Quasidiagonality and Amenability

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DEFINITION (Halmos)

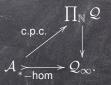
A C*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is quasidiagonal if there is an increasing net of finite rank projections $p_i \nearrow 1_{\mathcal{H}}$ strongly and which are approximately central with respect to \mathcal{A} .

An abstract C^* -algebra $\mathcal A$ is quasidiagonal if it has a faithful quasidiagonal representation.

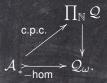
Equivalently (following Voiculescu):

There are c.p.c. maps $\phi_i : \mathcal{A} \to M_{r_i}$ which are approximately multiplicative and approximately isometric (in Norm).

For $\mathcal A$ separable: $\mathcal A$ is quasidiagonal iff there is a diagram



Equivalently, after choosing a free ultrafilter $\omega \in \beta \mathbb{N} \setminus \mathbb{N}$: \mathcal{A} is quasidiagonal iff there is a diagram



Note that Q_{ω} has a unique tracial state $\tau_{Q_{\omega}}$ (Ozawa).

For $\mathcal A$ separable and nuclear (using Choi–Effros): $\mathcal A$ is quasidiagonal iff there is an embedding

$$A > \stackrel{\iota}{\longrightarrow} Q_{\omega}.$$

If $\mathcal A$ is unital, the embedding ι may always be assumed to be unital, since $\mathcal Q$ is self-similar.

Voiculescu observed that any separable, unital, quasidiagonal $\mathcal A$ has a tracial state given by $\tau_{\mathcal Q_\omega}\circ\iota$.

Following N. Brown, we say a trace τ on \mathcal{A} is quasidiagonal, if

$$\tau = \tau_{\mathcal{Q}_{\omega}} \circ \phi$$

for some (not necessarily injective) *-homomorphism

$$\phi: \mathcal{A} \longrightarrow \mathcal{Q}_{\omega}.$$

The connection between quasidiagonality and amenability goes back to Rosenberg:

THEOREM / OBSERVATION

For a countable discrete group G, if $C_r^*(G)$ is quasidiagonal, then G is amenable.

Inspection of examples led to Rosenberg's

CONJECTURE

For a countable discrete group G, if G is amenable, then $C_r^*(G)$ is quasidiagonal.

QUESTIONS

Is every separable, stably finite, nuclear C*-algebra \mathcal{A} quasidiagonal? (QDQ; aka Blackadar–Kirchberg problem)

What if A is simple?

Is every trace on a separable, nuclear C^* -algebra \mathcal{A} quasidiagonal?

In 2013, Ozawa-Rørdam-Sato confirmed Rosenberg's conjecture for elementary amenable groups (a bootstrap type condition).

The argument uses classification of simple, nuclear C*-algebras in the sense of Elliott.

This was strong evidence that classification might be relevant for QDQ, while it was long known that quasidiagonality is relevant for classification:

- \bullet Quasidiagonality of cones (Voiculescu) appears in Kirchberg's $\mathcal{O}_2\text{-embedding}$ theorem.
- Popa introduced local quantisation for quasidiagonal C*-algebras with small projections.
- Lin developed a tracially large version to arrive at TAF classification.

• In 2013, Matui–Sato showed that for $\mathcal A$ separable, simple, unital, nuclear, monotracial and quasidiagonal, $\mathcal A \otimes \mathsf{UHF}$ is TAF.

(They also gave an answer to the Powers–Sakai conjecture about strongly continuous flows on UHF algebras, extending earlier work of Kishimoto.)

• In 2015, Elliott–Gong–Lin–Niu used work of Gong–Lin–Niu, Lin–Niu, Matui–Sato, W, ... to show that the class

{separable, unital, simple, nuclear, UCT C*-algebras with finite nuclear dimension and only quasidiagonal traces}

is classified by the Elliott invariant.

(Finite nuclear dimension is a notion of covering dimension for nuclear C^* -algebras. In the simple case, in large generality it is finite precisely for $\mathcal Z$ -stable C^* -algebras. $\mathcal Z$ -stability is the C^* -algebra analogue of being McDuff.)

Recall that a separable $\ensuremath{\mathcal{A}}$ is said to satisfy the UCT, if the sequence

$$0 \to \mathsf{Ext}(\mathsf{K}_*(\mathcal{A}),\mathsf{K}_{*+1}(\mathcal{B})) \to \mathsf{KK}(\mathcal{A},\mathcal{B}) \to \mathsf{Hom}(\mathsf{K}_*(\mathcal{A}),\mathsf{K}_*(\mathcal{B})) \to 0$$

is exact for any σ -unital \mathcal{B} .

UCT PROBLEM:

Does every separable nuclear C*-algebra satisfy the UCT?

THEOREM (Tikuisis–White–W, 2015) Let \mathcal{A} be a separable, nuclear C*-algebra satisfying the UCT. Then every faithful trace on \mathcal{A} is quasidiagonal.

Gabe:

Enough to assume the trace to be amenable and $\mathcal A$ exact.

COROLLARY

Rosenberg's conjecture holds: $C_r^*(G)$ is quasidiagonal for any discrete, amenable group G.

COROLLARY

The Blackadar–Kirchberg problem has an affirmative answer for simple UCT C*-algebras.

COROLLARY (using work by many hands)
Separable, unital, simple, nuclear, UCT C*-algebras with finite nuclear dimension are classified by the Elliott invariant.

In particular we have:

Separable, unital, simple, nuclear, UCT C*-algebras with at most one trace are classified up to \mathcal{Z} -stability by their ordered K-groups.

(The traceless case is Kirchberg–Phillips classification.)

Let us return to the main result and have a look at the proof.

THEOREM (Tikuisis–White–W, 2015) Let $\mathcal A$ be a separable, nuclear C*-algebra satisfying the UCT. Then every faithful trace on $\mathcal A$ is quasidiagonal.

We start with a lemma.

LEMMA

Let A be separable, unital and nuclear, and let $\tau \in T(A)$.

(i) There is a *-homomorphism

$$\Psi: \mathcal{C}_0((0,1]) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_{\omega}$$

such that

$$au_{\mathcal{Q}_{\omega}} \circ \Psi = \mathsf{Lebesgue} \otimes \tau.$$

(ii) There are *-homomorphisms

$$\dot{\Phi}: \textit{\textbf{C}}_{0}((0,1]) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_{\omega},$$

$$\dot{\Phi}: C_0([0,1)) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_{\omega},$$

$$\Theta: C([0,1]) \longrightarrow \mathcal{Q}_{\omega}$$

such that

$$\dot{\Phi}|_{\mathcal{C}_0((0,1])\otimes 1_{\mathcal{A}}},\ \dot{\Phi}|_{\mathcal{C}_0([0,1))\otimes 1_{\mathcal{A}}}\ \text{and}\ \Theta\ \text{are compatible}$$

and

$$egin{aligned} au_{\mathcal{Q}_{\omega}} \circ & \acute{\Phi} = \mathsf{Lebesgue} \otimes au, \ au_{\mathcal{Q}_{\omega}} \circ & \grave{\Phi} = \mathsf{Lebesgue} \otimes au, \ au_{\mathcal{Q}_{\omega}} \circ & \Theta = \mathsf{Lebesgue}. \end{aligned}$$

PROOF

(i) Let us assume for simplicity that τ is extremal. We then have

[Choi–Effros, Sato–White–W,
$$\overline{\Psi}$$
 c.p.c. \bot $\overset{\mathcal{Q}_{\omega}}{\bigvee}$ Kaplansky density $\overset{\mathcal{A}}{\longleftarrow} \overset{\mathcal{R}^{\omega}}{\sqcap}$

Now define

$$\Psi: \textbf{\textit{C}}_0((0,1]) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_{\omega}$$

by

$$\Psi(\mathsf{id}_{(0,1]}\otimes a):=\overline{\Psi}(a).$$

(ii) Find

$$\mu: {\it C}([0,1]) \longrightarrow {\it Q}_{\omega} \cap \overline{\Psi}({\it A})'$$

such that

$$au_{\mathcal{Q}_{\omega}} \circ \mu = \mathsf{Lebesgue}.$$

Define

$$\Phi(\mathrm{id}_{(0,1]}\otimes a):=\mu(\mathrm{id}_{(0,1]})\,\overline{\Psi}(a).$$

To find $\grave{\varphi}$ (and then automatically $\Theta),$ observe that

$$\acute{\Phi}(\mathsf{id}_{(0,1]} \otimes 1_{\mathcal{A}}) \sim_{\mathsf{u}} 1_{\mathcal{Q}_{\omega}} - \acute{\Phi}(\mathsf{id}_{(0,1]} \otimes 1_{\mathcal{A}})$$

(e.g. using a Cuntz semigroup argument).

By restricting $\dot{\Phi}$ and $\dot{\Phi}$, we obtain *-homomorphisms

$$\dot{\Lambda}, \dot{\Lambda}: C_0((0,1)) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_{\omega}.$$

Suppose for a moment that

$$\dot{\Lambda} \sim_{\mathsf{u}} \dot{\Lambda}.$$
 (†)

Let $R \in C([0,1]) \otimes M_2$ be a rotation matrix and define

$$\overline{\Lambda}: \textit{C}([0,1]) \otimes \mathcal{A} \longrightarrow \mathsf{M}_2(\mathcal{Q}_{\omega})$$

by

$$\overline{\Lambda} := \begin{pmatrix} \grave{\Phi} & \\ & 0 \end{pmatrix} + \begin{pmatrix} U & \\ & 1 \end{pmatrix} R \begin{pmatrix} \acute{\Lambda} & \\ & 0 \end{pmatrix} R^* \begin{pmatrix} U^* & \\ & 1 \end{pmatrix} + \begin{pmatrix} 0 & \\ & \acute{\Phi} \end{pmatrix}.$$

 $\overline{\Lambda}$ is a *-homomorphism implementing quasidiagonality of \mathcal{A} :

$$\overline{\Lambda}(\mathbf{1}_{[0,1]}\otimes \mathsf{id}_{\mathcal{A}}):\mathcal{A}\longrightarrow \mathsf{M}_2(\mathcal{Q}_\omega)\cong \mathcal{Q}_\omega$$

Even an approximate version of (\dagger) would suffice, but is still a lot to ask for (even though there is no K-theory obstruction). Enters stable uniqueness.

THEOREM (Lin; Dadarlat–Eilers) Given a finite subset $\mathcal{F} \subset C_0((0,1)) \otimes \mathcal{A}$ and $\epsilon > 0$, and a 'sufficiently full' *-homomorphism $\iota : C_0((0,1)) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_{\omega}$, there is $n \in \mathbb{N}$ such that

$$\dot{\Lambda} \oplus \iota^{\oplus n} \approx_{\mathsf{u}, \mathcal{F}, \epsilon} \dot{\Lambda} \oplus \iota^{\oplus n}.$$
(††)

We would now like to use Λ and Λ in place of ι , and apply the patching argument 2n+1 times along the interval.



In this pattern, we do not use the original Λ and Λ , but restrictions to 2n+1 small subintervals. However, these intervals depend on n, and n depends on the maps, hence the intervals!

We therefore need a version of stable uniqueness in which n only depends on \mathcal{A} , \mathcal{F} and ϵ , but not on the maps themselves.

Luckily, Dadarlat-Eilers do provide such a version.

The proof is by contradiction, producing a sequence of pairs of maps of the form

$$(\varphi_n)_{\mathbb{N}}, \ (\psi_n)_{\mathbb{N}}: \mathcal{C} \longrightarrow \prod_{\mathbb{N}} \mathcal{B}.$$

For each n,

$$\mathsf{KK}(\varphi_n) = \mathsf{KK}(\psi_n),$$

but one needs

$$\mathsf{KK}((\varphi_n)_{\mathbb{N}}) = \mathsf{KK}((\psi_n)_{\mathbb{N}}).$$

This is precisely where the UCT gets used, since it makes the map

$$\mathsf{KK}(\mathcal{C},\prod_{\mathbb{N}}\mathcal{B})\longrightarrow\prod_{\mathbb{N}}\mathsf{KK}(\mathcal{C},\mathcal{B})$$

injective.

QUESTIONS

Is the UCT really necessary for the quasidiagonality result?

Does quasidiagonality contribute towards the UCT?

Is there a direct proof of Rosenberg's conjecture, e.g. realising quasidiagonal approximations in terms of Følner sets?