}^{n-1}\times S^n$ by $\Gamma\times\mathbb{Z}^{n-1}$ on the left. Then we obtain a foliated S^n-bundle over $\Gamma\backslash\mathbb{H}\times T^{n-1}$ of which the total space is $\Gamma\backslash(\mathrm{SL}(2;\mathbb{R})\times T^{n-1}) \underset{\mathrm{SO}(2)}{\times} S^n$. We denote by $\mathcal{F}$ thus obtained foliation.

A trivialization of the foliated S^n-bundle over $\mathbb{H}\times\mathbb{R}^{n-1}$ is given as follows. We denote by $[g,u,w]$ the equivalence class represented by $(g,u,w)\in \mathrm{SL}(2;\mathbb{R})\times\mathbb{R}^{n-1}\times S^n$. Let ι be an embedding of $\mathbb{H}$ into $\mathrm{SL}(2;\mathbb{R})$ given by $\iota(x+\sqrt{-1}y) = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$. We define $F\colon \mathbb{H}\times\mathbb{R}^{n-1}\times S^n \to (\mathrm{SL}(2;\mathbb{R})\times\mathbb{R}^{n-1}) \underset{\mathrm{SO}(2)}{\times} S^n$ by $F(z,u,w) = [\iota(z),u,w]$. Then, F is a diffeomorphism and the leaf L_w of $\mathcal{F}$ which passes $(\sqrt{-1},0,w)\in \mathbb{H}\times\mathbb{R}^{n-1}\times S^n$ is given by

$$L_w = \{(z,u,\widetilde{\sigma}_{n+1}(\iota(z),u)^{-1}w) \mid (z,u)\in\mathbb{H}\times\mathbb{R}^{n-1}\}.$$

[†6]We slightly modified the construction in view of [7], §5.

Let $(z, u) = (x, y, u^3, \dots, u^{n+1})$ be the natural coordinates on $\mathbb{H} \times \mathbb{R}^{n-1}$. Then,

$$m_{(\sqrt{-1},0)}\left(\frac{\partial}{\partial x}\right) = -\sigma_{n+1}(Y),$$
$$m_{(\sqrt{-1},0)}\left(\frac{\partial}{\partial y}\right) = -\sigma_{n+1}(Z),$$
$$m_{(\sqrt{-1},0)}\left(\frac{\partial}{\partial u^i}\right) = -\sigma_{n+1}(t_i),$$

where $3 \le i \le n+1$. In general, $m_{(z,u)} = \widetilde{\sigma}_{n+1}(\iota(z), u)_* m_{(\sqrt{-1},0)}$. On the other hand, if we set $h = \operatorname{div}\left(g \cdot r\frac{\partial}{\partial r}\right) = r\frac{dg}{dr} + 2g$ then

1) $h = 2$ on the image of $S^{n-2} = \{o\} \times S^{n-2}$ in $S^n = (D^2_{1+\epsilon} \times S^{n-2}) \cup (S^1 \times D^{n-1}_{1+\epsilon})$.
2) $h = 0$ on $S^1 \times D^{n-1}_{1+\epsilon} \subset S^n$.

Therefore,

$$\begin{aligned}
&(m^*_{(\sqrt{-1},0)}\beta)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u^3}, \dots, \frac{\partial}{\partial u^{n+1}}\right)\\
&= (-1)^n \left(\int_r \left(1 - \frac{1}{2}h\right)^2 h^{n-2} dh\right)\left(\int_\theta \operatorname{div}\rho_2(Y)\, d(\operatorname{div}\rho_2(Z))\right)\\
&\quad\cdot \left(\int_{S^{n-2}} \sum_{i=3}^{n+1} (-1)^{i-3} f^i df^3 \wedge \cdots \wedge \widehat{df^i} \wedge \cdots \wedge df^{n+1}\right)\\
&= (-1)^n \frac{2^{n+1}\pi}{n(n^2-1)} \int_{S^{n-2}} \widetilde{f}^* \omega_{n-1},
\end{aligned}$$

where $\widetilde{f} = (f^3, \dots, f^{n+1})\colon S^{n-2} \to \mathbb{R}^{n-1}$, $\omega_{n-1} = \sum_{i=1}^{n-1} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n-1}$ and the symbol '$\widehat{\ }$' means omission. Note that if we set

$$V = \int_{S^{n-2}} \widetilde{f}^* \omega_{n-1},$$

then V is a generalization of the volume of the region bounded by $\widetilde{f}(S^{n-2})$. We have

$$\operatorname{gv}(W, \mathcal{F}) = (-1)^n \frac{2^{n+1}\pi\, V}{n(n^2-1)} \int_N \operatorname{vol}_N,$$

where $N = (\Gamma\backslash \mathrm{SL}(2;\mathbb{R})/\mathrm{SO}(2)) \times T^{n-1} = \Sigma \times T^{n-1}$ and vol_N denotes the volume form of N, so that $\operatorname{gv}(W, \mathcal{F})$ attains any value in $\mathbb{R}$ as f_i's vary.

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Remarks on the references. The paper [3] is the original of this translation. The papers [1] and [2] are cited in [3]. The rest is added by the translator.

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