# Quasidiagonality and Amenability

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DEFINITION (Halmos)
A $C^*$-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is quasidiagonal if there is an increasing net of finite rank projections $p_i \nearrow 1_{\mathcal{H}}$ strongly and which are approximately central with respect to $\mathcal{A}$.

An abstract $C^*$-algebra $\mathcal{A}$ is quasidiagonal if it has a faithful quasidiagonal representation.

Equivalently (following Voiculescu):
There are c.p.c. maps $\phi_i : \mathcal{A} \to M_{r_i}$ which are approximately multiplicative and approximately isometric (in Norm).

For $\mathcal{A}$ separable:
$\mathcal{A}$ is quasidiagonal iff there is a diagram

$$\begin{array}{ccc} & & \prod_{\mathbb{N}} \mathcal{Q} \\ & \overset{\text{c.p.c.}}{\nearrow} & \downarrow \\ \mathcal{A} & \underset{*-\text{hom}}{\rightarrowtail} & \mathcal{Q}_\infty . \end{array}$$

Equivalently, after choosing a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$:
$\mathcal{A}$ is quasidiagonal iff there is a diagram

$$\begin{array}{ccc} & & \prod_{\mathbb{N}} \mathcal{Q} \\ & \overset{\text{c.p.c.}}{\nearrow} & \downarrow \\ \mathcal{A} & \underset{*-\text{hom}}{\rightarrowtail} & \mathcal{Q}_\omega . \end{array}$$

Note that $\mathcal{Q}_\omega$ has a unique tracial state $\tau_{\mathcal{Q}_\omega}$ (Ozawa).

For $\mathcal{A}$ separable and nuclear (using Choi–Effros):
$\mathcal{A}$ is quasidiagonal iff there is an embedding

$$\mathcal{A} \overset{\iota}{\rightarrowtail} \mathcal{Q}_\omega.$$

If $\mathcal{A}$ is unital, the embedding $\iota$ may always be assumed to be unital, since $\mathcal{Q}$ is self-similar.

Voiculescu observed that any separable, unital, quasidiagonal $\mathcal{A}$ has a tracial state given by $\tau_{\mathcal{Q}_\omega} \circ \iota$.

Following N. Brown, we say a trace $\tau$ on $\mathcal{A}$ is quasidiagonal, if

$$\tau = \tau_{\mathcal{Q}_\omega} \circ \phi$$

for some (not necessarily injective) $^*$-homomorphism

$$\phi : \mathcal{A} \longrightarrow \mathcal{Q}_\omega.$$

The connection between quasidiagonality and amenability goes back to Rosenberg:

THEOREM / OBSERVATION
For a countable discrete group $G$, if $C^*_r(G)$ is quasidiagonal, then $G$ is amenable.

Inspection of examples led to Rosenberg's

CONJECTURE
For a countable discrete group $G$, if $G$ is amenable, then $C^*_r(G)$ is quasidiagonal.

QUESTIONS
Is every separable, stably finite, nuclear $C^*$-algebra $\mathcal{A}$ quasidiagonal? (QDQ; aka Blackadar–Kirchberg problem)

What if $\mathcal{A}$ is simple?

Is every trace on a separable, nuclear $C^*$-algebra $\mathcal{A}$ quasidiagonal?

In 2013, Ozawa–Rørdam–Sato confirmed Rosenberg's conjecture for elementary amenable groups (a bootstrap type condition).

The argument uses classification of simple, nuclear $C^*$-algebras in the sense of Elliott.

This was strong evidence that classification might be relevant for QDQ, while it was long known that quasidiagonality is relevant for classification:

• Quasidiagonality of cones (Voiculescu) appears in Kirchberg's $\mathcal{O}_2$-embedding theorem.

• Popa introduced local quantisation for quasidiagonal $C^*$-algebras with small projections.

• Lin developed a tracially large version to arrive at TAF classification.

• In 2013, Matui–Sato showed that for $\mathcal{A}$ separable, simple, unital, nuclear, monotracial and quasidiagonal, $\mathcal{A} \otimes$ UHF is TAF.

(They also gave an answer to the Powers–Sakai conjecture about strongly continuous flows on UHF algebras, extending earlier work of Kishimoto.)

• In 2015, Elliott–Gong–Lin–Niu used work of Gong–Lin–Niu, Lin–Niu, Matui–Sato, W, . . . to show that the class

$\{$separable, unital, simple, nuclear, UCT C$^*$-algebras with finite nuclear dimension and only quasidiagonal traces$\}$

is classified by the Elliott invariant.

(Finite nuclear dimension is a notion of covering dimension for nuclear C$^*$-algebras. In the simple case, in large generality it is finite precisely for $\mathcal{Z}$-stable C$^*$-algebras. $\mathcal{Z}$-stability is the C$^*$-algebra analogue of being McDuff.)

Recall that a separable $\mathcal{A}$ is said to satisfy the UCT, if the sequence

$$0 \to \mathrm{Ext}(\mathrm{K}_*(\mathcal{A}), \mathrm{K}_{*+1}(\mathcal{B})) \to \mathrm{KK}(\mathcal{A}, \mathcal{B}) \to \mathrm{Hom}(\mathrm{K}_*(\mathcal{A}), \mathrm{K}_*(\mathcal{B})) \to 0$$

is exact for any $\sigma$-unital $\mathcal{B}$.

UCT PROBLEM:
Does every separable nuclear $C^*$-algebra satisfy the UCT?

THEOREM (Tikuisis–White–W, 2015)
Let $\mathcal{A}$ be a separable, nuclear C$^*$-algebra satisfying the UCT.
Then every faithful trace on $\mathcal{A}$ is quasidiagonal.

Gabe:
Enough to assume the trace to be amenable and $\mathcal{A}$ exact.

COROLLARY
Rosenberg's conjecture holds: $C^*_r(G)$ is quasidiagonal for any discrete, amenable group $G$.

COROLLARY
The Blackadar–Kirchberg problem has an affirmative answer for simple UCT $C^*$-algebras.

COROLLARY (using work by many hands)
Separable, unital, simple, nuclear, UCT $C^*$-algebras with finite nuclear dimension are classified by the Elliott invariant.

In particular we have:

Separable, unital, simple, nuclear, UCT $C^*$-algebras with at most one trace are classified up to $\mathcal{Z}$-stability by their ordered K-groups.
(The traceless case is Kirchberg–Phillips classification.)

Let us return to the main result and have a look at the proof.

THEOREM (Tikuisis–White–W, 2015)
Let $\mathcal{A}$ be a separable, nuclear C$^*$-algebra satisfying the UCT.
Then every faithful trace on $\mathcal{A}$ is quasidiagonal.

We start with a lemma.

LEMMA

Let $\mathcal{A}$ be separable, unital and nuclear, and let $\tau \in \mathrm{T}(\mathcal{A})$.

(i) There is a $^*$-homomorphism

$$\Psi : C_0((0,1]) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_\omega$$

such that

$$\tau_{\mathcal{Q}_\omega} \circ \Psi = \text{Lebesgue} \otimes \tau.$$

(ii) There are $^*$-homomorphisms

$$\acute{\Phi} : C_0((0,1]) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_\omega,$$
$$\grave{\Phi} : C_0([0,1)) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_\omega,$$
$$\Theta : C([0,1]) \longrightarrow \mathcal{Q}_\omega$$

such that

$$\acute{\Phi}|_{C_0((0,1])\otimes 1_{\mathcal{A}}},\ \grave{\Phi}|_{C_0([0,1))\otimes 1_{\mathcal{A}}} \text{ and } \Theta \text{ are compatible}$$

and

$$\tau_{\mathcal{Q}_\omega} \circ \acute{\Phi} = \text{Lebesgue} \otimes \tau,$$
$$\tau_{\mathcal{Q}_\omega} \circ \grave{\Phi} = \text{Lebesgue} \otimes \tau,$$
$$\tau_{\mathcal{Q}_\omega} \circ \Theta = \text{Lebesgue}.$$

PROOF

(i) Let us assume for simplicity that $\tau$ is extremal.
We then have

[Choi–Effros, Sato–White–W, Kirchberg–Rørdam]

$$\begin{array}{ccc} & & \mathcal{Q}_\omega \\ & \overset{\overline{\Psi}\ \text{c.p.c.}\ \perp}{\nearrow} & \Big\downarrow \text{Kaplansky density} \\ \mathcal{A} & \xrightarrow[\text{nuclearity}]{\qquad} & \mathcal{R}^\omega \end{array}$$

Now define

$$\Psi : C_0((0,1]) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_\omega$$

by

$$\Psi(\mathrm{id}_{(0,1]} \otimes a) := \overline{\Psi}(a).$$

(ii) Find

$$\mu : C([0,1]) \longrightarrow \mathcal{Q}_\omega \cap \overline{\Psi}(\mathcal{A})'$$

such that

$$\tau_{\mathcal{Q}_\omega} \circ \mu = \text{Lebesgue}.$$

Define

$$\acute{\Phi}(\mathrm{id}_{(0,1]} \otimes a) := \mu(\mathrm{id}_{(0,1]})\, \overline{\Psi}(a).$$

To find $\grave{\Phi}$ (and then automatically $\Theta$), observe that

$$\acute{\Phi}(\mathrm{id}_{(0,1]} \otimes 1_{\mathcal{A}}) \sim_{\mathrm{u}} 1_{\mathcal{Q}_\omega} - \acute{\Phi}(\mathrm{id}_{(0,1]} \otimes 1_{\mathcal{A}})$$

(e.g. using a Cuntz semigroup argument). $\blacksquare$

By restricting $\acute{\Phi}$ and $\grave{\Phi}$, we obtain $^*$-homomorphisms

$$\acute{\Lambda}, \grave{\Lambda} : C_0((0,1)) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_\omega.$$

Suppose for a moment that

$$\acute{\Lambda} \sim_{\mathrm{u}} \grave{\Lambda}. \tag{$\dagger$}$$

Let $R \in C([0,1]) \otimes \mathrm{M}_2$ be a rotation matrix and define

$$\overline{\Lambda} : C([0,1]) \otimes \mathcal{A} \longrightarrow \mathrm{M}_2(\mathcal{Q}_\omega)$$

by

$$\overline{\Lambda} := \begin{pmatrix} \grave{\Phi} & \\ & 0 \end{pmatrix} + \begin{pmatrix} U & \\ & 1 \end{pmatrix} R \begin{pmatrix} \acute{\Lambda} & \\ & 0 \end{pmatrix} R^* \begin{pmatrix} U^* & \\ & 1 \end{pmatrix} + \begin{pmatrix} 0 & \\ & \acute{\Phi} \end{pmatrix}.$$

$\overline{\Lambda}$ is a $^*$-homomorphism implementing quasidiagonality of $\mathcal{A}$:

$$\overline{\Lambda}(1_{[0,1]} \otimes \mathrm{id}_{\mathcal{A}}) : \mathcal{A} \longrightarrow \mathrm{M}_2(\mathcal{Q}_\omega) \cong \mathcal{Q}_\omega$$

Even an approximate version of ($\dagger$) would suffice, but is still a lot to ask for (even though there is no K-theory obstruction). Enters stable uniqueness.

THEOREM (Lin; Dadarlat–Eilers)
Given a finite subset $\mathcal{F} \subset C_0((0,1)) \otimes \mathcal{A}$ and $\epsilon > 0$, and a 'sufficiently full' $^*$-homomorphism $\iota : C_0((0,1)) \otimes \mathcal{A} \longrightarrow \mathcal{Q}_\omega$, there is $n \in \mathbb{N}$ such that

$$\acute{\Lambda} \oplus \iota^{\oplus n} \approx_{\mathrm{u},\mathcal{F},\epsilon} \grave{\Lambda} \oplus \iota^{\oplus n}. \qquad (\dagger\dagger)$$

We would now like to use $\acute{\Lambda}$ and $\grave{\Lambda}$ in place of $\iota$, and apply the patching argument $2n+1$ times along the interval.

$$
\begin{array}{ccccccccccc}
\grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \acute{\Lambda} \\
\grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} \\
\grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} \\
\vdots & \sim_u & \vdots & \sim_u & \vdots & \sim_u & \vdots & \dots & \vdots & \sim_v & \vdots \\
\grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} \\
\grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} \\
\grave{\Lambda} & & \grave{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} \\
\grave{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda}
\end{array}
$$

In this pattern, we do not use the original $\grave{\Lambda}$ and $\acute{\Lambda}$, but restrictions to $2n+1$ small subintervals.
However, these intervals depend on $n$, and $n$ depends on the maps, hence the intervals!

We therefore need a version of stable uniqueness in which $n$ only depends on $\mathcal{A}$, $\mathcal{F}$ and $\epsilon$, but not on the maps themselves.

Luckily, Dadarlat–Eilers do provide such a version.

The proof is by contradiction, producing a sequence of pairs of maps of the form

$$(\varphi_n)_{\mathbb{N}},\ (\psi_n)_{\mathbb{N}} : \mathcal{C} \longrightarrow \prod_{\mathbb{N}} \mathcal{B}.$$

For each $n$,

$$\mathrm{KK}(\varphi_n) = \mathrm{KK}(\psi_n),$$

but one needs

$$\mathrm{KK}((\varphi_n)_{\mathbb{N}}) = \mathrm{KK}((\psi_n)_{\mathbb{N}}).$$

This is precisely where the UCT gets used, since it makes the map

$$\mathrm{KK}(\mathcal{C}, \prod_{\mathbb{N}} \mathcal{B}) \longrightarrow \prod_{\mathbb{N}} \mathrm{KK}(\mathcal{C}, \mathcal{B})$$

injective. ∎

QUESTIONS

Is the UCT really necessary for the quasidiagonality result?

Does quasidiagonality contribute towards the UCT?

Is there a direct proof of Rosenberg's conjecture, e.g. realising quasidiagonal approximations in terms of Følner sets?