Anomaly and global inconsistency in QFTs

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Why topological phases of matter?

- Topological materials are interesting by themselves.
    - Specific features for quantum many-body physics

- Useful tools to understand strongly-coupled QFTs.

  Very hard to tackle them directly

$\Rightarrow$ Systematic understanding on "symmetry" in quantum systems.

Anomaly matching

't Hooft anomaly matching = LSM theorem

for hep-th ... for cond-mat

Anomaly T : a d-dim QFT with symmetry G.

We define its partition function with G-gauge field A:

$$Z_T[A].$$

Under G-gauge transformations, $A \to A + \delta_\lambda A$,

$$Z_T[A+\delta_\lambda A] = \exp\left(i \int \alpha(\lambda, A)\right) Z_T[A].$$

local d-dim. functional of λ, A.

If $\alpha \neq \delta_\lambda(\mathcal{L}(A))$, α is an 't Hooft anomaly.

Anomaly matching

Inflow Argument $\Longrightarrow$ α must be the same for both UV/IR.

Trivially gapped ground state cannot match the nontrivial anomaly.

Assume that we have such a system. Then,

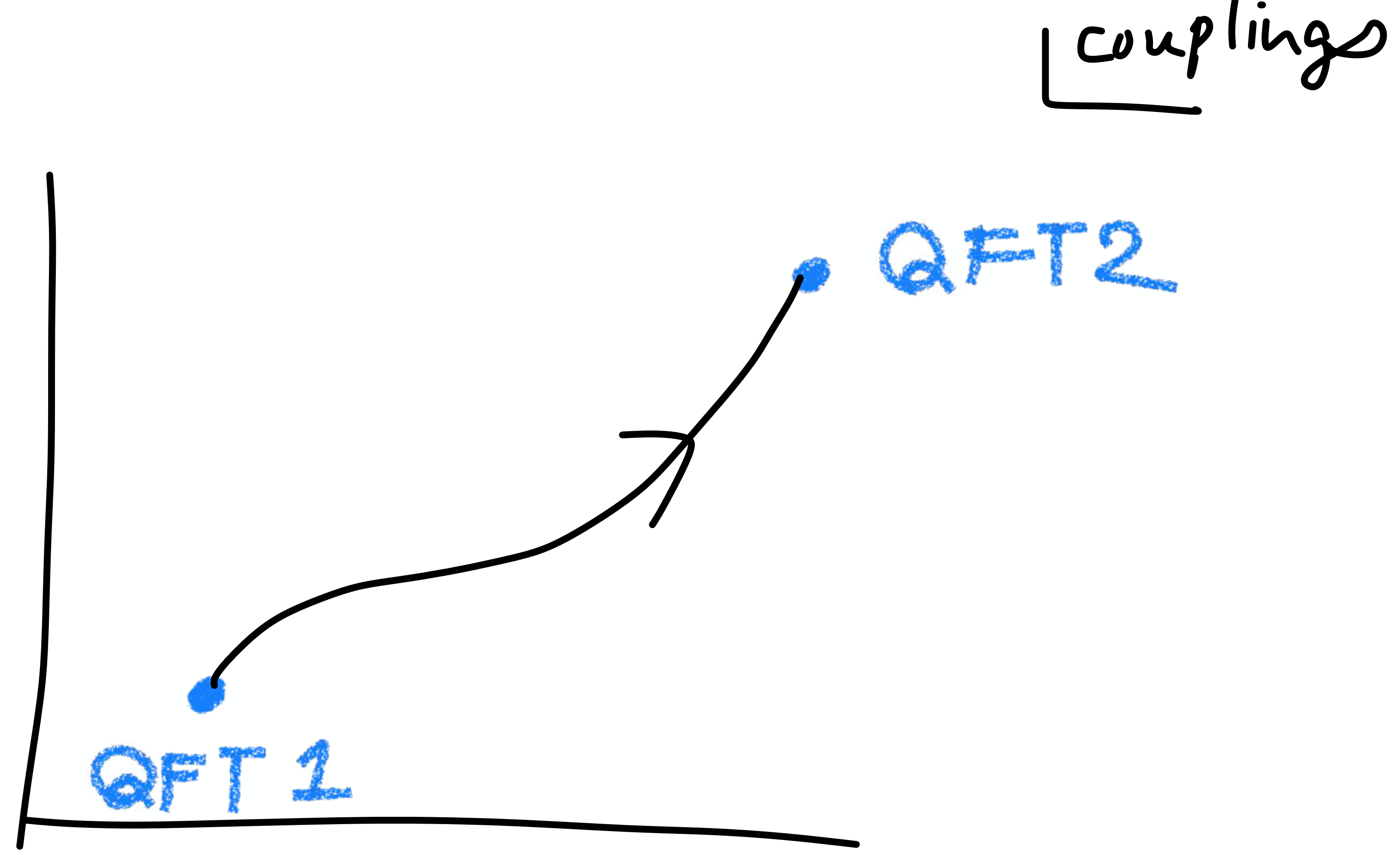
$$Z[A] = \exp\left(-\int \underbrace{\mathcal{L}(A)}_{\text{local } d\text{-dim. functional}}\right).$$

This is because every correlator drop exponentially fast due to the gap without any degeneracies.

$$\Rightarrow \quad \alpha(\lambda, A) = \delta_\lambda(\mathcal{L}(A)) \quad : \text{removable within } d\text{-dim.}$$

Space of couplings

Let's consider a continuous deformation of the local Lagrangian / Hamiltonian.



Do we encounter any quantum phase transition during such paths?

⇝ Especially if QFT 1 & 2 are trivially gapped, are they in the same SPT states?

2d $\mathbb{CP}^{N-1}$ sigma model

U(1) gauge field.

$$\mathcal{L} = \int \left(\frac{1}{g^2} \left| (\partial_\mu + i a_\mu) \vec{z} \right|^2 \right) + i \frac{\theta}{2\pi} \int da$$

$\vec{z} \in \mathbb{C}^N$, $|\vec{z}|^2 = 1$

For $N=2$, this is a low-energy theory of AF spin chains w/ $\theta = 2\pi S$ (Haldane).

Symmetry : • $\frac{SU(N)}{\mathbb{Z}_N}$ spin rotation.

(Spin variable $\mathbb{S} = \vec{z}^\dagger \cdot \mathbb{T} \cdot \vec{z}$)

• C : $z \leftrightarrow z^*$, $a \to -a$ charge conj. at $\theta \in \pi \mathbb{Z}$.

On closed, oriented 2d manifolds, $\int da \in 2\pi \mathbb{Z}$.

$$\Rightarrow \quad Z_{\theta+2\pi} = Z_{\theta}.$$

Now, introduce a background gauge field for $\frac{SU(N)}{\mathbb{Z}_N}$.

$$\frac{SU(N)}{\mathbb{Z}_N} \text{ gauge field} \sim \begin{cases} A: \ SU(N) \text{ 1-form gauge field} \\ B: \ \mathbb{Z}_N \text{ 2-form gauge field.} \end{cases}$$

With such a background,

$$Z_{\theta+2\pi}[A,B] = \underbrace{\exp\left(i\int B\right)}_{\text{similar to anomalous phase}} \cdot Z_{\theta}[A,B].$$

[Komargodski, Sharon, Thorngren, Zhou, '17]

Can we move from $\theta = 0$ to $\theta = 2\pi$ continuously?

If so, we should be able to promote θ to slowly-varying 2π-periodic scalar fields.

The partition func. is now

$$Z[\underset{\text{2π-periodic scalar}}{\theta}, \underbrace{A, B}_{\substack{\text{background} \\ SU(N)/\mathbb{Z}_N \text{ gauge field}}}] = \exp\left(-\int \mathcal{L}(\theta, A, B)\right).$$

$\mathcal{L}(\theta, A, B)$ must satisfy

$$\int \left(\mathcal{L}(\theta+2\pi, A, B) - \mathcal{L}(\theta, A, B)\right) = -i \int B.$$

No such local $\mathcal{L}(\theta, A, B)$ exists!

For example, naive guess

$$\mathcal{L}(\theta, A, B) = i\,\frac{\theta}{2\pi} \wedge B$$

does not work, because it violates the 1-form gauge invariance $B \to B + d\lambda^{(1)}$.

$\Rightarrow$ States at $\theta = 0$ and $\theta = 2\pi$ should be different as SPT with $\frac{SU(N)}{\mathbb{Z}_N}$ global symmetry.

(This is a rephrasement of the fact that $S = 1, 3, \cdots$ chains & $S = 2, 4, \cdots$ chains are different SPTs in cond-mat.)

Inflow from 3 dim.

We could not write down 2-dim. local Lagrangian $\mathcal{L}(\theta, A, B)$ with the desired properties.

In 3 dim, however, the topological Lagrangian

$$S_{3\text{-dim}} = i \int_{M_3} \frac{1}{2\pi} d\theta \wedge B$$

does a good job. [Komargodski, Sharon, Thorngren, Zhou '17] [Kikuchi, YT '17]

$$Z_{2d}[\theta, A, B] \times \exp\left(-S_{3\text{-dim}}\right)$$

has both 2π-periodicity in θ & all gauge invariances.

⇝ Very same with anomaly-inflow

Charge conjugation C at $\theta \in \pi\mathbb{Z}$

So far, we've totally forgotten about charg conj. $C: a \to -a$.

$$\mathcal{L} = i\frac{\theta}{2\pi}\int da + \cdots \xrightarrow{C} i\frac{(-\theta)}{2\pi}\int da + \cdots$$

Since $Z_{\theta+2\pi} = Z_\theta$, C is a good symmetry at $\theta \in \pi\mathbb{Z}$.

Using the advantage of C,

we can further constrain on possible phase diagrams.

$$\underline{Z_{\theta=\pi}[A,B]} \xrightarrow{C} Z_{\theta=-\pi}[A,B] = \underbrace{\exp\left(-i\int B\right)}_{\text{anomalous phase.}} \cdot \underline{Z_{\theta=\pi}[A,B]}.$$

anomalous phase. counter term?

Anomaly at $\theta = \pi$ for $N \in 2\mathbb{Z}_{>0}$

For even N, i.e. $\mathbb{CP}^{1,3,\cdots}$ sigma models,

$$Z_\pi[B] \xrightarrow{C} e^{-i\int B} Z_\pi[B]$$

is a genuine anomaly.

matching $\Longrightarrow$

- Spontaneous C-breaking $(N = 4, 6, \cdots)$
- Gapless excitations $(N = 2)$

No anomaly for $N = 2\mathbb{Z}_{\geq 0} + 1$

Unlike even N cases, there is no genuine anomaly for odd N.

$\frac{N \pm 1}{2}$ is integer !!

$\Rightarrow$ $Z_{\theta=\pi}[A,B] \times \exp\left(i \frac{N-1}{2} \int B\right)$ is C- inv. with $\frac{SO(N)}{\mathbb{Z}_N}$ gge-inv.

$$\left(\xrightarrow{C} \left(e^{-i\int B} Z_\pi[A,B]\right) \times \exp\left(-i \frac{N-1}{2} \int B\right) = Z_\pi[A,B] \cdot e^{i\frac{N-1}{2}\int B} \times \underbrace{e^{-iN\int B}}_{=1} \right)$$

Thus, for odd N, the ground state at $\theta = \pi$ can be a unique gapped one. (Even though C is broken spontaneously very likely ...)

Global inconsistency

Assume the system is trivially gapped both at $\theta=0$ and $\theta=\pi$.

↳ i.e. C at $\theta=0,\pi$ are both unbroken due to "unique gapped" condition.

Then, we should have a local effective action

$$Z[\theta, A, B] = \exp\left(-\int \mathcal{L}(\theta, A, B)\right).$$

For C-inv. at $\theta=0$,

$$\mathcal{L}(0, A, B) = 0.$$

For C-inv. at $\theta=\pi$,

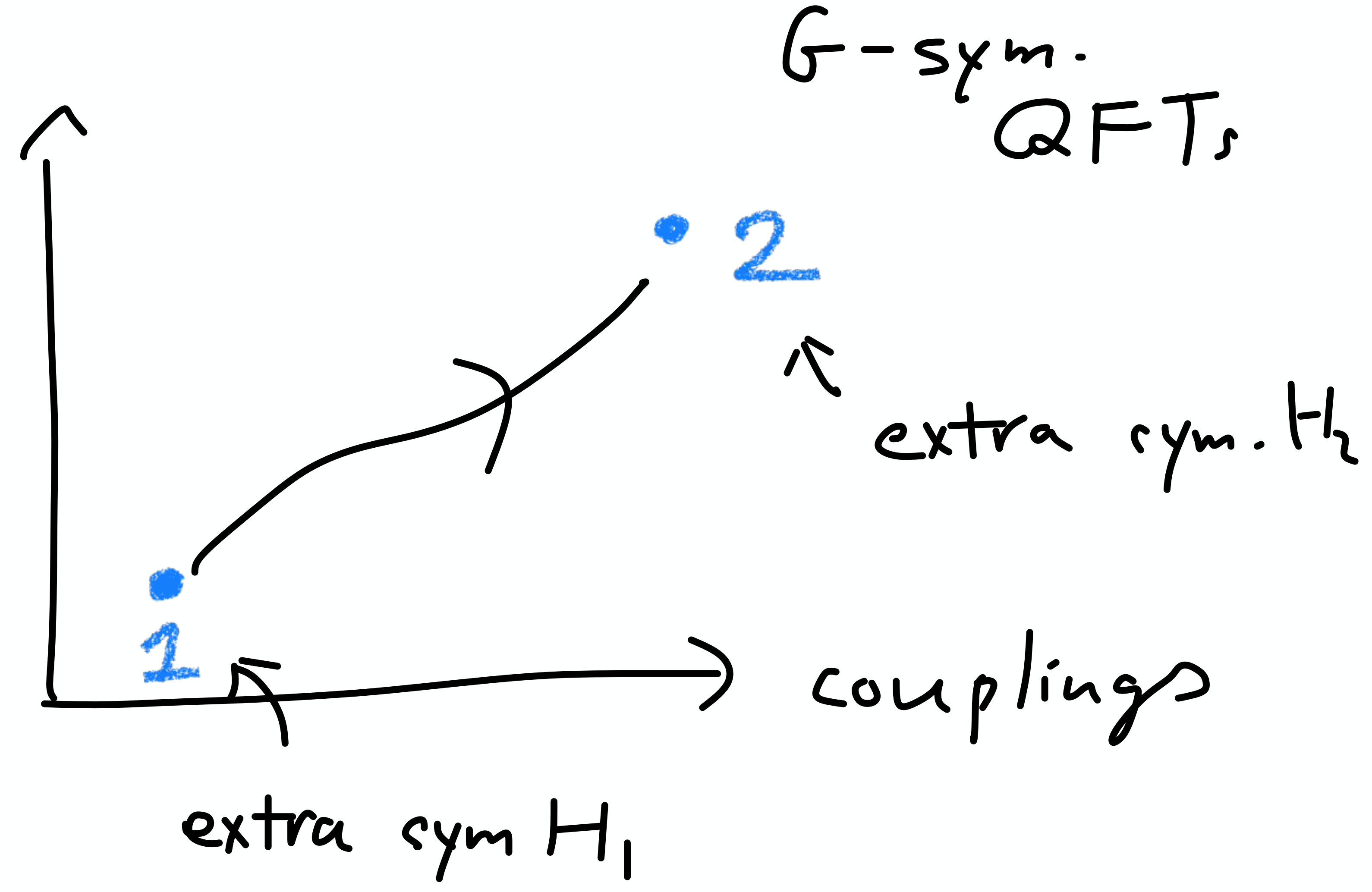
$$\mathcal{L}(\pi, A, B) = i\,\frac{N+1}{2}\int B.$$

← Again, no local 2d Lagrangian with this property.

(up to real parts)

QFT 1 : sym. $G \tilde{\times} H_1$ without anomaly

QFT 2 : sym. $G \tilde{\times} H_2$ without anomaly



If $H_{1,2}$ require different discrete θ-angles of G-gauge fields, (← global inconsistency)

then QFT_1 and QFT_2 cannot be in the same SPT.

[Gaiotto, Kapustin, Komargodski, Seiberg '17] [Kikuchi, YT '17] [YT, Sulejmanpasic '18]

2 possibilities for matching condition:

- QFT_1 or QFT_2 has degenerate ground states (like anomaly matching)
- QFT_1 and QFT_2 are different SPTs with sym. G.

Application : 4d gauge theories

SU(N) Yang-Mills theory

$$\mathcal{L} = \frac{1}{g^2}\int \mathrm{tr}(f \wedge * f) + i\frac{\theta}{8\pi^2}\int \mathrm{tr}(f \wedge f)$$

f: SU(N) gauge field strength.

This model shows a very similar behavior with that of 2d $\mathbb{CP}^{N-1}$ sigma model.

$$\frac{SU(N)}{\mathbb{Z}_N} \text{ spin rotation} \longleftrightarrow \mathbb{Z}_N \text{ 1-form sym.}$$

$$C \text{ at } \theta = 0, \pi \longleftrightarrow CP \text{ at } \theta = 0, \pi$$

B : $\mathbb{Z}_N$ 2-form gauge field for $\mathbb{Z}_N^{(1)}$ ← 1-form sym.

Then, we can find [Gaiotto, Kapustin, Komargodski, Seiberg '17]

$$Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B\wedge B} \cdot Z_\theta[B].$$

$\Longrightarrow$
- $\underline{N\in 2\mathbb{Z}}$ $\quad$ $\mathbb{Z}_N^{(1)}$ & T is anomalous at $\theta=\pi$.
- $\underline{N\in 2\mathbb{Z}+1}$ $\quad$ No anomaly.

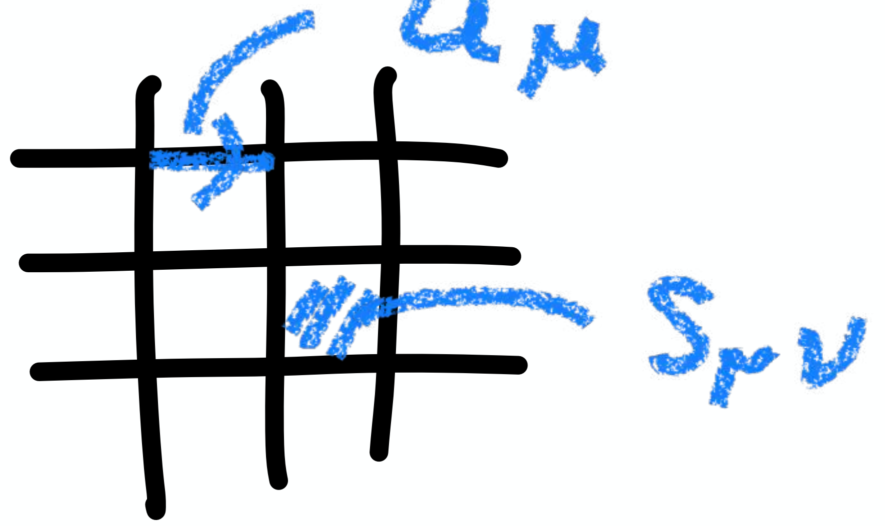
$\rightsquigarrow$ global inconsistency bet. $\theta=0,\pi$.

Easier model to understand physics?

Lattice U(1) gauge theory w/ confinement-deconfinement

(Banks, Myerson, Kogut '77
Savit '77 ...)

$$\begin{cases} a_\mu : \mathbb{R}\text{-valued link variable} \\ s_{\mu\nu} : \mathbb{Z}\text{-valued plaquette variable} \end{cases}$$



$\Rightarrow$ $f = da - 2\pi s$: field strength

$$\mathcal{L} = \frac{1}{g^2}\int f \wedge * f + iN \int \underline{n_\mu}\, a_\mu$$

world-line of electric charges

* Theory inherits magnetic particles

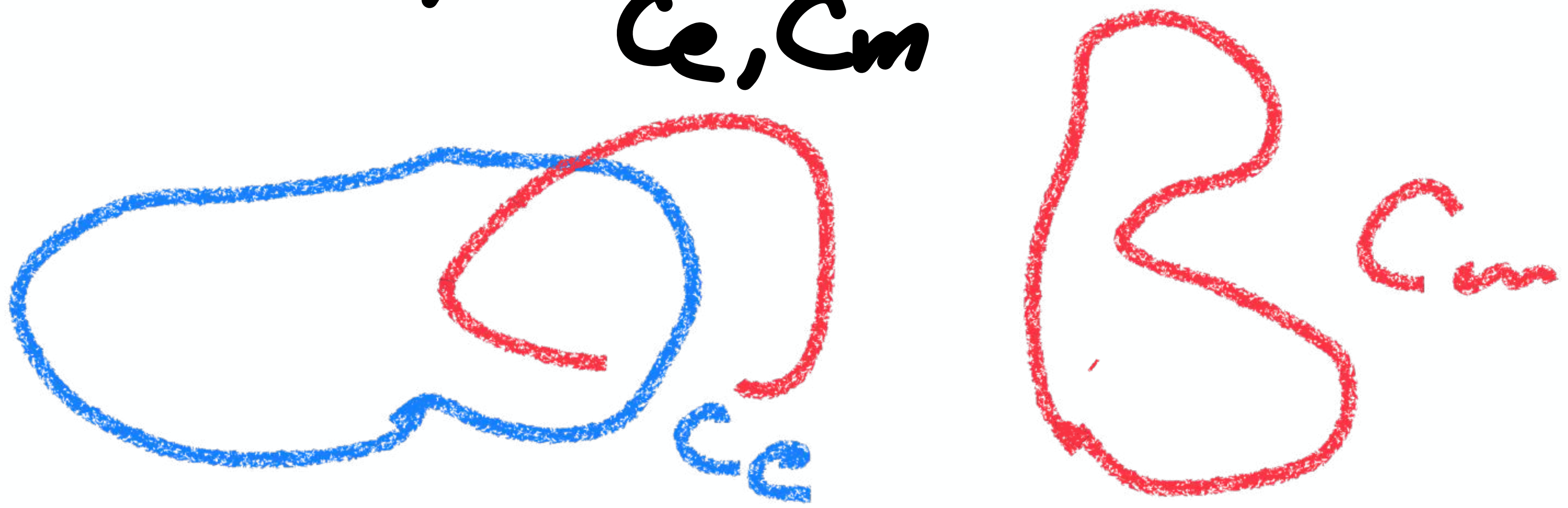
$$m = \frac{1}{2\pi} * df = * ds .$$

Cardy-Rabinovici ('82) : θ-angle via Witten effect "$n_\mu \Rightarrow n_\mu + \frac{\theta}{2\pi} m_\mu$".

Cardy - Rabinovici model

In a (formal) continuum description,

$$Z = \int \mathcal{D}a \, \exp \int \left(-\frac{1}{2g^2}|f|^2 + i\frac{N\theta}{8\pi^2}\int f \wedge f \right) \times \sum_{\substack{\text{world-lines} \\ C_e, C_m}} \overset{\text{Wilson line}}{W^N(C_e)} \cdot \overset{\text{'t Hooft line}}{H(C_m)}$$



Witten effect

$$(Nn, m) \xrightarrow{\theta} \left(N\left(n + \frac{\theta}{2\pi}m\right), m\right).$$

Vacuum energy of CR model

If a particle of (n,m) condenses,

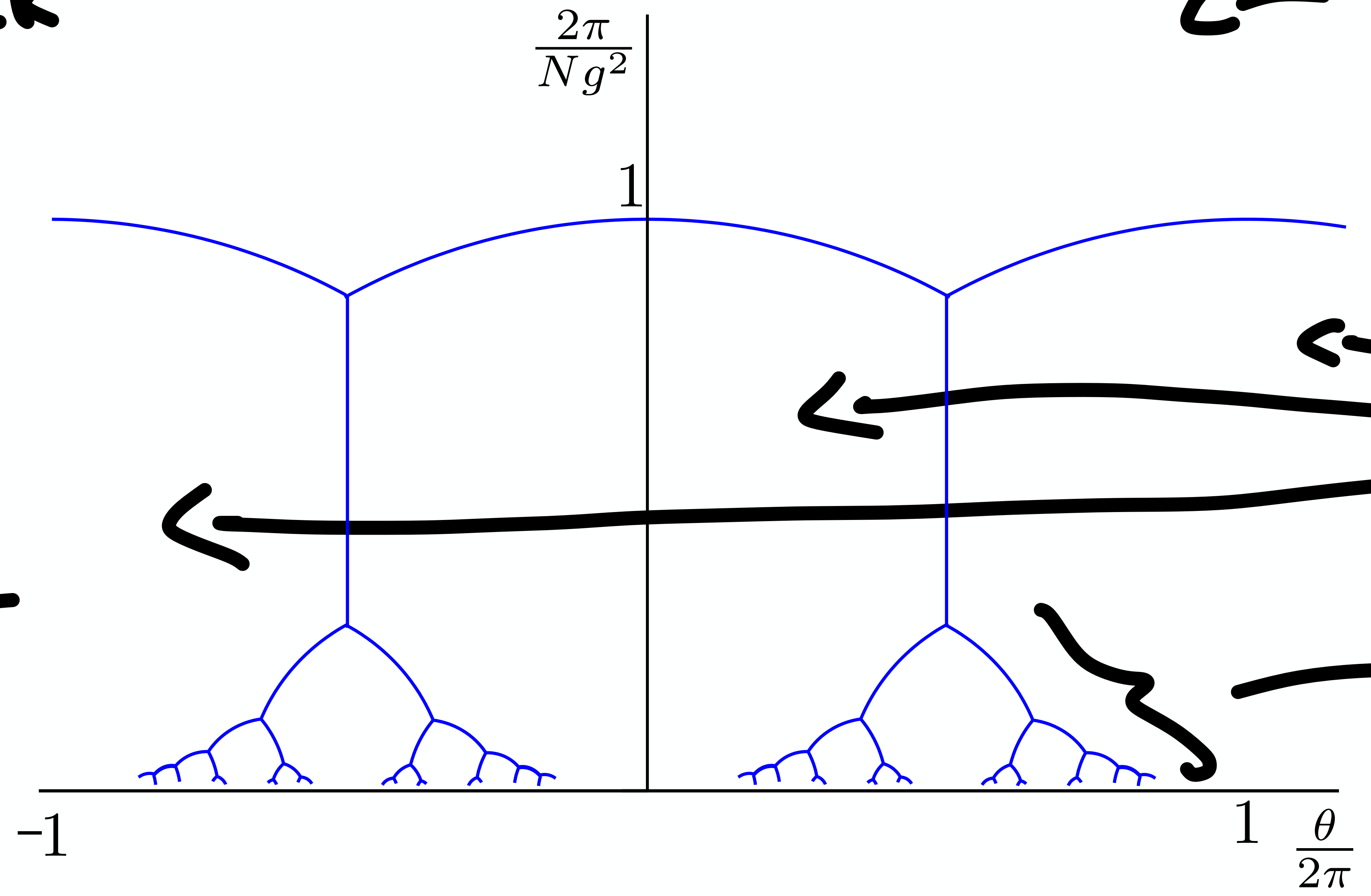
$$F_{(n,m)} \sim g^2 \left(N\left(n+\frac{\theta}{2\pi}m\right)\right)^2 + \left(\frac{2\pi}{g}\right)^2 m^2$$

$$= \frac{N}{\mathrm{Im}(\tau)} |n+m\tau|^2 \qquad \left(\tau = \frac{\theta}{2\pi} + i\,\frac{2\pi}{Ng^2}\right).$$

Predicted phases

weak $\uparrow$

$\leftarrow$ Higgs



$\downarrow$ strong

$\leftarrow$ Confinement

Oblique confinement

't Hooft anomaly of CR model.

This model has the same anomaly with pure YM [GKKS '17]

B : Gauge field for $\mathbb{Z}_N$ 1-form symmetry

$$Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B\wedge B} Z_\theta[B]$$

This anomaly can be used to constrain the possible phase diagrams.

$\theta \to \theta + 2\pi$: $\left(\theta\text{-term is } i\frac{N\theta}{8\pi^2}\int f\wedge f \right)$.

$$\Delta S = 2\pi i \frac{N}{8\pi^2}\int (f-B)\wedge(f-B)$$

$$= 2\pi i \left(\underbrace{\frac{N}{8\pi^2}\int f\wedge f}_{\in N\mathbb{Z}} - \underbrace{\frac{N}{4\pi^2}\int f\wedge B}_{\in \mathbb{Z}} + \frac{N}{8\pi^2}\int B\wedge B \right)$$

$$= i\frac{N}{4\pi}\int B\wedge B.$$

(Under this transformation, $H \to H W^{-N}$ by Witten eff.,
so $\{n_\mu, m_\mu\} \to \{n_\mu - m_\mu, m_\mu\}$.)

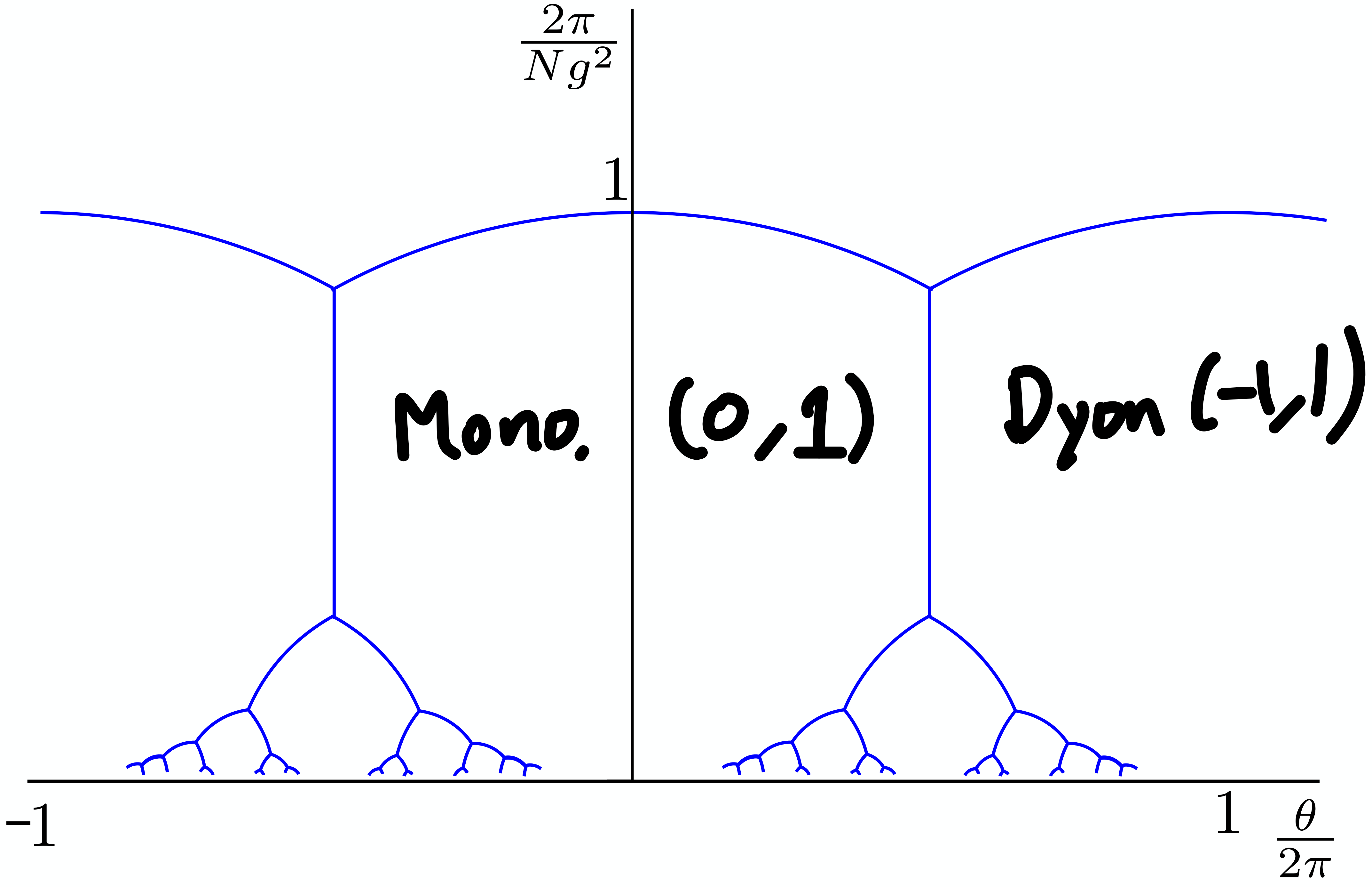
Anomaly matching in confined phase

$\theta \approx 0$: Monopole condensation

$$Z_0[B] \sim 1$$

$\theta \approx 2\pi$: Dyon condensation

$$Z_{2\pi}[B] \sim e^{i\frac{N}{4\pi}\int B\wedge B}$$



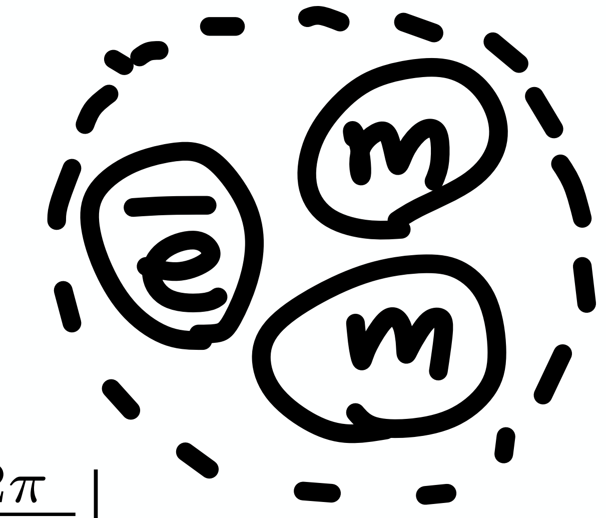
These two vacua are different as SPT phases with $\mathbb{Z}_N$ 1-form sym.

Exotic condensation : Oblique confinement.

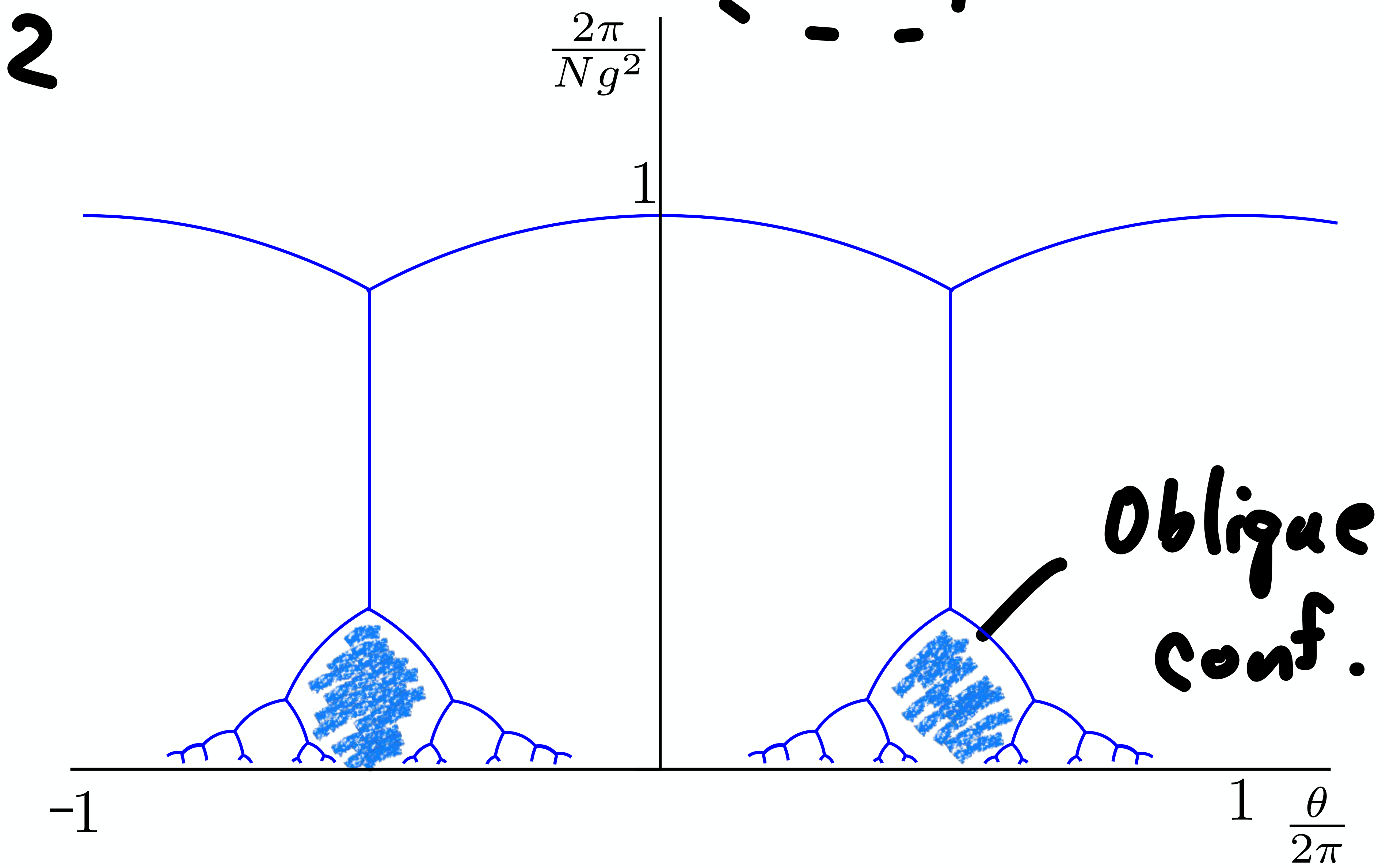
Around $\theta = \pi$, exotic condensation may be possible :

$$F_{\text{mono}} = F_{(0,1)} \sim N^2 g^2 \Big(\underbrace{\frac{\theta}{2\pi}}_{=\frac{1}{2} \text{ at } \theta = 0}\Big)^2$$

Consider a composite particle with $(-1, 2)$



$$F_{\text{oblique}} = F_{(-1,2)} \sim N^2 g^2 \Big(\underbrace{-1 + 2 \cdot \frac{\theta}{2\pi}}_{\substack{=0 \\ \text{at } \theta = \pi}}\Big)^2$$



Low-energy property of oblique confinement

$N \in 2\mathbb{Z}$ $\mathbb{Z}_N^{(1)} \xrightarrow{\text{SSB}} \mathbb{Z}_{N/2}^{(1)}$

Only $W^{\frac{N}{2}}(C)$ obeys the perimeter law.

$N \in 2\mathbb{Z}+1$ All nontrivial Wilson loops are confined.

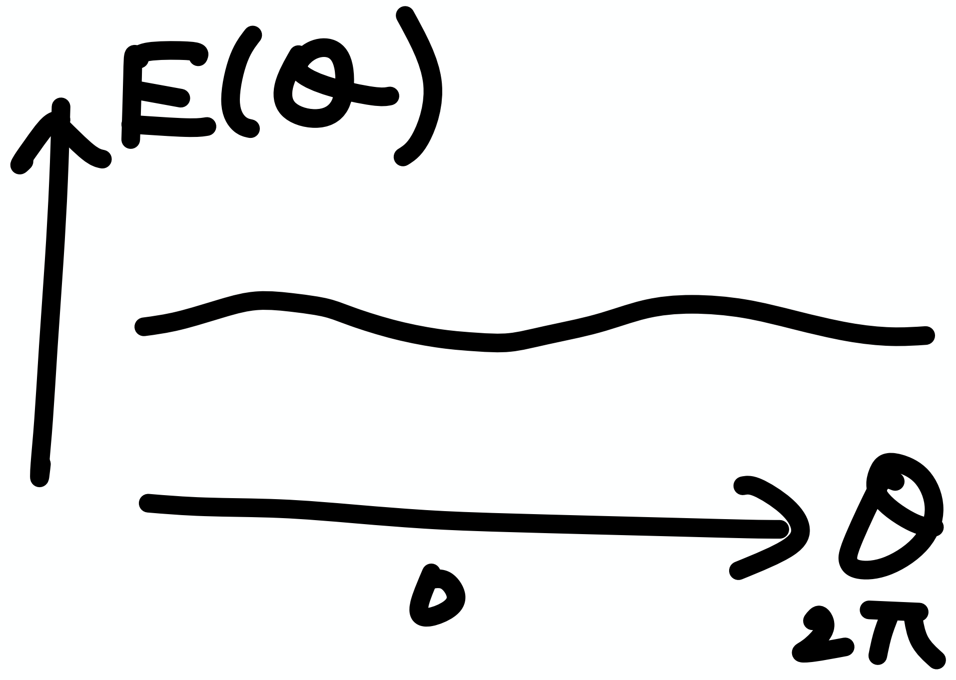
It's an SPT with $\mathbb{Z}_N^{(1)}$,

$$Z[B] \sim \exp\left(i\,\frac{N-1}{2}\cdot\frac{N}{4\pi}\int B\wedge B\right).$$

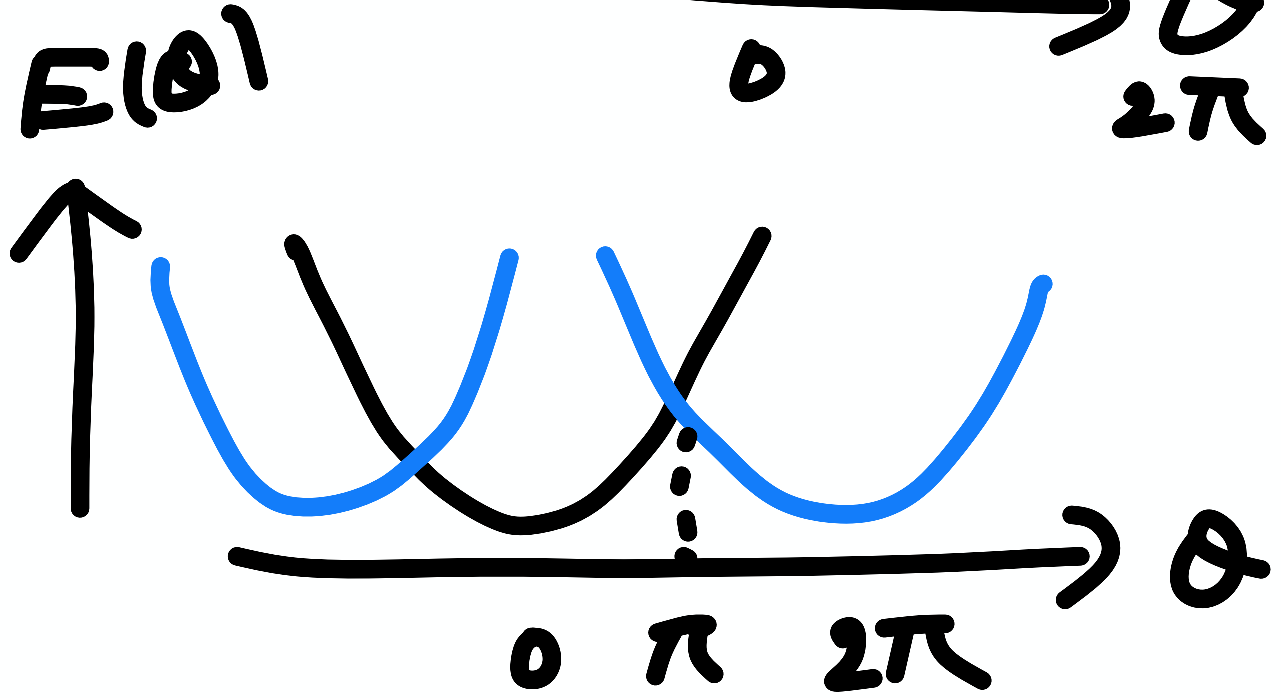
(cf. 't Hooft '81 for SU(2) YM) [Honda, YT '20]

Possible scenarios to match the anomaly & global inconsistency

- Higgs (or, Coulomb) $\quad \mathbb{Z}_N^{(1)} \to 1$



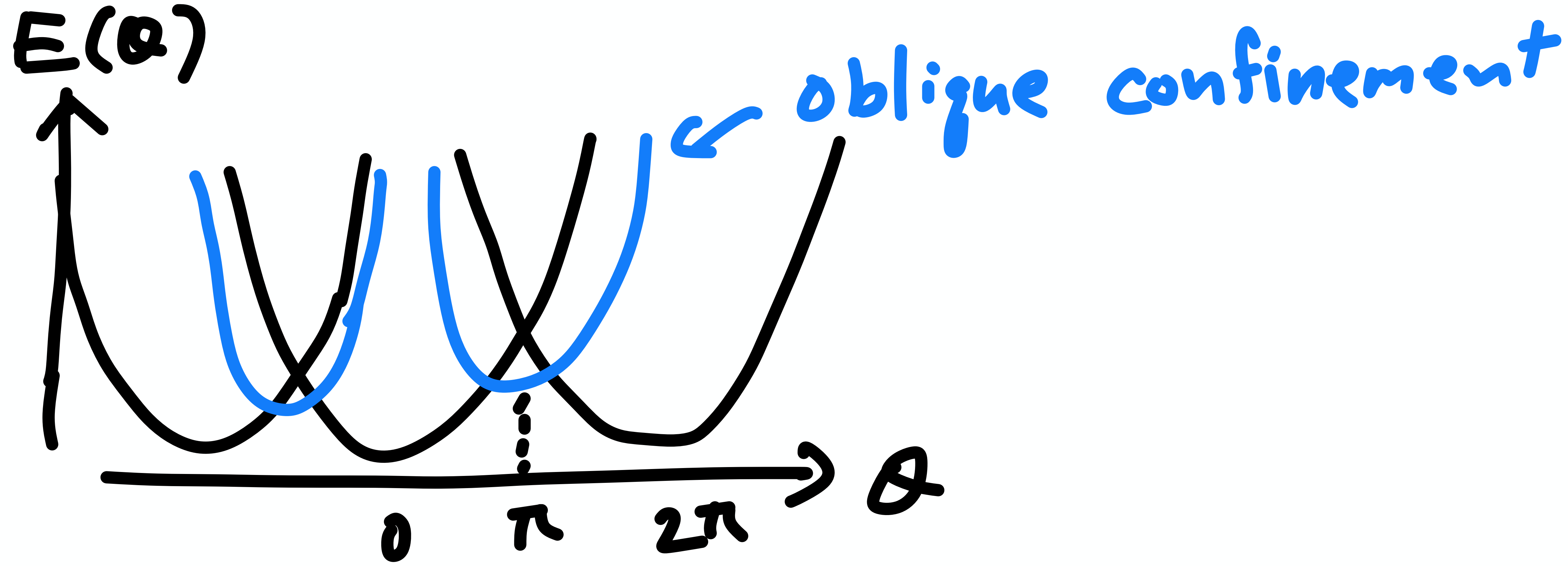
- Confinement $\quad \cancel{CP}$



- Oblique confinement

$\mathbb{Z}_N^{(1)} \to \mathbb{Z}_{N/2}^{(1)}$ (N : even)

$\mathbb{Z}_N^{(1)}$, CPV (N : odd)



Thanks for the attention !!