Introduction to differential cohomology

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Motivation : Topological terms in Lagrangans

Differential cohomology is a mathematical framework which refines generalized cohomology with differential geometric data on manifolds. They are deeply related with physics. See [FMS07], [Fre00], [HS05] and [HTY20] for example.
For mathematical accounts, see [BS12] and [Bun12] for example.

Differential cohomology accounts for "topological terms" in Lagrangians in physics. Examples of "topological terms" are,

- Holonomy for $U(1)$-connections,
- Chern-Simons invariants,
- Wess-Zumino-Witten terms,
- Reduced eta invariants.

Mathematically, they are called secondary invariants.

Let us look at the following examples of "topological terms".

- Holonomy for $U(1)$-connection.
  Let $(L, \nabla) \to X$ be a hermitian line bundle with $U(1)$-connection over a manifold. For a closed curve $f\colon S^1 \to X$ in X, we get its *holonomy* $\mathrm{Hol}(L, \nabla)(f) \in \mathbb{R}/\mathbb{Z}$.
  If L is trivialized and $\nabla = d + A$ for $A \in \Omega^1(X; \sqrt{-1}\mathbb{R})$, we have
  $$\mathrm{Hol}(L, \nabla)(f) = \int_{S^1} f^* \frac{A}{2\pi\sqrt{-1}} \pmod{\mathbb{Z}}.$$
- Chern-Simons invariants.
  Let $(E, \nabla) \to X$ be a hermitian vector bundle with connection. For $f\colon M^3 \to X$ with M : 3-dimensional closed oriented manifold, we get its *Chern-Simons invariant* $\mathrm{CS}(E, \nabla)(f) \in \mathbb{R}/\mathbb{Z}$.
  If E is trivialized and $\nabla = d + A$ for $A \in \Omega^1(X; \mathfrak{u}(n))$, we have
  $$\mathrm{CS}(E, \nabla)(f) = \int_M f^* \frac{\mathrm{Tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A)}{4\pi} \pmod{\mathbb{Z}}.$$

Poroperties of “topological terms”

(A) They are expressed as

$$\int_{M^{n-1}} f^*\alpha \quad (\mathrm{mod}\ \mathbb{Z})$$

for some $\alpha \in \Omega^{n-1}(X)/\mathrm{im}(d)$ when the topology is trivial. But in the presence of nontrivial topology, they CANNOT be expressed by differential forms.

$$\mathrm{Hol}(L, \nabla)(f) = \int_{S^1} f^* \frac{A}{2\pi\sqrt{-1}} \quad (\mathrm{mod}\ \mathbb{Z}),$$

$$\mathrm{CS}(E, \nabla)(f) = \int_M f^* \frac{\mathrm{Tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A)}{4\pi} \quad (\mathrm{mod}\ \mathbb{Z}).$$

But for general X, we cannot take such A globally.

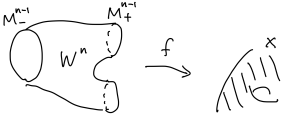
Problem

What is the object $\widehat{x}$ giving the topological terms by “$\int_{M^{n-1}} f^*\widehat{x}$” for general X? Where does it live?

(B) They are NOT topological invariants. Rather, they depend on the geometry. The variation under bordisms is measured by

$$\int_{W^n} f^* R(\widehat{x}) \pmod{\mathbb{Z}}$$

for some $R(\widehat{x}) \in \Omega^n_{\mathrm{clo}}(X)$ ("field strength").

If we have a bordism like 

, we have

$$\mathrm{Hol}(L, \nabla)(f|_{M^1_+}) - \mathrm{Hol}(L, \nabla)(f|_{M^1_-}) = \int_{W^2} f^* \frac{F_\nabla}{2\pi\sqrt{-1}},$$

$$\mathrm{CS}(E, \nabla)(f|_{M^3_+}) - \mathrm{CS}(E, \nabla)(f|_{M^3_-}) = \int_{W^4} f^* \mathrm{ch}_2(F_\nabla).$$

Moreover, when the topology is trivial, we have $R(\alpha) = d\alpha$.

(C) The “field strength” $R(\widehat{x})$ is integral ($R(\widehat{x}) \in \Omega^n_{\mathrm{clo}}(X)_{\mathbb{Z}}$), i.e., for all $f \colon W^n \to X$ where W is oriented and closed (compact without boundary), we have

$$\int_{W^n} f^* R(\widehat{x}) \in \mathbb{Z}.$$

Actually this follows from (B).
Called “Dirac charge quantization”.

(D) If we know the value "$\int_{M^{n-1}} f^* \widehat{x}$" for all $f \colon M^{n-1} \to X$, we can recover the topology, including torsions.

Indeed,

- The collection of values of $\mathrm{Hol}(L, \nabla)(f)$ for all f recovers L up to isomorphism (i.e., $c_1(L) \in H^2(X; \mathbb{Z})$), not just $c_1(F_\nabla) \in H^2(X; \mathbb{R})$.
- The collection of values of $\mathrm{CS}(E, \nabla)(f)$ for all f recovers $\mathrm{ch}_2(E) \in H^4(X; \mathbb{Z})$, not just $\mathrm{ch}_2(F_\nabla) \in H^4(X; \mathbb{R})$.

The properties (A) – (D) suggests that, $\widehat{x}$ is an element in a group ?? fitting into a commutative diagram like

$$\begin{tikzcd}
 & & & & 0 \\
 & & H^n(X;\mathbb{Z}) \arrow[urr] \arrow[dr, "\otimes\mathbb{R}"] & & \\
 & ?? \arrow[ur, "I"] \arrow[dr, "R"] & & H^n(X;\mathbb{R}) & \\
\Omega^{n-1}(X) \arrow[ur, "a"] \arrow[rr, "d"] & & \Omega^n_{\mathrm{clo}}(X)_{\mathbb{Z}} \arrow[ur, "\mathrm{Rham}"] & &
\end{tikzcd}$$

where the diagonal sequence is exact.

- (A) corresponds to a and the exactness at ?? (Trivial topology implies $\widehat{x}$ comes from $\Omega^{n-1}(X)$).
- (B) and (C) correspond to R.
- (D) corresponds to I.

The answer : differential cohomology

Actually, the ordinary differential cohomology $\widehat{H}^n(X;\mathbb{Z})$ is such a group. We have the ordinary differential cohomology hexagon

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{n-1}(X;\mathbb{R}/\mathbb{Z}) \arrow[rr, "\mathrm{Bock}"] \arrow[dr] & & H^n(X;\mathbb{Z}) \arrow[ur] \arrow[dr, "\otimes\mathbb{R}"] & \\
H^{n-1}(X;\mathbb{R}) \arrow[ur] \arrow[dr] & & \widehat{H}^n(X;\mathbb{Z}) \arrow[ur, "I"] \arrow[dr, "R"] & & H^n(X;\mathbb{R}) \\
& \Omega^{n-1}(X)/\Omega^{n-1}_{\mathrm{clo}}(X)_{\mathbb{Z}} \arrow[rr, "d"] \arrow[ur, "a"] & & \Omega^n_{\mathrm{clo}}(X)_{\mathbb{Z}} \arrow[ur, "\mathrm{Rham}"] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

which is commutative and diagonal sequences are exact. We have "higher holonomy function" for oriented closed $(n-1)$-dimensional manifolds,

$$\int_M : \widehat{H}^n(M^{n-1};\mathbb{Z}) \to \mathbb{R}/\mathbb{Z},$$

which satisfy all the required properties.

The main message of these lectures are,

The answer to Problem 1

We can interpret $\widehat{x}$ as an element in (generalized) differential cohomology theories $\widehat{E}^*(X)$.
The "topological terms" are interpreted as the images of integration maps in differential cohomology.

In the examples of Hol and CS, we use $E = H\mathbb{Z}$.
But for some cases we should use other cohomology theories such as $E = K, KO$. The choices correspond to different "charge quantization conditions".

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   - Differential characters
   - Differential cocycles

3. Differential *K*-theory
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Hermitian line bundles

Recall the following classical fact.

Theorem

For any CW-complex X, we have

$$H^2(X;\mathbb{Z}) \simeq \{L \to X : \textit{Hermitian line bundle}\}/\sim_{\mathrm{isom}}$$

The corresponding class $c_1(L) \in H^2(X;\mathbb{Z})$ is called the *first Chern class*.

Connections and Curvatures

Given $L \to X$, how do we detect $c_1(L) \in H^2(X;\mathbb{Z})$? One way is to take a connection.

Assume X is a (smooth) manifold. Take a $U(1)$-connection ∇ on L (locally, $\nabla = d + A$ for $A \in \Omega^1(X;\sqrt{-1}\mathbb{R})$).

The curvature is $F_\nabla := \nabla^2 \in \Omega^2_{\mathrm{clo}}(X;\sqrt{-1}\mathbb{R})$ (locally, $F_\nabla = dA$).

We have

$$c_1(L)_\mathbb{R} = c_1(F_\nabla) := \frac{1}{2\pi\sqrt{-1}}[F_\nabla] \in H^2(X;\mathbb{R}).$$

Here $c_1(L)_\mathbb{R}$ is the image of $c_1(F_\nabla)$ under the $\mathbb{R}$-ification $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R})$. I.e., the curvature recovers $c_1(L)$ *up to torsion*.

In particular,

$$c_1(F_\nabla) \in \Omega^2_{\mathrm{clo}}(X)_\mathbb{Z} \text{ (closed forms with } \mathbb{Z}\text{-periods).}$$

Physically : "Dirac charge quantization".

Flat line bundles

However, there are nontrivial line bundles which cannot be detected by the curvature : flat ones.
Example : $X = \mathbb{RP}^2 = S^2/\mathbb{Z}_2$. The trivial bundle $\mathbb{C} \times S^2 \to S^2$ admits a $\mathbb{Z}_2$-action by $-(z, x) \mapsto (-z, -x)$, preserving the trivial connection d.
Taking quotient we get $L \to \mathbb{RP}^2$ with a flat connection ∇.
L is nontrivial : $c_1(L) = -1 \in H^2(\mathbb{RP}^2; \mathbb{Z}) \simeq \mathbb{Z}_2$.
The nontriviality is detected by the holonomy. $\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}_2$ and the holonomy of (L, ∇) gives the nontrivial element

$$\mathrm{Hol}(L, \nabla) \in \mathrm{Hom}(\pi_1(\mathbb{RP}^2), U(1)) \simeq \mathbb{Z}_2.$$

Holonomy

Holonomy $\mathrm{Hol}(L,\nabla)$ remembers the isomorphism (=gauge equivalence) class of (L,∇).
Fix an orientation on S^1. Holonomy function for (L,∇) :

$$\mathrm{Hol}(L,\nabla)\colon C^\infty(S^1,X)\to U(1)$$

In the case $\nabla = d + A$ we have $\mathrm{Hol}(L,\nabla)(f) = \exp(\int_{S^1} f^*A)$.

Theorem

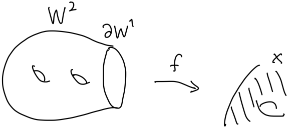
Assume we have (L_1,∇_1) and (L_2,∇_2) on X. We have

$$\mathrm{Hol}(L_1,\nabla_1) = \mathrm{Hol}(L_2,\nabla_2) \Rightarrow (L_1,\nabla_1)\simeq(L_2,\nabla_2).$$

In particular, $\mathrm{Hol}(L,\nabla)$ remembers $c_1(L)\in H^2(X;\mathbb{Z})$ completely.

Characterization of holonomy

Holonomy functions cannot be arbitrary maps $C^\infty(S^1, X) \to \mathbb{R}/\mathbb{Z}$. What is the condition?

If we have 

, we have

$$\mathrm{Hol}(L, \nabla)(f|_{\partial W}) \equiv \int_W f^* c_1(F_\nabla) \pmod{\mathbb{Z}}.$$

Conversely, the equation

$$\varphi(f|_{\partial W}) \equiv \int_W f^* \omega \pmod{\mathbb{Z}}$$

can be regarded as a *compatibility condition* for a pair (ω, φ) consisting of $\omega \in \Omega^2_{\mathrm{clo}}(X)$ and $\varphi\colon C^\infty(S^1, X) \to \mathbb{R}/\mathbb{Z}$. $\mathrm{Hol}(L, \nabla)$ should arise as φ for such a pair.

The first definition of $\widehat{H}^2(X;\mathbb{Z})$: Geometric model

Let X be a manifold. Let us define the *geometric model* of $\widehat{H}^2(X;\mathbb{Z})$ by

Definition

$$\widehat{H}^2_{\mathrm{geom}}(X;\mathbb{Z})$$
$$:= \{(L,\nabla) \to X : \text{Hermitian line bundle with } U(1)\text{-connection}\}/\sim_{\mathrm{isom}}$$

We define *structure maps*

$$R\colon \widehat{H}^2_{\mathrm{geom}}(X;\mathbb{Z}) \to \Omega^2_{\mathrm{clo}}(X), \quad [L,\nabla] \mapsto \frac{1}{2\pi\sqrt{-1}}F_\nabla$$
$$I\colon \widehat{H}^2_{\mathrm{geom}}(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}), \quad [L,\nabla] \mapsto c_1(L)$$
$$a\colon \Omega^1(X)/\mathrm{im}(d) \to \widehat{H}^2_{\mathrm{geom}}(X;\mathbb{Z}), \quad \alpha \mapsto [X\times\mathbb{C}, d+2\pi\sqrt{-1}\alpha].$$

For a smooth map $\phi\colon X \to Y$ between manifolds, we get the *pullback*

$$\phi^*\colon \widehat{H}^2_{\mathrm{geom}}(Y;\mathbb{Z}) \to \widehat{H}^2_{\mathrm{geom}}(X;\mathbb{Z}), \quad [L,\nabla] \mapsto [\phi^*L, \phi^*\nabla].$$

The hexagon for $\widehat{H}^2_{\mathrm{geom}}$

We get the commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^1(X;\mathbb{R}/\mathbb{Z}) \arrow[rr, "\mathrm{Bock}"] \arrow[dr] & & H^2(X;\mathbb{Z}) \arrow[ur] \arrow[dr, "\otimes\mathbb{R}"] & \\
H^1(X;\mathbb{R}) \arrow[ur] \arrow[dr] & & {\color{red}\widehat{H}^2(X;\mathbb{Z})} \arrow[ur, "{\color{red}I}"] \arrow[dr, "{\color{red}R}"] & & H^2(X;\mathbb{R}) \\
& \Omega^1(X)/\Omega^1_{\mathrm{clo}}(X)_{\mathbb{Z}} \arrow[rr, "d"] \arrow[ur, "{\color{red}a}"] & & \Omega^2_{\mathrm{clo}}(X)_{\mathbb{Z}} \arrow[ur, "\mathrm{Rham}"] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

The diagonal sequences are exact.

This implies that $\left(\widehat{H}^2_{\mathrm{geom}}(-;\mathbb{Z}), R, I, a\right)$ is a *differential extension* of $H^2(-;\mathbb{Z})$.

Pros and cons of $\widehat{H}^2_{\mathrm{geom}}(X;\mathbb{Z})$

Advantage :

- Intuitive.

Disadvantage :

- Hard to analyze directly.
- Difficult to generalize to $\widehat{H}^n_{\mathrm{geom}}(X;\mathbb{Z})$.

We seek for alternative definitions.

The second definition of $\widehat{H}^2(X;\mathbb{Z})$: Cheeger-Simons' model

Let us abstractize the property of the pair of curvature and holonomy as follows.

Definition (Second differential characters [CS85])

A *second differential character* on X is a pair (ω, φ) consisting of

- A closed 2-form $\omega \in \Omega^2_{\mathrm{clo}}(X)$,
- A group homomorphism $\varphi \colon Z_{\infty,1}(X;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$,

such that, for any $c \in C_{\infty,2}(X;\mathbb{Z})$ we have

$$\varphi(\partial c) \equiv \int_c \omega \pmod{\mathbb{Z}}. \tag{6}$$

Here $C_{\infty,n}$ and $Z_{\infty,n}$ is the group of smooth singular chains and cochains (a slight generalization of "oriented M^n with $f\colon M \to X$ with/without boundaries")
(6) automatically implies $\omega \in \Omega^2_{\mathrm{clo}}(X)_{\mathbb{Z}}$. (why?)

Definition (The Cheeger-Simons' model [CS85])

Let us define

$$\widehat{H}^2_{\mathrm{CS}}(X;\mathbb{Z}) := \{(\omega,\varphi) : \text{second differential character on } X\}.$$

Theorem

We have an isomorphism

$$\widehat{H}^2_{\mathrm{geom}}(X;\mathbb{Z}) \simeq \widehat{H}^2_{\mathrm{CS}}(X;\mathbb{Z}),$$

by mapping $[L,\nabla]$ *to* $(c_1(F_\nabla), \mathrm{Hol}(L,\nabla))$.

The first definition of $\widehat{H\mathbb{Z}}^*$: Differential characters

The definition of $\widehat{H}^2_{\mathrm{CS}}(X;\mathbb{Z})$ easily generalize as follows.

Definition (The Cheeger-Simons' model [CS85])

Let n be a nonnegative integer. An n-th *differential character* on X is a pair (ω, φ) consisting of

- A closed n-form $\omega \in \Omega^n_{\mathrm{clo}}(X)$,
- A group homomorphism $\varphi\colon Z_{\infty,n-1}(X;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$,

such that, for any $c \in C_{\infty,n}(X;\mathbb{Z})$ we have

$$\varphi(\partial c) \equiv \int_c \omega \pmod{\mathbb{Z}}.$$

Definition (The Cheeger-Simons' model [CS85])

Let us define

$$\widehat{H}^n_{\mathrm{CS}}(X;\mathbb{Z}) := \{(\omega, \varphi) : n\text{-th differential character on } X\}.$$

Structure maps

For a smooth map $\phi \colon X \to Y$ between manifolds, we get the *pullback*

$$\phi^* \colon \widehat{H}^n_{\mathrm{CS}}(Y;\mathbb{Z}) \to \widehat{H}^n_{\mathrm{CS}}(X;\mathbb{Z}), \quad (\omega, \varphi) \mapsto (\phi^*\omega, \phi^*\varphi).$$

We define *structure maps*

$$\begin{aligned} R \colon \widehat{H}^n_{\mathrm{CS}}(X;\mathbb{Z}) &\to \Omega^n_{\mathrm{clo}}(X), \quad (\omega, \varphi) \mapsto \omega \\ I \colon \widehat{H}^n_{\mathrm{CS}}(X;\mathbb{Z}) &\to H^n(X;\mathbb{Z}), \quad (\omega, \varphi) \mapsto [\omega - \varphi_{\mathbb{R}} \circ \partial] \\ a \colon \Omega^{n-1}(X)/\mathrm{im}(d) &\to \widehat{H}^n_{\mathrm{CS}}(X;\mathbb{Z}), \quad \alpha \mapsto (d\alpha, \int \alpha \pmod{\mathbb{Z}}). \end{aligned}$$

Here $\varphi_{\mathbb{R}}$ is any $\mathbb{R}$-valued lift of φ.
(Check : I is well-defined.)

The hexagon for $\widehat{H}^*_{\mathrm{CS}}$

We get the commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{n-1}(X;\mathbb{R}/\mathbb{Z}) \arrow[rr, "\mathrm{Bock}"] \arrow[dr] & & H^n(X;\mathbb{Z}) \arrow[ur] \arrow[dr, "\otimes\mathbb{R}"] & \\
H^{n-1}(X;\mathbb{R}) \arrow[ur] \arrow[dr] & & \widehat{H}^n_{\mathrm{CS}}(X;\mathbb{Z}) \arrow[ur, "I"] \arrow[dr, "R"] & & H^n(X;\mathbb{R}) \\
& \Omega^{n-1}(X)/\Omega^{n-1}_{\mathrm{clo}}(X)_{\mathbb{Z}} \arrow[rr, "d"] \arrow[ur, "a"] & & \Omega^n_{\mathrm{clo}}(X)_{\mathbb{Z}} \arrow[ur, "\mathrm{Rham}"] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

The diagonal sequences are exact.

This implies that $\left(\widehat{H}^*_{\mathrm{CS}}(-;\mathbb{Z}), R, I, a\right)$ is a *differential extension* of $H^*(-;\mathbb{Z})$.

Exercises

$\widehat{H}^n(\mathrm{pt};\mathbb{Z})$ are :

$$\begin{aligned}
&\widehat{H}^0(\mathrm{pt};\mathbb{Z}) = H^0(\mathrm{pt};\mathbb{Z}) \simeq \mathbb{Z},\\
&\widehat{H}^1(\mathrm{pt};\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z},\\
&\widehat{H}^n(\mathrm{pt};\mathbb{Z}) = 0 \ (n \geq 2).
\end{aligned}$$

We have

$$\begin{aligned}
&\widehat{H}^0(X;\mathbb{Z}) = H^0(\mathrm{pt};\mathbb{Z}),\\
&\widehat{H}^1(X;\mathbb{Z}) \simeq C^\infty(X,\mathbb{R}/\mathbb{Z}).
\end{aligned}$$

The higher holonomy function

M^{n-1} : closed oriented $(n-1)$-dimensional manifold. We define the *higher holonomy function* denoted by $\int_M$,

$$\int_M : \widehat{H}^n_{\mathrm{CS}}(M;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}, \quad (\omega, \varphi) \mapsto \varphi(\mathrm{id}\colon M \to M).$$

Note that $\widehat{H}^1_{\mathrm{CS}}(M;\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$, so it is like *integration*.
One important property is :

Proposition (The Bordism formula)

Suppose $(W^n, \partial W)$ is an oriented compact n-dimensional manifold. For any $\widehat{x} \in \widehat{H}^n(W;\mathbb{Z})$, we have

$$\int_{\partial W} \widehat{x}|_{\partial W} \equiv \int_W R(\widehat{x}) \pmod{\mathbb{Z}}$$

This is clear by the definition of differential characters.

Actually, for fiber bundle $p\colon N \to X$ whose fibers are oriented and closed manifold, we can define the *differential integration map*

$$\int_{N/X} \colon \widehat{H}^n(N;\mathbb{Z}) \to \widehat{H}^{n-r}(X;\mathbb{Z}), \tag{12}$$

where $r = \dim N - \dim X$.

Differential integration is a refinement of integrations in $H\mathbb{Z}^*$ and Ω^* in the sense that the following diagram commutes.

$$\begin{tikzcd}
\Omega^{n-1}(N)/\mathrm{im}(d) \arrow[r, "a"] \arrow[d, "\int_{N/X}"] & \widehat{H}^n(N;\mathbb{Z}) \arrow[r, "I"] \arrow[d, "\int_{N/X}"] \arrow[rr, bend left, "R"] & H^n(N;\mathbb{Z}) \arrow[d, "\int_{N/X}"] & \Omega^n_{\mathrm{clo}}(N) \arrow[d, "\int_{N/X}"] \\
\Omega^{n-r-1}(X)/\mathrm{im}(d) \arrow[r, "a"] & \widehat{H}^{n-r}(X;\mathbb{Z}) \arrow[r, "I"] \arrow[rr, bend right, "R"'] & H^{n-r}(X;\mathbb{Z}) & \Omega^{n-r}_{\mathrm{clo}}(X)
\end{tikzcd}$$

.

Application : Chern-Simons invariants

An example of differential character is constructed from Chern-Simons invariants. The basic setting ($G = U(n)$) is :
Let $(E, \nabla) \to X$ be a hermitian vector bundle with connection. For $f\colon M^3 \to X$ with M : 3-dimensional closed oriented manifold, set

$$\begin{aligned}\mathrm{CS}(E, \nabla)(f\colon M \to X) &:= \mathrm{CS}(f^*E, f^*\nabla) \\ &= \int_M f^*\mathrm{Tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A) \pmod{\mathbb{Z}}.\end{aligned}$$

Here $\mathrm{CS}(f^*E, f^*\nabla) \in \mathbb{R}/\mathbb{Z}$ is the *Chern-Simons invariant*.
The second Chern character form is

$$\mathrm{ch}_2(F_\nabla) = \mathrm{Tr}((dA \wedge A + A \wedge A)^2) \in \Omega^4_{\mathrm{clo}}(X).$$

We get

$$(\mathrm{ch}_2(F_\nabla), \mathrm{CS}(E, \nabla)) \in \widehat{H}^4_{\mathrm{CS}}(X; \mathbb{Z}).$$

Definition of the Chern-Simons invariants

Actually, the definition of the Chern-Simons invariants uses $\widehat{H\mathbb{Z}}^*$.
Generally, take a compact Lie group G (gauge group).
Fix $n \in 2\mathbb{Z}$ and $\lambda \in H^n(BG;\mathbb{Z})$: the *level* (If G is simple and simply connected, $H^4(BG;\mathbb{Z}) \simeq \mathbb{Z}$).
The characteristic polynomial for $\lambda \in H^n(BG;\mathbb{Z})$ is its $\mathbb{R}$-ification,

$$\lambda_{\mathbb{R}} \in H^n(BG;\mathbb{R}) \simeq (\mathrm{Sym}^{n/2}\mathfrak{g}^*)^G.$$

Let $(P,\nabla) \to X$ be a principal G-bundle with connection.
The characteristic form associated to $\lambda_{\mathbb{R}}$ is

$$\lambda_{\mathbb{R}}(F_\nabla) \in \Omega^n_{\mathrm{clo}}(X).$$

Choose a *n-classifying manifold for G-connection $B^n_\nabla G$* (appropriate approximation of BG by manifold with "universal connection" ∇_{univ}). There exists a unique element $\widehat{\lambda} \in \widehat{H}^n(B^n_\nabla G;\mathbb{Z})$ such that

$$\begin{aligned} I(\widehat{\lambda}) &= \lambda \in H^n(B^n_\nabla G;\mathbb{Z}) \simeq H^n(BG;\mathbb{Z}), \\ R(\widehat{\lambda}) &= \lambda_{\mathbb{R}}(F_{\nabla_{\mathrm{univ}}}). \end{aligned}$$

(Why? Hint : use $n \in 2\mathbb{Z}$.)
Let $(P,\nabla) \to M^{n-1}$ be a principal G-bundle with connection with closed oriented M. Take a classifying map $f\colon M \to B^n_\nabla G$ of (P,∇).

Definition (The Chern-Simons invariant)

The *Chern-Simons invariant with level λ* of (P,∇) is

$$\mathrm{CS}_\lambda(P,\nabla) := \int_{M^{n-1}} f^*\widehat{\lambda} \in \mathbb{R}/\mathbb{Z}. \tag{14}$$

(14) does not depend on the choice of $B^n_\nabla G$.

Let $(P, \nabla) \to X$ be a principal G-bundle with connection.
For $f \colon M^{n-1} \to X$ with M : $(n-1)$-dimensional closed oriented manifold, set

$$\mathrm{CS}_\lambda(P, \nabla)(f \colon M \to X) := \mathrm{CS}_\lambda(f^* P, f^* \nabla).$$

Proposition

We get an element

$$(\lambda_{\mathbb{R}}(F_\nabla), \mathrm{CS}_\lambda(P, \nabla)) \in \widehat{H}^n_{\mathrm{CS}}(X; \mathbb{Z}).$$

It satisfies ($f \colon X \to BG$: a classifying map for P)

$$f^* \lambda = I\left(\lambda_{\mathbb{R}}(F_\nabla), \mathrm{CS}_\lambda(P, \nabla)\right) \in H^n(X; \mathbb{Z}).$$

Pros and cons of the Cheeger-Simons model

Advantages :

- More algebraic than $\widehat{H}^2_{\mathrm{geom}}$.
- The higher holonomy can be directly evaluated.

Disadvantages :

- Not realized in terms of cochain complexes (as opposed to H^*_{dR}, H^*_{sing}...).
  For example, what is the "trivialization" of a differential character? (c.f., We can talk about trivializations of (L, ∇).)
- Does not generalize to other cohomology theories (actually, the *Anderson self-duality* of $H\mathbb{Z}$ is hidden behind the definition of $\widehat{H}^*_{\mathrm{CS}}(-;\mathbb{Z})$.).

The second definition of $\widehat{H\mathbb{Z}}^*$: Differential cocycles

Let X be a manifold. An n-th *differential cocycle* on X is an element

$$(c, h, \omega) \in Z^n_\infty(X; \mathbb{Z}) \times C^{n-1}_\infty(X; \mathbb{R}) \times \Omega^n_{\mathrm{clo}}(X)$$

such that

$$\omega - c_\mathbb{R} = \delta h. \tag{16}$$

Here C^*_∞ and Z^*_∞ denotes the groups of smooth singular cochains and cocycles. We introduce the equivalence relation $\sim$ on differential cocycles by setting

$$(c, h, \omega) \sim (c + \delta b, h - b_\mathbb{R} - \delta k, \omega)$$

for some $(b, k) \in C^{n-1}_\infty(X; \mathbb{Z}) \times C^{n-2}_\infty(X; \mathbb{R})$.

Definition ($\widehat{H}^*_{\mathrm{HS}}(X; \mathbb{Z})$ [HS05])

Set

$$\widehat{H}^n_{\mathrm{HS}}(X; \mathbb{Z}) := \{(c, h, \omega) : \text{differential } n\text{-cocycle on } X\}/\sim$$

$\widehat{H}^*_{\mathrm{HS}}(X;\mathbb{Z}) \simeq \widehat{H}^*_{\mathrm{CS}}(X;\mathbb{Z})$

Proposition

We have an isomorphism

$$\widehat{H}^n_{\mathrm{HS}}(X;\mathbb{Z}) \simeq \widehat{H}^n_{\mathrm{CS}}(X;\mathbb{Z})$$

by mapping $[c, h, \omega]$ *to* $(\omega, h \mod \mathbb{Z})$.

The corresponding structure maps for $\widehat{H}^*_{\mathrm{HS}}(-;\mathbb{Z})$ are

$$\begin{aligned} R&\colon \widehat{H}^n_{\mathrm{HS}}(X;\mathbb{Z}) \to \Omega^n_{\mathrm{clo}}(X), \quad [c,h,\omega] \mapsto \omega \\ I&\colon \widehat{H}^n_{\mathrm{HS}}(X;\mathbb{Z}) \to H^n(X;\mathbb{Z}), \quad [c,h,\omega] \mapsto [c] \\ a&\colon \Omega^{n-1}(X)/\mathrm{im}(d) \to \widehat{H}^n_{\mathrm{HS}}(X;\mathbb{Z}), \quad \alpha \mapsto [0,\alpha,d\alpha]. \end{aligned}$$

Thus $\left(\widehat{H}^n_{\mathrm{HS}}(-;\mathbb{Z}), R, I, a\right)$ is a *differential extension* of $H^n(-;\mathbb{Z})$.

The differential chain complexes

Actually, $\widehat{H}^*_{\mathrm{HS}}(-;\mathbb{Z})$ can be realized as the cohomology group of the *differential cochain complex*.
Fix $k \in \mathbb{Z}$ and define the cochain complex $\widehat{C}(k)^*(X)$ by

$$\widehat{C}(k)^n(X) := \begin{cases} C^n_\infty(X;\mathbb{Z}) \times C^{n-1}_\infty(X;\mathbb{R}) & n \leq k-1 \\ C^n_\infty(X;\mathbb{Z}) \times C^{n-1}_\infty(X;\mathbb{R}) \times \Omega^n(X) & n \geq k \end{cases}$$

with the differential

$$d(c,h,\omega) := (\delta c, \omega - c_{\mathbb{R}} - \delta h, d\omega).$$

Let $\widehat{H}(k)^n(X)$ be the n-th cohomology group of $\widehat{C}(k)^*(X)$, i.e.,

$$\widehat{H}(k)^n(X) := \widehat{Z}(k)^n(X)/d\widehat{C}(k)^{n-1}(X),$$

where $\widehat{Z}(k)^n(X) := \ker d \subset \widehat{C}(k)^n(X)$.

We have

Proposition

$$\widehat{H}(k)^n(X) \simeq \begin{cases} H^{n-1}(X;\mathbb{R}/\mathbb{Z}) & n \leq k-1 \\ \widehat{H}^n_{\mathrm{HS}}(X) & n = k \\ H^n(X;\mathbb{Z}) & n \geq k+1. \end{cases}$$

One advantage of having the cochain complex is that we can talk about trivializations. Let us look at second differential cocycles.
We have $H^2_{\mathrm{HS}}(X;\mathbb{Z}) \simeq H^2_{\mathrm{geom}}(X;\mathbb{Z}) = \{(L,\nabla)\}/\sim_{\mathrm{isom}}$.
Given (L,∇), let us fix $\widehat{x} \in \widehat{Z}(2)^2(X) = \widehat{Z}(1)^2(X)$ representing it.
We can consider two types of trivializations of (L,∇).

- Topological trivialization, i.e., a section s of L (with $|s| = 1$).
  The choices of such s are in bijection with the set

  $$\{\widehat{y} \in \widehat{C}(1)^1(X) \mid d\widehat{y} = \widehat{x}\}/d\widehat{C}(1)^0(X), \tag{20}$$

  which is a torsor over
  $\widehat{Z}(1)^1(X)/d\widehat{C}(1)^0(X) = \widehat{H}^1(X;\mathbb{Z}) \simeq C^\infty(X;\mathbb{R}/\mathbb{Z})$.
- Flat trivialization, i.e., a flat section s of (L,∇).
  The choices of such s are in bijection with the set

  $$\{\widehat{y} \in \widehat{C}(2)^1(X) \mid d\widehat{y} = \widehat{x}\}/d\widehat{C}(2)^0(X), \tag{21}$$

  which is a torsor over $\widehat{Z}(2)^1(X)/d\widehat{C}(2)^0(X) = H^0(X;\mathbb{R}/\mathbb{Z})$.

1. Introduction

2. Ordinary differential cohomology
    - Hermitian line bundles with connection
    - Differential characters
    - Differential cocycles

3. Differential K-theory
    - Review : Topological K-theory
    - Chern-Weil constructions
    - The model by vector bundles with connections

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Review : Topological K-theory

K-theory is a generalized cohomology theory which is important in both math and physics.

There are various models for K^*, for example there are models in terms of

- Vector bundles,
- Families of Fredholm operators,
- "Gradations" on Clifford modules.

The vector bundle model of K^*

$K^0(X)$ classifies stable equivalence classes of complex vector bundles over X.
Let X be a finite CW-complex. Let $\mathrm{Vect}(X)$ be the set of isomorphism classes $[E]$ of complex vector bundles over X, with the abelian monoid structure by $\oplus$.
$K^0(X)$ is defined to be the Grothendieck group associated to $\mathrm{Vect}(X)$.
This means that $K^0(X)$ is a group whose elements are formal differences

$$[E_+] - [E_-] \in K^0(X)$$

and we have

$$[E] = [F] \text{ in } K^0(X) \text{ if } E \oplus G \simeq F \oplus G \text{ for some } G.$$

For a finite CW-pair (X, Y) (i.e., $Y \subset X$), the *relative K^0-group $K^0(X, Y)$* is defined by taking the Grothendieck group of the abelian monoid of isomorphism classes of triples

$$(E_+, E_-, \sigma),$$

where E_+ and E_- are complex vector bundles over X and $\sigma \colon E_+|_Y \simeq E_-|_Y$. We set $K^{-n}(X, Y) := K^0(\Sigma^n(X/Y), \mathrm{pt})$, in particular we have

$$K^{-n}(X) := K^0(\Sigma^n(X^+), \mathrm{pt}) = K^0(S^n \times X, \mathrm{pt} \times X).$$

Some facts on K^*

Bott periodicity. We have

$$K^n(X) \simeq K^{n+2}(X).$$

K-groups on pt:

$$K^0(\mathrm{pt}) \simeq \mathbb{Z}, \quad K^1(\mathrm{pt}) = 0.$$

The *(topological) Chern character*. We have a natural transformation

$$\mathrm{Ch}\colon K^n(X) \to H^{2\mathbb{Z}+n}(X;\mathbb{R}) = H^n(X; K^*(\mathrm{pt}) \otimes \mathbb{R})$$

If X is a manifold, taking a unitary connection ∇ on E we have

$$\mathrm{Ch}([E]) = \left[\mathrm{Tr}(e^{F_\nabla/(2\pi\sqrt{-1})})\right] \in H^{2\mathbb{Z}}_{\mathrm{dR}}(X;\mathbb{R}).$$

Chern-Weil constructions

Let X be a manifold and (E, ∇) be a harmitian vector bundle with unitary connection over X. Let $F_\nabla \in \Omega^2_{\mathrm{clo}}(X; \mathrm{End}(E))$ be the curvature.
We define the *Chern chacater form* by

$$\mathrm{Ch}(F_\nabla) := \mathrm{Tr}(e^{F_\nabla/(2\pi\sqrt{-1})}) \in \Omega^{2\mathbb{Z}}_{\mathrm{clo}}(X).$$

Its de Rham cohomology class represents the topological Chern character of $[E]$,

$$\mathrm{Ch}([E]) = [\mathrm{Ch}(F_\nabla)] \in H^{2\mathbb{Z}}(X; \mathbb{R}).$$

In particular, the cohomology class does not depend on the choice of ∇, i.e., if we have two connections ∇_0 and ∇_1, we have

$$\mathrm{Ch}(F_{\nabla_1}) - \mathrm{Ch}(F_{\nabla_0}) \in \mathrm{Im}(d).$$

Chern-Simons forms

For two connections ∇_0 and ∇_1 on E, we have $\mathrm{Ch}(F_{\nabla_1}) - \mathrm{Ch}(F_{\nabla_0}) \in \mathrm{Im}(d)$. Why?
Take a homotopy $\nabla_{[0,1]}$ between ∇_0 and ∇_1.
Define the *Chern-Simons form* for the homotopy $\nabla_{[0,1]}$ by

$$\mathrm{CS}(F_{\nabla_{[0,1]}}) := \int_{[0,1]} \mathrm{Ch}(F_{\nabla_{[0,1]}}) \in \Omega^{2\mathbb{Z}-1}(X).$$

We have the transgression formula

$$\mathrm{Ch}(F_{\nabla_1}) - \mathrm{Ch}(F_{\nabla_0}) = d\mathrm{CS}(F_{\nabla_{[0,1]}}).$$

The Chern-Simons form depends on the choice of homotopy only up to $\mathrm{Im}(d)$ (again, checked by taking a homotopy between homotopies). Thus

$$\mathrm{CS}(\nabla_0, \nabla_1) := \left[\mathrm{CS}(F_{\nabla_{[0,1]}})\right] \in \Omega^{2\mathbb{Z}-1}(X)/\mathrm{Im}(d)$$

is well-defined.

The first definition of $\widehat{K}^*$: Vector bundles with connections

Freed and Lott [FL10] gave a model $\widehat{K}^*_{\mathrm{FL}}$ of differential K-theory in terms of vector bundles with connections.

Let X be a manifold. Rhoughly speaking, $\widehat{K}^0_{\mathrm{FL}}(X)$ is a group of hermitian vector bundles with connections,

$$[E, \nabla] \in \widehat{K}^0_{\mathrm{FL}}(X).$$

The functor R is given by the Chern character forms,

$$R \colon \widehat{K}^0_{\mathrm{FL}}(X) \to \Omega^{2\mathbb{Z}}_{\mathrm{clo}}(X), \quad [E, \nabla] \mapsto \mathrm{Ch}(F_\nabla).$$

The functor a accounts for the Chern-Simons forms,

$$a \colon \Omega^{2\mathbb{Z}-1}(X)/\mathrm{im}(d) \to \widehat{K}^0_{\mathrm{FL}}(X), \quad \mathrm{CS}(\nabla_0, \nabla_1) \mapsto [E, \nabla_1] - [E, \nabla_0].$$

$d = R \circ a$ follows by the transgression formula

$$\mathrm{Ch}(F_{\nabla_1}) - \mathrm{Ch}(F_{\nabla_0}) = d\mathrm{CS}(\nabla_0, \nabla_1).$$

Definition of $\widehat{K}^0_{\mathrm{FL}}$

Definition (The model of $\widehat{K}^0$ by vector bundle with connection [FL10])

Let X be a manifold. Define $\widehat{\mathrm{Vect}}(X)$ to be the set of isomorphism classes of triples

$$(E, \nabla, \alpha), \tag{23}$$

where (E, ∇) is a hermitian vector bundle with a unitary connection on X and $\alpha \in \Omega^{2\mathbb{Z}-1}(X)/\mathrm{Im}(d)$. We introduce the abelian monoid structure by

$$[E, \nabla, \alpha] + [E', \nabla', \alpha'] := [E \oplus E', \nabla \oplus \nabla', \alpha + \alpha'].$$

We introduce the following relation $\sim$ on $\widehat{\mathrm{Vect}}(X)$,

$$[E, \nabla_1, \alpha] \sim [E, \nabla_0, \mathrm{CS}(\nabla_0, \nabla_1) + \alpha].$$

Define $\widehat{K}^0_{\mathrm{FL}}(X)$ to be the Grothendieck group associated to $\widehat{\mathrm{Vect}}(X)/\sim$.

Structure maps

We define *structure maps*

$$\begin{aligned} R &\colon \widehat{K}^0_{\mathrm{FL}}(X) \to \Omega^{2\mathbb{Z}}_{\mathrm{clo}}(X), \quad [E, \nabla, \alpha] \mapsto \mathrm{Ch}(F_\nabla) + d\alpha \\ I &\colon \widehat{K}^0_{\mathrm{FL}}(X) \to K^0(X), \quad [E, \nabla, \alpha] \mapsto [E] \\ a &\colon \Omega^{2\mathbb{Z}-1}(X)/\mathrm{im}(d) \to \widehat{K}^0_{\mathrm{FL}}(X), \quad \alpha \mapsto [0, 0, \alpha]. \end{aligned}$$

The well-definedness of R follows by the transgression formula

$$\mathrm{Ch}(F_{\nabla_1}) - \mathrm{Ch}(F_{\nabla_0}) = d\mathrm{CS}(\nabla_0, \nabla_1).$$

The hexagon for $\widehat{K}^0_{\mathrm{FL}}$

We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& K^{-1}(X;\mathbb{R}/\mathbb{Z}) \arrow[rr, "\mathrm{Bock}"] \arrow[dr] & & K^0(X) \arrow[ur] \arrow[dr, "\mathrm{Ch}"] & \\
H^{2\mathbb{Z}-1}(X;\mathbb{R}) \arrow[ur] \arrow[dr] & & \widehat{K}^0_{\mathrm{FL}}(X) \arrow[ur, "I"] \arrow[dr, "R"] & & H^{2\mathbb{Z}}(X;\mathbb{R}) \\
& \Omega^{2\mathbb{Z}-1}(X)/\Omega^{2\mathbb{Z}-1}_{\mathrm{clo}}(X)_K \arrow[ur, "a"] \arrow[rr, "d"] & & \Omega^{2\mathbb{Z}}_{\mathrm{clo}}(X)_K \arrow[ur, "\mathrm{Rham}"] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

The diagonal sequences are exact.
This implies that $\left(\widehat{K}^0_{\mathrm{FL}}, R, I, a\right)$ is a *differential extension* of K^0.

$\widehat{K}^1_{\mathrm{FL}}$

Freed and Lott [FL10] constructed a model $\widehat{K}^1_{\mathrm{FL}}$ of $\widehat{K}^1$.
Elements of $\widehat{K}^1_{\mathrm{FL}}(X)$ are represented by quintuples

$$(E, \nabla, U, \alpha)$$

where

- (E, ∇) is a hermitian vector bundle with a unitary connection on X,
- U is a unitary automorphism on E,
- $\alpha \in \Omega^{2\mathbb{Z}-1}(X)/\mathrm{Im}(d)$.

The equivalence relations are given by transgression forms as before.
We have the structure maps and the hexagon as before. We also set $\widehat{K}^{2n}_{\mathrm{FL}} := \widehat{K}^0_{\mathrm{FL}}$ and $\widehat{K}^{2n-1}_{\mathrm{FL}} := \widehat{K}^1_{\mathrm{FL}}$.

Integrations in K^* and $\widehat{K}^*$

$\widehat{K}^*$ also has the differential integration maps.

First we recall the (topological) integrations in K^*. For fiber bundles $p\colon N \to X$ whose fibers are closed manifold and equipped with a fiberwise Spin^c structure g_p[1], we have the (topological) integration map,

$$(p, g_p)_*\colon K^n(N) \to K^{n-r}(X),$$

where $r = \dim N - \dim X$.
c.f. for $H\mathbb{Z}^*$ we only require fiberwise orientation and get

$$\int_{N/X}\colon H^n(N;\mathbb{Z}) \to H^{n-r}(X;\mathbb{Z}),$$

For more on integrations (a.k.a. pushforward, Gysin maps, ...) in generalized cohomology theories, see [Rud98] for example.

---

[1] Or more generally, proper Spin^c-oriented maps (p, g_p)

Topological integration in $K^* =$ Atiyah-Singer's index

In particular if (M^{2n}, g) is a closed even dimensional manifold with a Spin^c structure, the integration map along $p_M \colon M \to \mathrm{pt}$ gives the homomorphism

$$(p_M, g)_* \colon K^0(M) \to K^{-2n}(\mathrm{pt}) \simeq K^0(\mathrm{pt}) \simeq \mathbb{Z}. \tag{24}$$

By the Atiyah-Singer's index theorem, the map (24) is given by

$$(p_M, g)_*[E] = \mathrm{Index}(\not{D}_{E,\nabla}),$$

where $\not{D}_{E,\nabla} \colon C^\infty(M; \$ \otimes E) \to C^\infty(M; \$ \otimes E)$ is the Dirac operator twisted by (E, ∇).

In general for $(p \colon N \to X, g_p)$, the integration map is given by taking the family index of fiberwise twisted Dirac operators.

Differerntial integration in $\widehat{K}^*$ = reduced eta invariants

In order to define differential integrations in $\widehat{K}^*$, we need geometric Spin^c structures, i.e., Spin^c structures with Spin^c-connections compatible with Levi-Civita connections[2].
For fiber bundles $p\colon N \to X$ whose fibers are closed manifold and equipped with a fiberwise geometric Spin^c structure $\widehat{g}_p$, we have the differential integration map,

$$(p, \widehat{g}_p)_*\colon \widehat{K}^n(N) \to \widehat{K}^{n-r}(X),$$

where $r = \dim N - \dim X$.

---

[2] Actually we can drop the compatibility with Levi-Civita connections.

In particular if $(M^{2n-1}, \widehat{g})$ is a closed odd dimensional manifold with a geometric Spin^c structure, the differential integration map along $p_M \colon M \to \mathrm{pt}$ gives the homomorphism

$$(p_M, \widehat{g})_* \colon \widehat{K}^0(M) \to \widehat{K}^{-2n+1}(\mathrm{pt}) \simeq \widehat{K}^1(\mathrm{pt}) \simeq \mathbb{R}/\mathbb{Z}. \qquad (25)$$

Fact ([FL10])

The differential integration map (25) is given by

$$(p_M, \widehat{g})_*[E, \nabla, \alpha] = \overline{\eta}(\not{D}_{E,\nabla}) + \int_M \alpha \wedge \mathrm{Todd}(M, \widehat{g}) \pmod{\mathbb{Z}}.$$

Here the reduced eta invariant $\overline{\eta}(\not{D}_{E,\nabla})$ is given by

$$\overline{\eta}(\not{D}_{E,\nabla}) := \frac{\eta(\not{D}_{E,\nabla}) + \dim\ker(\not{D}_{E,\nabla})}{2} \in \mathbb{R}. \qquad (26)$$

The bordism formula and the APS index theorem

The Atiyah-Patodi-Singer's index theorem is an index theorem for compact manifolds with boundaries.

Fact (Atiyah-Patodi-Singer, [APS76])

Suppose $(W^{2n}, \partial W, \widehat{g})$ *is a compact even dimensional manifold with a geometric Spin*^{c} *structure. Let* (E, ∇) *be a hermitian vector bundle on* W*. Assuming collar structure on everything, we have*

$$\mathrm{Index}_{\mathrm{APS}}(\not{D}_{E,\nabla}) = \int_W \mathrm{Ch}(F_\nabla) \wedge \mathrm{Todd}(W, \widehat{g}) - \overline{\eta}(\not{D}_{(E,\nabla)|\partial W})$$

Here $\mathrm{Index}_{\mathrm{APS}}(\not{D}_{E,\nabla})$ is the Fredholm index with respect to the "APS boundary condition". In particular we have $\mathrm{Index}_{\mathrm{APS}}(\not{D}_{E,\nabla}) \in \mathbb{Z}$. Thus we get

$$\overline{\eta}(\not{D}_{(E,\nabla)|\partial W}) \equiv \int_W \mathrm{Ch}(F_\nabla) \wedge \mathrm{Todd}(W, \widehat{g}) \pmod{\mathbb{Z}}. \tag{27}$$

The APS index theorem, in particular (27), implies the following Bordism formula.

Proposition (The bordism formula)

Suppose $(W^{2n}, \partial W, \widehat{g})$ *is a compact even dimensional manifold with a geometric Spin*^{c} *structure. For any* $\widehat{x} \in \widehat{K}^0(W)$, *we have*

$$(p_{\partial W}, \widehat{g}|_{\partial W})_* \widehat{x}|_{\partial W} \equiv \int_W R(\widehat{x}) \wedge \mathrm{Todd}(W, \widehat{g}) \pmod{\mathbb{Z}} \tag{29}$$

Indeed, if we can represent $\widehat{x} = [E, \nabla, 0] \in \widehat{K}^0_{\mathrm{FL}}(W)$, we see (29) = (27).
Then the general case follows by the Stokes theorem (check!).
Actually the bordism formula also holds in the case dim W is odd and $\widehat{x} \in \widehat{K}^1(W)$.

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Generalized differential cohomology

So far we have seen the differential ordinary cohomology $\widehat{H\mathbb{Z}}^*$ and the differential K-theory $\widehat{K}^*$.
Actually we can talk about differential extensions $\widehat{E}^*$ of any generalized cohomology theory E^*.
Here we explain the axiomatic approach given by Bunke and Schick [BS12].
The idea is to generalize the hexagon as

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& E^{n-1}(X;\mathbb{R}/\mathbb{Z}) \arrow[rr, "\mathrm{Bock}"] \arrow[dr] & & E^n(X) \arrow[ur] \arrow[dr, "\mathrm{ch}"] & \\
H^{n-1}(X;V^\bullet) \arrow[ur] \arrow[dr] & & \widehat{E}^n(X) \arrow[ur, "I"] \arrow[dr, "R"] & & H^n(X;V^\bullet) \\
& \Omega^{n-1}(X;V^\bullet)/\Omega^{n-1}_{\mathrm{clo}}(X;V^\bullet)_E \arrow[ur, "a"] \arrow[rr, "d"] & & \Omega^n_{\mathrm{clo}}(X;V^\bullet)_E \arrow[ur, "\mathrm{Rham}"] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

The axiom by Bunke and Schick

Suppose we are given

- a generalized cohomology theory E,
- a $\mathbb{Z}$-graded vector space $V^\bullet$ over $\mathbb{R}$ (universal choice : $E^\bullet(\mathrm{pt}) \otimes \mathbb{R}$),
- a natural transformation $\mathrm{ch} \colon E^*(X) \to H^*(X; V^\bullet)$.

Definition (The axiom of differential cohomology, [BS12])

A differential extension of (E^*, ch) is a quintuple $(\widehat{E}^*, R, I, a)$ such that $\widehat{E}^* \colon \mathrm{Mfd}^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$ is a functor, and R, I and a are natural transformations fitting into the following commutative diagram where the diagonal sequence is exact.

$$\begin{tikzcd}
 & & & & & 0 \\
 & & & E^n(X) \arrow[urr] \arrow[dr, "\mathrm{ch}"] & & \\
 & & \widehat{E}^n(X) \arrow[ur, "I"] \arrow[dr, "R"] & & H^n(X; V^\bullet) & \\
 & \Omega^{n-1}(X; V^\bullet)/\Omega^{n-1}_{\mathrm{clo}}(X; V^\bullet)_E \arrow[ur, "a"] \arrow[rr, "d"] & & \Omega^n_{\mathrm{clo}}(X; V^\bullet)_E \arrow[ur, "\mathrm{Rham}"] \arrow[drr] & & \\
0 \arrow[ur] & & & & & 0
\end{tikzcd}$$

Remarks

- Hopkins and Singer [HS05] constructed a differential extension $\widehat{E}^*_{\mathrm{HS}}$ of each (E^*, ch).
- Given (E^*, ch), the uniqueness of its differential extension is highly nontrivial. Bunke and Schick [BS10] investigate into this uniqueness problem. They show the uniqueness under some (very mild) assumptions. As far as I heard, there is no known conterexample for the uniqueness.
- When we take the universal choice $V^\bullet = E^\bullet(\mathrm{pt}) \otimes \mathbb{R}$,

  $$\widehat{E}^n_{\mathrm{flat}}(X) := \ker\left(R \colon \widehat{E}^n(X) \to \Omega^n_{\mathrm{clo}}(X; V^\bullet)_E\right)$$

  is called the flat theory. It is a homotopy invariant functor, but it is not known that we have $E^{n-1}(X; \mathbb{R}/\mathbb{Z}) \simeq \widehat{E}^n_{\mathrm{flat}}(X)$ in general [BS10].
- There are variations on the axioms, such as multiplicative differential extensions when E is multiplicative.

The Hopkins-Singer's model $\widehat{E}^*_{\mathrm{HS}}$

Hopkins and Singer [HS05] constructed a differential extension $\widehat{E}^*_{\mathrm{HS}}$ of each (E^*, ch).
For this, we represent E^* by an Ω-spectrum $E = \{E_n\}_{n \in \mathbb{Z}}$ and take a singular cocycle $\iota \in Z^0(E; V^\bullet)$ representing $\mathrm{ch} \in H^0(E; V^\bullet)$.
An element in $\widehat{E}^n_{\mathrm{HS}}(X)$ is represented by a *differential function*

$$(c, h, \omega) \colon X \to (E_n, \iota_n),$$

consists of a continuous map $c \colon X \to E_n$, a singular cochain $h \in C^{n-1}(X; V^\bullet)$ and $\omega \in \Omega^n_{\mathrm{clo}}(X; V^\bullet)$, such that

$$\omega - c^* \iota_n = \delta h.$$

We introduce an equivalence relation on differential functions coming from differential functions on $X \times [0, 1]$.
Taking $E = H\mathbb{Z}$ and $\iota \in Z^0(H\mathbb{Z}; \mathbb{Z})$ to be $\mathbb{Z}$-valued fundamental cocycle, we recover $H^*_{\mathrm{HS}}(-; \mathbb{Z})$ explained before.

Differential extensions $\widehat{I\Omega^G_{\mathrm{dR}}}$ of the Anderson duals

In Yonekura-Y [YY21], we constructed a differential extension $\widehat{I\Omega^G_{\mathrm{dR}}}$ of the *Anderson dual to G-bordism theory* $I\Omega^G$.
The motivation comes from the classification of invertible QFT's (a.k.a invertible phases), in particular the conjecture by Freed-Hopkins [FH21]; an element in $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$ can be regarded as an invertible QFT on G-manifolds.
The construction is analogous to the Cheeger-Simons' differential character group $H^*_{\mathrm{CS}}(X;\mathbb{Z})$.
Here G is a tangential structure group such as SO, Spin, etc.

For simplicity here we assume G is oriented.

An element in $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$ is represented by a pair (ω, h) consisting of

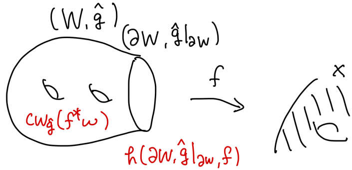
- $\omega \in \Omega^n_{\mathrm{clo}}(X; (\mathrm{Sym}\mathfrak{g}^*)^G)$,
- h is a *partition function*, which is a map assigning

$$h(M^{n-1}, \widehat{g}, f) \in \mathbb{R}/\mathbb{Z}$$

  to each closed $(n-1)$-dimensional differential G-manifold with a map $f \in C^\infty(M, X)$. We require the additivity under disjoint unions.

We require the following compatibility condition for (ω, h).

If we have



, we have

$$h(\partial W, \widehat{g}|_{\partial W}, f|_{\partial W}) \equiv \int_W \mathrm{cw}_{\widehat{g}}(f^*\omega) \pmod{\mathbb{Z}}.$$

Differential integrations and $\widehat{I\Omega^G_{\mathrm{dR}}}$

Important examples of elements in $(\widehat{I\Omega^G_{\mathrm{dR}}})^n(X)$ comes from differential integrations.
First we consider the case of $\widehat{H\mathbb{Z}}$. Let us fix $\widehat{x} \in \widehat{H}^n(X;\mathbb{Z})$. Then we can construct the element $(\omega_{\widehat{x}}, h_{\widehat{x}}) \in (\widehat{I\Omega^{\mathrm{SO}}_{\mathrm{dR}}})^n(X)$ by

$$\begin{aligned} \omega_{\widehat{x}} &:= R(\widehat{x}), \\ h_{\widehat{x}}(M^{n-1}, \widehat{g}, f) &:= \int_M f^*\widehat{x} \quad \text{(higher holonomy of } f^*\widehat{x}\text{)}. \end{aligned}$$

The compatibility condition follows by the bordism formula.
For example if $\widehat{x} = [L, \nabla] \in \widehat{H}^2(X;\mathbb{Z})$, we have
$(\omega_{\widehat{x}}, h_{\widehat{x}}) = (c_1(F_\nabla), \mathrm{Hol}(L, \nabla))$.

Next we consider the case of $\widehat{K}$. Let us fix $\widehat{x} \in \widehat{K}^n(X)$. Then we can construct the element $(\omega_{\widehat{x}}, h_{\widehat{x}}) \in (\widehat{I\Omega^{\mathrm{Spin}^c}_{\mathrm{dR}}})^n(X)$ by

$$\omega_{\widehat{x}} := R(\widehat{x}) \otimes \mathrm{Todd},$$
$$h_{\widehat{x}}(M^{n-1}, \widehat{g}, f) := (p_M, \widehat{g})_* f^* \widehat{x}.$$

Again the compatibility condition follows by the bordism formula.
For example if $\widehat{x} = 1 \in \widehat{K}^{2n}(\mathrm{pt}) \simeq \mathbb{Z}$, we have $(\omega_{\widehat{x}}, h_{\widehat{x}}) = (\mathrm{Todd}, \overline{\eta})$.

In this way we get natural transformations

$$\widehat{H}^n(X; \mathbb{Z}) \to (\widehat{I\Omega^{\mathrm{SO}}_{\mathrm{dR}}})^n(X),$$
$$\widehat{K}^n(X) \to (\widehat{I\Omega^{\mathrm{Spin}^c}_{\mathrm{dR}}})^n(X).$$

Actually these are differential refinements of the combinations of Anderson dual to multiplicative genera (universal orientation $M\mathrm{SO} \to H\mathbb{Z}$ and the Atiyah-Bott-Shapiro orientation $M\mathrm{Spin}^c \to K$, resp.) and the Anderson self-dualities of $H\mathbb{Z}$ and of K [Yam21].

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