Nonperturbative anomaly inflow for fermions (and p-form fields)

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- [1909.08775] with Edward Witten
- ([2003.11550] with Chang-Tse Hsieh and Yuji Tachikawa)

Basic setup

I will use the language of relativistic quantum field theory.
There are two kinds of fields.

- Dynamical fields
- Background fields

Dynamical fields are integration variables in path integrals.
Background fields are not integrated.

Which fields are dynamical and which fields are background depend on the physical problem of interest.

Basic setup

Example:
A dynamical electron Ψ moving in a background gravitational potential g and electromagnetic field B.

$$Z(g,B) = \int [D\Psi] e^{-S(\Psi,g,B)}$$

- $S(\Psi,g,B)$: functional of (Ψ,g,B) , called action
- $\displaystyle\int [D\Psi]$: integration over Ψ, called path integral
- $Z(g,B)$: functional of (g,B), called partition function

Basic setup

Depending on physical interests,
one may further perform integration over B and/or g;

E.g. $$Z(g) = \int [DB]\, Z(g, B)$$

However, in this talk I will only integrate over fermions (or what is called p–form fields).
(g, B) will be background fields.

Basic setup

Einstein's general relativity says that, roughly speaking

gravity = curved manifolds with metric $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$

Usually the metric g has Lorentz signature $(D-1,1)$ in D–spacetime dimensions.

However, after a formal manipulation called Wick rotation, we can take g to be positive definite.

Basic setup

The partition function is more precisely a functional of the following form. For a dynamical fermion Ψ,

$$Z(X, g, B, \cdots) = \int [D\Psi] e^{-S_X(\Psi, g, B, \cdots)}$$

- X : a D–dimensional compact manifold
- g : a Riemann metric on X
- B : a gauge field for a group G

  (a connection of a G–bundle)
- $\cdots$: possible other background fields

Basic setup

For simplicity of notation,
I will often omit $g, B, \cdots$ and write like

$$Z(X, g, B, \cdots) = Z(X)$$

In other words, the symbol X is supposed to contain all information of the background fields.

Contents

Introduction

(1) Chiral fermions

“Chiral” fermions are physically very important.

- Edge (boundary) modes of topological material, such as quantum Hall systems, topological insulators, and so on.

- All fermions of the standard model of particle physics.

- Various chiral fermions in various dimensions in string theory and M–theory.

Introduction

A chiral fermion is described by the following ingredient:

- D_μ : covariant derivative (coordinates x^μ, $\mu = 1,\cdots,d$)

  (It acts on some Clifford module bundle)

- γ^μ : gamma matrices satisfying Clifford algebra

  $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$

- Chirality operator $\bar{\gamma}$, such that $\bar{\gamma}^2 = 1$ and $\{\bar{\gamma}, \gamma^\mu\} = 0$

  (It is a $\mathbb{Z}_2$-grading of the Clifford module)

Introduction

A chiral fermion χ has a definite eigenvalue under $\bar{\gamma}$ in the sense that under the projection operator

$$P_{\pm} = \frac{1 \pm \bar{\gamma}}{2}$$

We have

$$P_{+}\chi = \chi, \quad P_{-}\chi = 0,$$

The action is

$$S = \int \bar{\chi} \mathscr{D} \chi$$

$\mathscr{D} = (\gamma^{\mu} D_{\mu}) P_{+} = P_{-}(\gamma^{\mu} D_{\mu})$: chiral Dirac operator

Introduction

The meaning of the action is that the partition function on a d-dimensional closed manifold W is given by

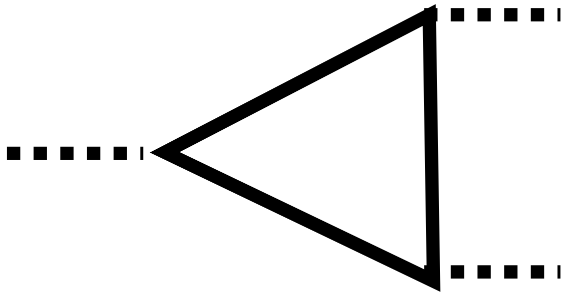
$$Z(W) \sim \mathrm{Det}(\mathscr{D})$$

Its absolute value $|Z(W)|$ can be made well-defined (by some regularization).

But it is difficult to make the phase of $Z(W)$ well-defined. Sometimes there is a obstruction to making $Z(W)$ well-defined (as a number in $\mathbb{C}$) : anomaly.

Introduction

Perturbative case (i.e. topologically trivial bundles);
Well-studied in quantum field theory.



(Feynman diagram leading to anomaly)

$\longrightarrow$ ambiguity in the phase of $Z(W)$

Nonperturbative anomalies were not completely clear, before [Witten,2015] (based on theorems of [Dai-Freed,1994])

Introduction

(2) Chiral (self-dual) p-form dynamical fields

$$a = \frac{1}{p!} a_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

It is, very roughly speaking, a differential p-form.

(A little more precisely, a differential cohomology element)

Field strength (curvature)

$$f = da : (p+1)\text{-form}$$

Introduction

We restrict to spacetime dimension $d = 2p + 2$

“Chiral” means its field strength (curvature) $f = da$ satisfies self-duality equation (in Lorentz signature)

$$\star f = f \quad (\star : \text{Hodge star})$$

It can appear in condensed matter physics, such as the edge of (integer or fractional) quantum Hall system for which $p = 0$: chiral boson.

But probably they are more important in string theory. So I skip it in this workshop.

Introduction

This talk:

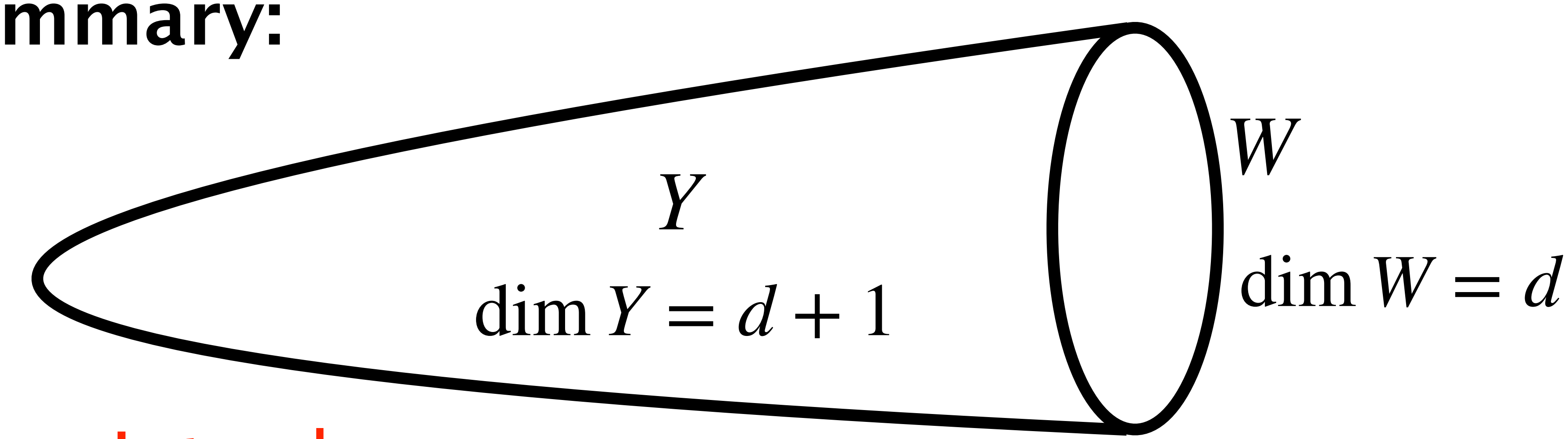
Nonperturbative formulation of chiral fields.

In particular, nonperturbative description of anomalies of chiral fields.

- I need to skip many details. Please see our paper for them.
- I focus on some conceptual (rather than technical) aspects.
- I will not distinguish our contributions and other peoples'.

Introduction

Short summary:



Introduce bulk theory on Y

Anomalous chiral theory on W

- The total system has a well-defined partition function.

- When the bulk theory is trivial, the partition function does not depend on Y. Bulk = Anomaly

Contents

- Introduction
- A formulation of chiral theories
- Anomalies
- Summary

Chiral fermion

First I discuss chiral fermions.

I construct d–dimensional chiral fermions as boundary modes of $(d+1)$–dimensional massive fermions in one–higher dimension.

(In the case of fermions, it is similar to what is called domain wall fermions.)

Massive fermion

Let us consider a massive fermion Ψ in $d+1$-dimensions

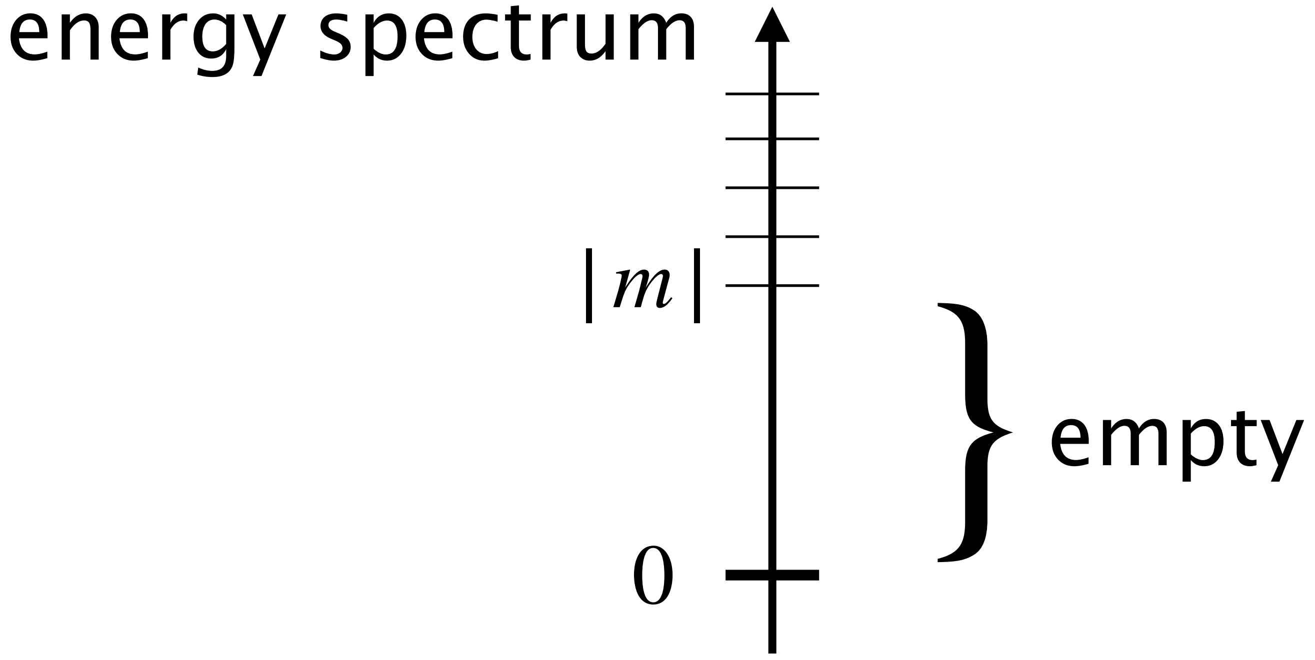
$$S = \int_Y \bar{\Psi}(\gamma^\mu D_\mu + m)\Psi$$

- $m \in \mathbb{R}$: a parameter, called mass parameter
- No chirality operator ($\mathbb{Z}_2$-grading of Clifford module) assumed in $d+1$-dimensions.
- The partition function is roughly (but not precisely)

$$Z(Y) \sim \mathrm{Det}(\gamma^\mu D_\mu + m)$$

Massive fermion

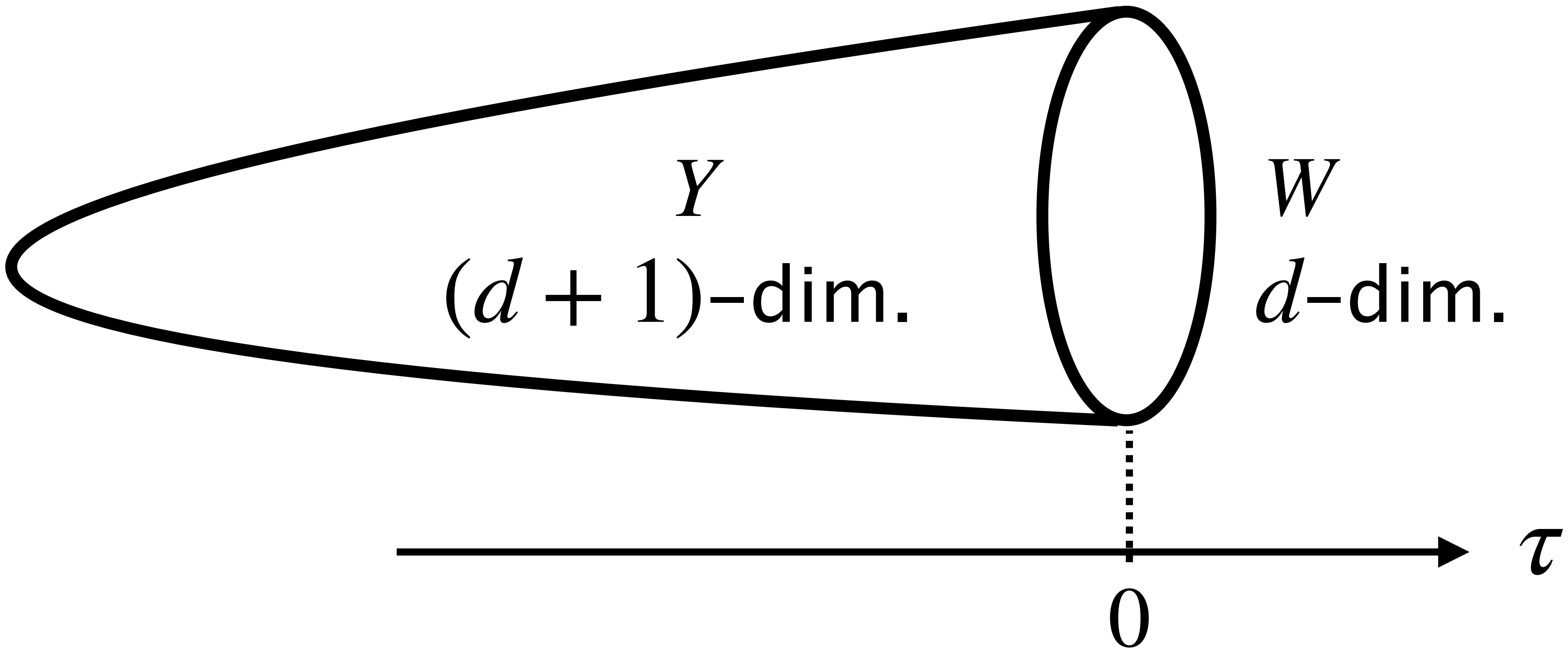
If we take a limit $|m| \to \infty$, the theory is almost empty on a manifold without boundary in the following sense.



However, let us consider it on a manifold with boundary. We will see a localized chiral fermion on the boundary.

Spacetime with boundary

Spacetime Y with boundary W



Near the boundary, the spacetime Y is of the form

$$Y \supset (-\epsilon, 0] \times W$$

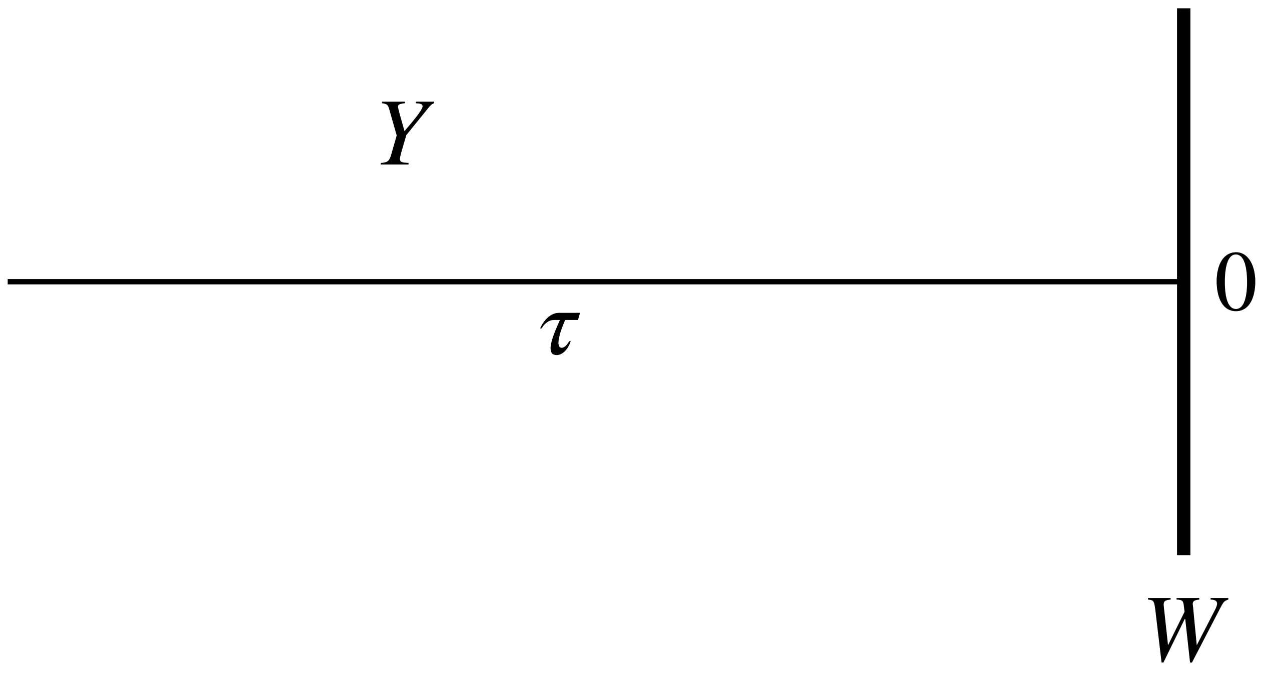
$\tau \in (-\epsilon, 0]$: coordinate orthogonal to the boundary

Spacetime with boundary

Locally, it is just as follows.

$$W = \mathbb{R}^d = \{(x^1, \cdots, x^d)\}$$

$$Y = \mathbb{R}^{d+1}_{-} := \{(\tau, x^1, \cdots, x^d); \quad \tau \leq 0\}$$



The boundary condition

We impose a local boundary condition L defined by

$$\mathsf{L}: \ (1-\gamma^{\tau})\Psi|_{\tau=0} = 0$$

γ^{τ} : gamma matrix in the τ direction

We restrict boundary modes to the eigenspace with eigenvalue $\gamma^{\tau} = +1$

Near the boundary

Near the boundary, the Dirac operator in $d+1$-dim. is represented as

$$\gamma^\mu D_\mu = \gamma^\tau \left(\partial_\tau + \mathscr{D}_W\right) \qquad \left(\partial_\tau = \frac{\partial}{\partial \tau} \right)$$

$$\mathscr{D}_W = \sum_{\mu \neq \tau} \gamma^\tau \gamma^\mu D_\mu$$

Dirac equation

$$\left(\partial_\tau + \mathscr{D}_W + m\gamma^\tau\right) \Psi = 0$$

It admits a localized solution

if the mass parameter is negative: $m < 0$

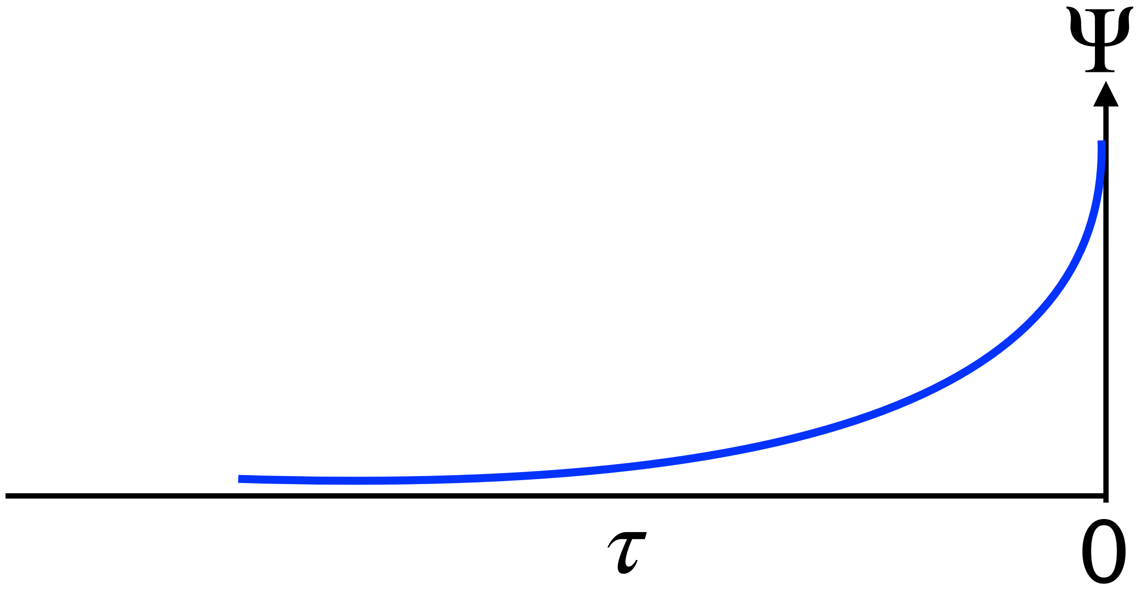
Localized chiral fermion

$$\left(\partial_\tau + \mathscr{D}_W + m\gamma^\tau\right)\Psi = 0, \quad \mathsf{L} : (1-\gamma^\tau)\Psi|_{\tau=0} = 0$$

Normalizable solutions for $m < 0$:

$$\Psi = \chi \exp(-m\tau), \quad \gamma^\tau \chi = \chi, \quad \mathscr{D}_W \chi = 0$$

Recall: $\tau \leq 0$. Exponentially localized near $\tau = 0$



Localized chiral fermion

- The operator $\mathscr{D}_W$ is a Dirac operator on boundary W.

- γ^τ can be regarded as a chirality operator on the boundary W

$$\bar{\gamma} := \gamma^\tau \text{ on } W = \partial Y$$

$$(\gamma^\tau)^2 = 1, \quad \{\gamma^\tau, \gamma^\mu\} = 0 \quad (\mu \neq \tau)$$

(The Clifford moduli bundle restricted to $\partial Y = W$ has a $\mathbb{Z}_2$-grading given by γ^τ)

Localized chiral fermion

χ is a localized chiral fermion:

- Massless Dirac equation on W : $\mathscr{D}_W \chi = 0$
- Chiral fermion : $\gamma^\tau \chi = +\chi$

Remark:

Any physically sensible fermions with spin $\dfrac{1}{2}$ can be realized as a boundary mode in this way.

[Witten–KY,2019]

Chiral p-form

Next I discuss chiral p–form fields.

I construct d–dimensional chiral p–form fields
as boundary modes of $(d+1)$–dimensional
massive (p+1)–form fields in one–higher dimension.

(For p–form fields, it is not possible in the
usual domain wall formulation with varying mass parameter.)

Massive (p+1)-form

Let's consider a massive (p+1)-form A in $d+1$-dimensions

$$S = 2\pi \int_Y \left(-\frac{1}{2e^2} dA \wedge \star dA + \frac{k}{2} A \wedge dA + C \wedge dA \right)$$

- $2p+3 = d+1$ to allow the term $A \wedge dA$
- $k \in \mathbb{Z}$: integer parameter
- $e^2 \in \mathbb{R}_+$: positive parameter
- C : background (p+1)-form field

Massive (p+1)-form field

$$S = 2\pi \int_Y \left(-\frac{1}{2e^2} dA \wedge \star dA + \frac{k}{2} A \wedge dA + C \wedge dA \right)$$

$$Z(Y) = \int [DA] e^{-S} : \quad \text{functional of } (g, C)$$

The field A is massive, with mass given by $|k|e^2$.

If we set $|k| = 1$ and take a limit $e^2 \to \infty$, the theory is almost empty.

(For $|k| > 1$ there is nontrivial topological field theory.)

The boundary condition

Again, we put this theory on a manifold Y with boundary W as in the case of fermions.

We impose a local boundary condition L defined by

$$\mathsf{L} : \; A|_W = 0$$

$A|_W$: restriction of the (p+1)-form field to $W = \partial Y$

Equations of motion

The equation of motion is, for $C = 0$,

$$\frac{1}{e^2} d(\star F) = kF \qquad (F = dA : (p+2)\text{–form})$$

We want to solve the equation with the boundary condition $F|_W = 0$.

Equations of motion

$$\frac{1}{e^2} d(\star F) = kF, \qquad F|_W = 0$$

Ansatz:

$F \propto e^{|k|e^2\tau} d\tau \wedge f$ (exponentially localized)

$f :$ $(p+1)$–form independent of τ

The equation of motion:

$$\star f = \frac{k}{|k|} f, \qquad d(\star f) = 0$$

Localized chiral p-form

$$\star f = \frac{k}{|k|} f, \qquad d(\star f) = 0$$

The first equation is

- Self-dual eq. if $k > 0$
- Anti-self-dual eq. if $k < 0$

The second equation is the usual equation for p-form fields.

We have obtained a chiral theory as a boundary mode of the bulk massive theory.

Definition of chiral theory

Chiral theories are realized on a boundary of massive theories in one-higher dimension with a local boundary condition.

I regard them as a definition of chiral theories in d-dimensions.

Contents

Definition of chiral theory

- Abstractly, let $\mathscr{T}$ be an almost empty theory:
  (invertible field theory)

Its Hilbert space is always one-dimensional on any closed manifold without boundary (in a large mass gap limit).

For example, $\mathscr{T}$ = [massive fermion]

- Suppose $\mathscr{T}$ admits a local boundary condition L.

For example, L : $(\gamma^{\tau} - 1)\Psi|_{W} = 0$

Definition of chiral theory

$Z(Y, \mathsf{L})$: The partition function of the theory $\mathcal{T}$ on a manifold Y with boundary $\partial Y = W$ with the local boundary condition L.

We want to define the chiral theory on W as

$$\text{“}Z_{\text{chiral}}(W) = Z(Y, \mathsf{L})\text{”}$$

This is based on the intuition that all physical modes are localized near the boundary (since bulk is almost empty).

Definition of chiral theory

$$\text{“}Z_{\text{chiral}}(W) = Z(Y, \mathsf{L})\text{”}$$

For this definition to be valid, the right hand side needs to be independent of a choice of Y with $\partial Y = W$.

So we compute the ratio

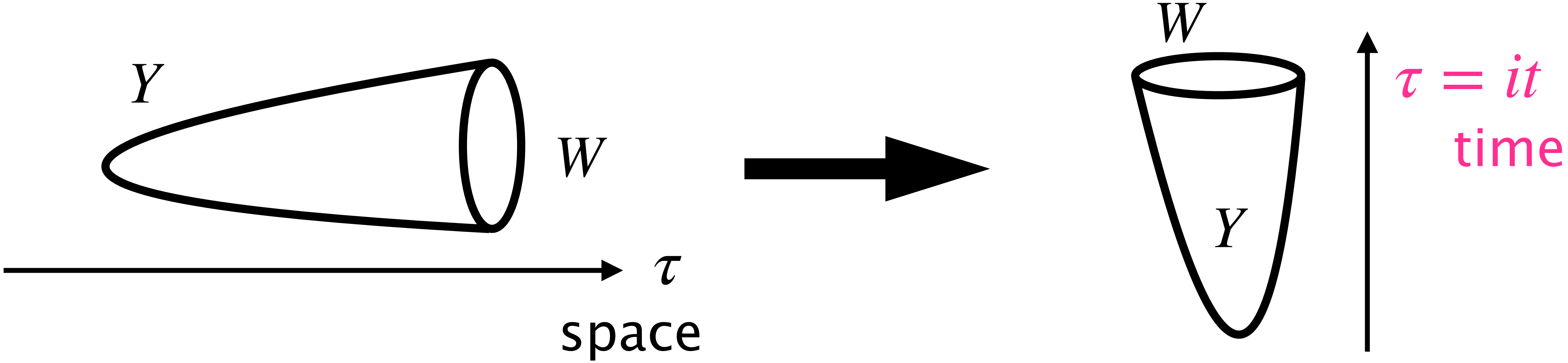
$$\frac{Z(Y, \mathsf{L})}{Z(Y', \mathsf{L})}$$

Y, Y' are manifolds with the same boundary W.

It will turn out to be generically nontrivial, and that is the anomaly.

Wick rotation

For computation, it is crucial to change our point of view :
Perform Wick rotation to see $\tau = it$ as a Wick rotated time.



The right figure can be seen as a transition amplitude from nothing to W. This gives a physical state

$$|Y\rangle \in \mathscr{H}_W \qquad (\mathscr{H}_{\mathscr{W}} : \text{Hilbert space on } W\,)$$

Local b.c. as a physical state

The local boundary condition L can also be seen as defining some physical state.

L : spatial boundary condition

↓ The change of viewpoint

$$|\mathsf{L}\rangle \in \mathscr{H}_W$$

For example,

$$\langle \mathsf{L} | (1-\gamma^{\tau})\Psi = 0$$

Z(Y,L) as state overlap

The partition function with the boundary condition L is now computed as a state overlap

$$\boxed{\begin{aligned} Z(Y, \mathsf{L}) &= \int [D\Psi] e^{-S} \\ &= \langle \mathsf{L} \,|\, Y \rangle \end{aligned}}$$

$|\, Y\rangle$: obtained by the bulk path integral

$\langle L\,|$: boundary condition

Splitting bulk and boundary

Recall that the theory $\mathscr{T}$ has only the ground state $|\Omega\rangle$ on closed manifolds. W is a closed manifold. Thus

$$\mathbf{1} = |\Omega\rangle\langle\Omega| \quad \text{(the complete set of states)}$$

So we get

$$\begin{aligned} Z(Y, \mathsf{L}) &= \langle \mathsf{L} | Y \rangle \\ &= \underbrace{\langle \mathsf{L} | \Omega \rangle}_{\text{Determined by boundary } W} \underbrace{\langle \Omega | Y \rangle}_{\text{Determined by bulk } Y} \end{aligned}$$

Computation of ratio

Now the ratio is computed as

$$\frac{Z(Y, \mathsf{L})}{Z(Y', \mathsf{L})} = \frac{\langle \mathsf{L} | \Omega \rangle \langle \Omega | Y \rangle}{\langle \mathsf{L} | \Omega \rangle \langle \Omega | Y' \rangle} = \frac{\langle \Omega | Y \rangle}{\langle \Omega | Y' \rangle}$$

I claim that (up to a local counterterm)

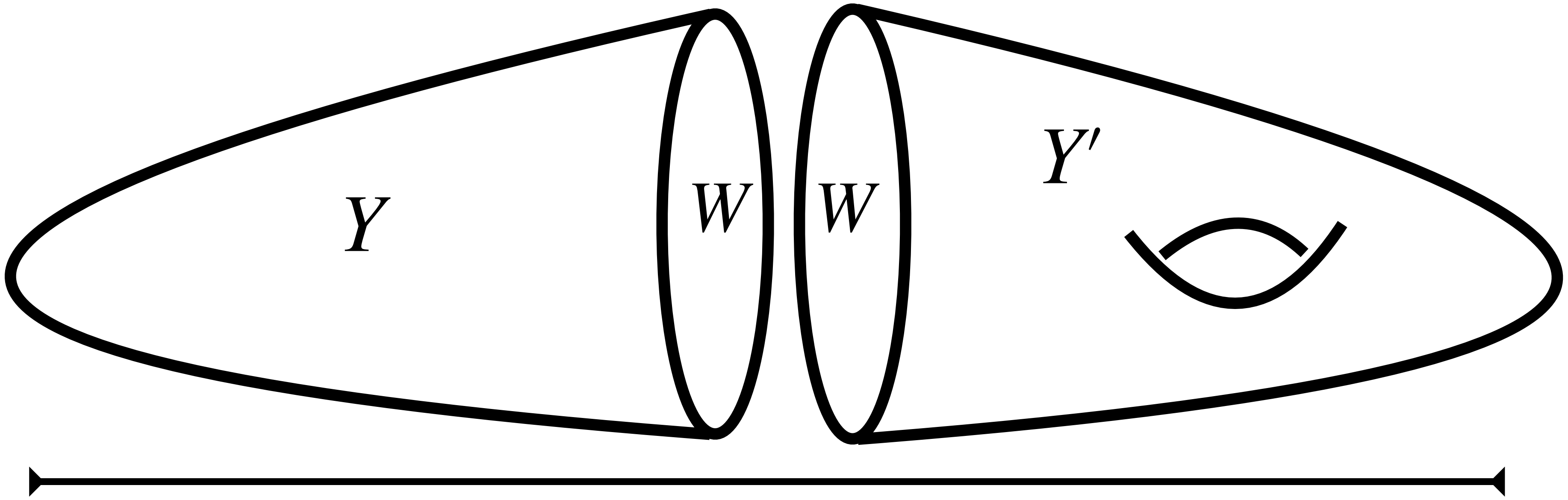
$$| \langle \Omega | Y \rangle |^2 = 1$$

Accepting this, we get

$$\frac{\langle \Omega | Y \rangle}{\langle \Omega | Y' \rangle} = \langle Y' | \Omega \rangle \langle \Omega | Y \rangle = \langle Y' | Y \rangle = Z(Y_{\text{closed}})$$

Y_{closed} : closed manifold obtained by gluing Y and Y'

Computation of ratio



Glue them together along the boundary :

closed manifold Y_{closed}

$\langle Y' | Y \rangle = Z(Y_{\text{closed}})$:

The partition function of the theory $\mathcal{T}$ on Y_{closed}

Anomaly

The obstruction to defining $Z_{\text{chiral}}(W) := Z(Y, \mathsf{L})$

$$\frac{Z(Y, \mathsf{L})}{Z(Y', \mathsf{L})} = Z(Y_{\text{closed}})$$: bulk theory partition function

This is the nonperturbative description of anomalies.

No Anomaly $\iff$ $Z(Y_{\text{closed}}) = 1$ for any Y_{closed} (up to local counterterm)

(See [Witten–KY,2019] for more details.)

Example

A $d+1=4$ fermion with time reversal symmetry.

(Time reversal = it can be put on non-orientable manifold)

As an example, take $Y_{\text{closed}} = \mathbb{RP}^4$

$$Z(\mathbb{RP}^4) = \exp\left(2\pi i \cdot \frac{N}{16}\right)$$

[Witten,2015]
[Hsieh-Cho-Ryu,2015]

$N \in \mathbb{N}$: the number of fermions

- $\mathbb{Z}_{16}$ classification of $4=3+1$-dimensional topological superconductors.
- Boundary modes: anomalous "chiral" fermions

Contents

Summary

- d-dim. chiral theory is formulated by $(d+1)$-dim. bulk theory by imposing a local boundary condition on manifolds with boundary.

- The anomaly of the d-dim. theory is the partition function of the $(d+1)$-dim. bulk theory on closed manifolds without boundary.