Vertex operator algebras and category theory in conformal field theory

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Outline

I Some motivation from lattice models

conformal field theory

II

vertex operator algebra

III

braided tensor cat. of VOA - modules

& commutative algebra in (two copies of) cat. of VOA-modules

IV

(arrows need extra conditions to work)

Part I

Conformal field theory

"A conformal field theory is a quantum field theory that is covariant under conformal transformations"

here :

- two-dimensional
- Euclidean (as opposed to Minkowski)
- oriented
- genus - zero
- discrete spectrum

Motivation from lattice models

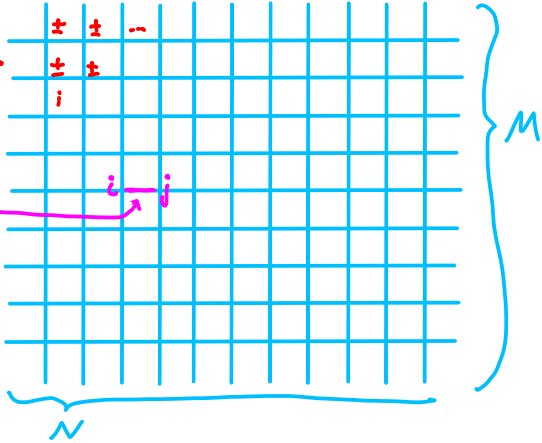
E.g. 2d Ising model

Λ : lattice points

$\sigma : \Lambda \to \{\pm 1\}$ spin config.

$$E[\sigma] = -\sum_{\langle ij \rangle} \sigma_i \sigma_j$$

equal spins $\hat{=}$ low energy

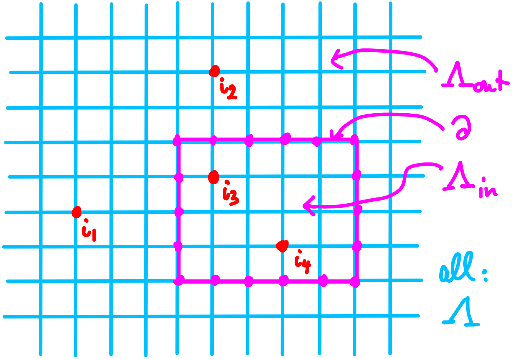


partition function $\quad Z_{MN} = \sum_{\sigma} e^{-\beta E[\sigma]} \qquad \beta > 0$

correlation function $\quad \langle \sigma_i \sigma_j \cdots \rangle_{MN} = \frac{1}{Z_{MN}} \sum_{\sigma} (\sigma_i \sigma_j \cdots) e^{-\beta E[\sigma]}$

... motivation from lattice models

correlation function $\langle \sigma_i \sigma_j \cdots \rangle_{MN} = \frac{1}{Z_{MN}} \sum_{\sigma} (\sigma_i \sigma_j \cdots) e^{-\beta E[\sigma]}$



$$\langle \sigma_{i_1} \cdots \sigma_{i_4} \rangle_{MN} = \frac{1}{Z_\Lambda} \sum_{\delta : \partial \to \{\pm 1\}} Z_{out}(\delta) \cdot Z_{in}(\delta)$$

$$Z_{in}(\delta) = \sum_{\substack{\sigma : \Lambda_{in} \to \{\pm 1\} \\ \sigma|_\partial = \delta}} \sigma_{i_3} \sigma_{i_4} \, e^{-\beta E[\sigma]}$$

$$Z_{out}(\delta) = \sum_{\substack{\sigma : \Lambda_{out} \to \{\pm 1\} \\ \sigma|_\partial = \delta}} \sigma_{i_1} \sigma_{i_2} \, e^{-\beta E[\sigma]}$$

(for $E[\sigma]$ in Z_{out}, links in ∂ are not summed over to avoid overcounting)

... motivation from lattice models

partition function $\quad Z_{MN} = \sum_{\sigma} e^{-\beta E[\sigma]} \qquad \beta > 0$

correlation function $\quad \langle \sigma_i \sigma_j \cdots \rangle_{MN} = \frac{1}{Z_{MN}} \sum_{\sigma} (\sigma_i \sigma_j \cdots) e^{-\beta E[\sigma]}$

$$Z_{M,N} \xrightarrow{M,N\to\infty} \infty$$

but

$$\langle \sigma_i \sigma_j \cdots \rangle := \lim_{M,N\to\infty} \langle \sigma_i \sigma_j \cdots \rangle_{MN} \text{ exists}$$

... motivation from lattice models

fun facts about 2pt. correlator

set $s := \sinh(2\beta)$ $\quad (s=1 \Leftrightarrow \beta = \ln\sqrt{1+\sqrt{2}})$

$$\langle \sigma_{0,0}\, \sigma_{n,n} \rangle = \begin{cases} s<1 : & (\text{const})\, n^{-1/2}\, \boxed{s^{2n}}\, (1+O(n^{-1})) \\ s>1 : & (\text{const})\, (1 + O(n^{-2}\boxed{s^{-4n}})) \\ s=1 : & (\text{const})\, \boxed{n^{-1/4}}\, (1+O(n^{-2})) \end{cases}$$

$\square$ exponential decay $\quad$ $\square$ long range interaction

... motivation from lattice models

Continuum model of "long range behaviour at critical point":

for $x_1, \dots, x_k \in \mathbb{C}$:

$$\langle \sigma(x_1) \cdots \sigma(x_k) \rangle_{\text{cont}} := \lim_{\lambda \to \infty} \lambda^{k\Delta} \langle \sigma_{[\lambda x_1]} \cdots \sigma_{[\lambda x_k]} \rangle$$

at $\sinh(2\beta) = 1$

nearest lattice point

where here

$$\Delta = \tfrac{1}{8}$$

... motivation from lattice models

$$\langle \sigma(x_1) \cdots \sigma(x_k) \rangle_{cont} := \lim_{\lambda \to \infty} \lambda^{k\Delta} \langle \sigma_{[\lambda x_1]} \cdots \sigma_{[\lambda x_k]} \rangle$$

where here $\Delta = 1/8$

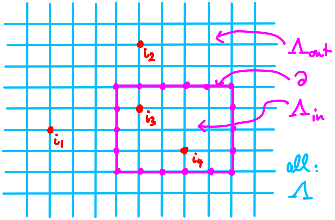
Scale covariance : for $\mu > 0$

$$\boxed{\begin{aligned} \langle \sigma(\mu x_1) \cdots \sigma(\mu x_k) \rangle_{cont} &= \lim_{\lambda \to \infty} \lambda^{k\Delta} \langle \sigma_{[\lambda \mu x_1]} \cdots \rangle \\ &\overset{\lambda' = \lambda\mu}{=} \lim_{\lambda' \to \infty} \mu^{-k\Delta} \lambda'^{k\Delta} \langle \sigma_{[\lambda' x_1]} \cdots \rangle \\ &= \mu^{-k\Delta} \langle \sigma(x_1) \cdots \sigma(x_k) \rangle_{cont} \end{aligned}}$$

... motivation from lattice models

Two observations we want to model

1) Cutting computation of correlators into smaller pieces



2) Scale covariance

$$\langle \sigma(\mu x_1) \cdots \sigma(\mu x_k) \rangle_{cont}$$
$$= \mu^{-k\Delta} \langle \sigma(x_1) \cdots \sigma(x_k) \rangle_{cont}$$

Full conformal symmetry later

Part II

Scale covariant field theory

Data (part 1): $\mathcal{F}$, a $\mathbb{C}$-v.sp. "space of fields"

$\Delta \in \operatorname{End} \mathcal{F}$ "generator of scale transformations"

$\mathbb{1} \in \mathcal{F}$ "vacuum vector"

Properties (part 1) • $\mathcal{F} = \bigoplus_{\delta \in \mathbb{R}} \mathcal{F}_\delta$ could use $\mathbb{C}$ instead

where $\mathcal{F}_\delta$ is eigenspace of Δ of eigenvalue δ.

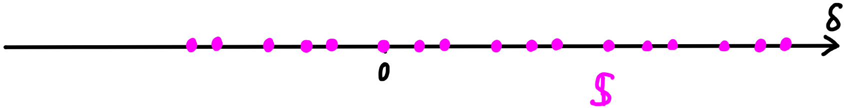
could use generalised eigenspace

... scale covariant field theory

- all F_δ are finite dimensional,

  the set $\mathcal{S} = \{ \delta \in \mathbb{R} \mid F_\delta \neq 0 \}$ (the spectrum) is bounded below,

  and there is $\varepsilon > 0$ s.th. $\delta, \delta' \in \mathcal{S}$, $\delta \neq \delta'$ $\Rightarrow$ $|\delta - \delta'| \geq \varepsilon$.



In particular, $F_{\leq d} := \bigoplus_{\delta \leq d} F_\delta$ is finite dim. for all $d \in \mathbb{R}$

... scale covariant field theory

Write $\overline{\mathcal{F}} = \prod_{\delta \in \mathbb{R}} \mathcal{F}_\delta$ (the direct product completion of $\mathcal{F} = \bigoplus \mathcal{F}_\delta$)

$P_{\leq d} : \overline{\mathcal{F}} \longrightarrow \mathcal{F}_{\leq d}$ projection

(also use for $\mathcal{F} \longrightarrow \mathcal{F}_{\leq d}$)

$\mathcal{M}_n = \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \ (i \neq j) \}$

(configuration space of n ordered points in $\mathbb{C}$)

... scale covariant field theory

Data (part 2) : For $n = 1, 2, \ldots$ a function ("n-pt. correlator")

$$\mu_n : \mathcal{M}_n \times \mathcal{F}^{\otimes n} \longrightarrow \mathcal{F}$$

s.t.

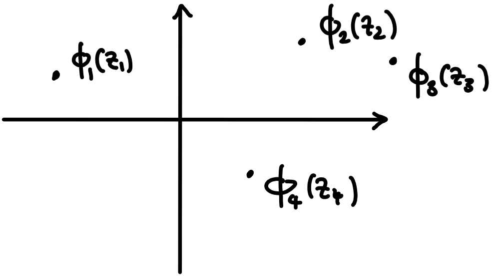
- linear in $\mathcal{F}^{\otimes n}$
- $P_{\leq d} \circ \mu_n$ is smooth in $\mathcal{M}_n$ for all d.

... scale covariant field theory

Notation : $\vec{z} \in \mathcal{M}_n$, $\phi_i \in F$, $\alpha \in (F_{\leq d})^*$

$$\alpha \circ \mu_n\big((z_1, \dots, z_n), \phi_1 \otimes \cdots \otimes \phi_n\big)$$
$$=: \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_\alpha$$

field   position   out-state



... scale covariant field theory

Properties (part 2)

(permutation invariance)

$$\mu_n\big((z_1,\dots,z_n),\ \phi_1\otimes\cdots\otimes\phi_n\big) = \mu_n\big((z_{\sigma 1},\dots,z_{\sigma n}),\ \phi_{\sigma 1}\otimes\cdots\otimes\phi_{\sigma n}\big)$$

for all $\sigma\in S_n$.

(unit)

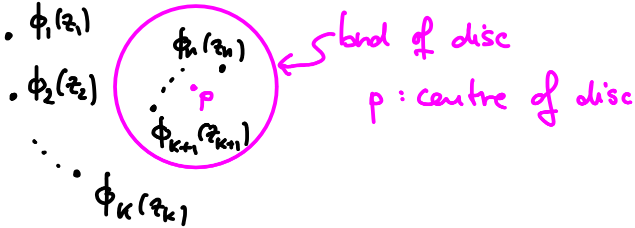
$$\mu_{n+1}\big((z_1,\dots,z_n,z_{n+1}),\ \phi_1\otimes\cdots\otimes\phi_n\otimes 1\big) = \mu_n\big((z_1,\dots,z_n),\ \phi_1\otimes\cdots\otimes\phi_n\big) \text{ for } n\geq 1,$$

and $\mu_1(0,\phi)=\phi$

(but $\mu_1(z,\phi)\neq\phi$ in general)

... scale covariant field theory

(factorisation) For each disc w/o any z_i on its bnd.



$$\mu_n\left((z_1,\dots,z_n),\ \phi_1\otimes\cdots\otimes\phi_n\right)$$

$$= \lim_{d\to\infty} \mu_{k+1}\left((z_1,\dots,z_k,p),\ \phi_1\otimes\cdots\otimes\phi_k\otimes P_{\le d}\Psi\right)$$

where $\Psi = \mu_{n-k}\left((z_{k+1}-p,\dots,z_n-p),\ \phi_{k+1}\otimes\cdots\otimes\phi_n\right)\in\overline{\mathcal{F}}$

... scale covariant field theory

(scale covariance) For $\lambda > 0$

$$\lambda^{\Delta} \mu_n\big((z_1, \dots, z_n), \phi_1 \otimes \cdots \otimes \phi_n\big)$$
$$= \mu_n\big((\lambda z_1, \dots, \lambda z_n), \lambda^{\Delta}\phi_1 \otimes \cdots \otimes \lambda^{\Delta}\phi_n\big)$$

eg for $\phi_i \in F_{\delta_i}$, $\alpha \in (F_\delta)^*$ get

$$\langle \phi_1(\lambda z_1) \cdots \phi_n(\lambda z_n) \rangle_\alpha = \lambda^{\delta - \delta_1 - \dots - \delta_n} \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_\alpha$$

(aside: if Δ is not diagonalisable, get $\log \lambda$ terms $\rightsquigarrow$ log CFT)

... scale covariant field theory

Some consequences:

- $\exists!\ \partial, \bar\partial \in \operatorname{End} F$ s.th. $\partial, \bar\partial : F_\delta \longrightarrow F_{\delta+1}$ and

$$\frac{d}{dz}\mu_n\big((z, w_1, \dots), \phi \otimes \psi_1 \otimes \cdots\big)$$
$$= \mu_n\big((z, w_1, \dots), \partial\phi \otimes \psi_1 \otimes \cdots\big)$$

dito $\frac{d}{d\bar z} \cdots = \cdots \bar\partial\phi \cdots$

Sketch of proof:

- use fact. to reduce to $\frac{d}{dz}\mu_1(z, \phi)\big|_{z=0} =: \partial\phi \in F$

(⚠ subtlety: need to show $\frac{d}{dz}$ and $\lim\limits_{\alpha \to \infty}$ commute.)

- use scale cov. and $\phi \in F_\delta$ to compare

$$\lambda^\Delta \frac{d}{dz}\mu_1(z, \phi)\Big|_{z=0} = \frac{d}{dz}\mu_1(\lambda z, \lambda^\delta \phi)$$

Proof: Show for $\frac{d}{dx}$ and $D_x : \mathcal{F} \to \mathcal{F}$.

1) Determine D_x:

$$D_x \phi := \frac{d}{dx} \mu_1(x + i0, \phi)\Big|_{x=0} \in \mathcal{F}$$

Action of Δ: Let $\phi \in \mathcal{F}_\delta$

$$\lambda^\Delta D_x \phi = \lambda^\Delta \frac{d}{dx} \mu_1(x, \phi) = \frac{d}{dx} \mu_1(\lambda x, \lambda^\delta \phi)$$

$$= \lambda^\delta \lambda \frac{d}{dr} \mu_1(r, \phi) = \lambda^{\delta+1} D_x \phi$$

Thus $D_x \phi \in \mathcal{F}_{\delta+1}$.

2) Reduce general case to μ_1 by factorisation:

Have $\mu_n((z+\varepsilon, w_1, \dots), \phi \otimes \psi_1 \otimes \cdots)$

$$= \lim_{d \to \infty} \mu_n((z, w_1, \dots), P_{\leq d} \mu_1(\varepsilon, \phi) \otimes \psi_1 \cdots)$$

Need to show that we can take $\frac{\partial}{\partial \varepsilon}$ past $\lim\limits_{d \to \infty}$.

use aside:

Let $P_{=\delta}$ be projection $P_{=\delta} : \mathcal{F} \longrightarrow \mathcal{F}_\delta$. Factorisation is equiv. to, for all $\alpha \in (\mathcal{F}_{\leq r})^*$

$$\alpha \circ \mu_n(\vec{z}, u \otimes v) = \lim_{d \to \infty} \sum_{\delta \leq d} a_\delta$$

$$a_\delta = \alpha \circ \mu_K((\vec{z}_{[1 \dots K]}, p), u \otimes P_{=\delta}\, \mu(\vec{z}_{[k+1 \dots n]} - p, v))$$

<u>Lemma</u> $\sum_\delta a_\delta$ converges absolutely

Pf: Write $a_\delta(\lambda) = P_{=\delta}\, \lambda^\Delta \mu((z_{K+1}, \dots, z_n), \phi_{K+1} \otimes \cdots \phi_n)$

$$= P_{=\delta}\, \mu((\lambda z_{K+1}, \dots), \lambda^\Delta \phi_{K+1} \otimes \cdots)$$

But also $a_\delta(\lambda) = \lambda^\delta a_\delta$. There exists $\Lambda > 1$ s.th.

$\Lambda z_{k+1}, \ldots, \Lambda z_n$ are still inside the same disc. Thus also

$$\lim_{d\to\infty} \sum_{\delta\le d} a_\delta \Lambda^\delta$$

exists. Hence there is $S>0$ s.th. $|a_\delta \Lambda^\delta| < S$ for all δ. Then

$$\sum_\delta |a_\delta| = \sum_\delta |a_\delta \Lambda^\delta| \cdot \Lambda^{-\delta} \le S \sum_\delta \Lambda^{-\delta}$$

and the last sum converges by the spectral gap condition. $\square$

Now for $\phi \in \mathcal{F}_k$: $\quad P_{=\delta}\, \mu_1(\varepsilon, \phi) = P_{=\delta}\, \varepsilon^{\Delta-k} \mu_1(1,\phi)$

$$= \varepsilon^{\delta-k} P_{=\delta}\, \mu_1(1,\phi)$$

If $\delta > k$, this holds for $\varepsilon = 0$, too. (Take $\lim_{\varepsilon\to 0}$ on both sides.)

Let $\alpha \in (P_{\leq r})^*$ and

$$a_\delta(\varepsilon) = \alpha \circ \mu_n((z, w_1, \dots), P_{=\delta}\, \mu_1(\varepsilon, \phi) \otimes \psi_1 \cdots)$$

Then $\sum_\delta a_\delta(\varepsilon_0)$ converges absolutely for some $\varepsilon_0 > 0$.

Since for all ε with $0 \leq \varepsilon \leq \varepsilon_0$ and $\delta > \kappa$

$$|a_\delta(\varepsilon)| = \left(\frac{\varepsilon}{\varepsilon_0}\right)^{\delta - \kappa} \cdot |a_\delta(\varepsilon_0)| \leq |a_\delta(\varepsilon_0)|.$$

Thus $\sum_\delta a_\delta(\varepsilon)$ splits into a finite sum $(\delta < \kappa)$ and a sum which converges normally on $0 < \varepsilon \leq \varepsilon_0$ $(\delta \geq \kappa)$.

For normally convergent sums, $\sum_\delta$ and $\frac{d}{d\varepsilon}$ commute.

$\square$

... scale covariant field theory

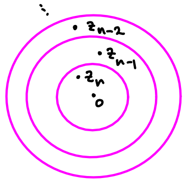
- Let $m : \mathbb{C}^* \times \mathcal{F}^{\otimes 2} \longrightarrow \overline{\mathcal{F}}$

$$(z, \phi \otimes \psi) \mapsto \mu_2((z,0), \phi \otimes \psi)$$

Then m determines μ_n for all n

Sketch of proof:

- For $|z_1| > \dots > |z_n|$ use factorisation



$$\mu_{n+1}\left((z_1, \dots, z_n, 0), \phi_1 \otimes \cdots \phi_n \otimes \psi\right)$$

$$= \lim_{d \to \infty} \mu_n\big((z_1, \dots, z_{n-1}, 0),$$
$$\phi_1 \otimes \cdots \otimes \phi_{n-1} \otimes P_{\leq d}\, m(z_n, \phi_n \otimes \psi)\big)$$

- For general z_i use continuity.

... scale covariant field theory

- $\Delta \mathbb{1} = 0, \quad \partial \mathbb{1} = 0 = \bar{\partial} \mathbb{1}$

Sketch of proof:

- $\mu_1(z, \mathbb{1}) = \mu_2((z,0), \mathbb{1}\otimes\mathbb{1}) = \mu_1(0, \mathbb{1}) = \mathbb{1}$

  implies derivative

- Write $\mathbb{1} = \sum_{\delta \in I} a_\delta$, where I finite and $a_\delta \in \mathcal{F}_\delta$.

  Insert in

$$\lambda^\Delta \mu_2((x,0), \mathbb{1}\otimes\mathbb{1}) = \mu_2((\lambda x, 0), \lambda^\Delta \mathbb{1} \otimes \lambda^\Delta \mathbb{1})$$

$$= \sum_{\delta \in I} \lambda^\delta a_\delta$$

and compare x-dependence

Boring example

A : finite dim. comm. assoc. algebra $/\mathbb{C}$

Then take

$$F = F_0 = A$$

$$\Delta = 0$$

$$1 \in A \quad \text{unit}$$

$$\mu_n\big((z_1, \dots, z_n), a_1 \otimes \cdots \otimes a_n\big) = a_1 \cdot \ldots \cdot a_n$$

Slightly less boring example

A : comm. assoc. algebra $/\mathbb{C}$
$\mathbb{Z}$-graded & bounded below

D : derivation on A of deg 1

Take

$\mathcal{F} = A \quad (\mathbb{Z}\text{-graded}) \qquad 1 \in A$

$\Delta = \deg$

$$\mu_n\big((z_1, \dots, z_n), a_1 \otimes \cdots \otimes a_n\big) = \prod_{i=1}^{n} (e^{z_i D})(a_i)$$

E.g. $A = \mathbb{C}[x^{-1}]$ or $\mathbb{C}[x^{-1}]/\langle x^{-1}\rangle$, $D = \frac{d}{dx}$, $\Delta = -\deg$

Holomorphic integral subtheory

Let $V := \operatorname{span}\{\phi \in \mathcal{F}_\delta \text{ for } \delta \in \mathbb{Z} \mid \bar{\partial}\phi = 0\}$

$$= \ker \bar{\partial} \cap \bigoplus_{\delta \in \mathbb{Z}} \mathcal{F}_\delta$$

For $\phi_1, \dots, \phi_n \in V$ have

$$\frac{d}{d\bar{z}_i} \mu_n\big((z_1, \dots, z_n), \phi_1 \otimes \cdots \otimes \phi_n\big) = 0$$

<u>Lemma</u> $\mu_n\big((z_1, \dots), \phi_1 \otimes \cdots\big) \in \overline{V}$

... holomorphic integral subtheory

Sketch of proof:

- $\mu_n \in \ker \bar\partial$:

$$0 = \frac{d}{d\bar\varepsilon} \mu_n\big((z_1+\epsilon, \dots, z_n+\epsilon), \phi_1 \otimes \cdots \big) \Big|_{\varepsilon=0}$$

$$= \underbrace{\frac{d}{d\bar\varepsilon} \lim_{d\to\infty}}_{\text{show}} \mu_1\big(\varepsilon, \underbrace{P_{\leq d}}_{\text{becomes } \bar\partial} \mu_n\big((z_1, \dots, z_n), \phi_1 \otimes \cdots \big)\big) \Big|_{\varepsilon=0}$$

- $\mu_n \in \prod_{\delta \in \mathbb{Z}} F_\delta$: For $\lambda > 0$, $\phi_i \in V_{\delta_i}$

$$P_{=\delta}\, \lambda^{\Delta} \mu_n\big((z_1, \dots), \phi_1 \otimes \cdots \big)$$

$$\lambda^{\delta} P_{=\delta} \mu_n(\cdots) = \lambda^{\overbrace{\delta_1 + \cdots + \delta_n}^{\in \mathbb{Z}}} \underbrace{\mu_n\big((\lambda z_1, \dots, \lambda z_n), \phi_1 \otimes \cdots \big)}_{\text{hol. in } \lambda \in \mathbb{C}^*}$$

$\hookleftarrow$ hol. in $\lambda \in \mathbb{C}^*$ iff $\delta \in \mathbb{Z}$

$\mathbb{Z}$-graded vertex algebras

Notation : V : $\mathbb{C}$-v.sp. $\quad x$: formal variable

$V[[x^{\pm 1}]]$: formal Laurent series $\sum_{m \in \mathbb{Z}} v_m x^m$, $v_m \in V$

<u>Definition</u> A <u>$\mathbb{Z}$-graded vertex algebra</u> consists of

- a $\mathbb{Z}$-graded $\mathbb{C}$-v.sp $V = \bigoplus_{m \in \mathbb{Z}} V_m$ s.th.

  $\dim V_m < \infty$, $\quad V_m = 0$ for $m \ll 0$.

- a linear map $Y : V \longrightarrow (\operatorname{End} V)[[x^{\pm 1}]]$
  $$u \longmapsto Y(u,x)$$

- a linear map $D : V \to V$ of degree 1

- $\mathbb{1} \in V_0$

... $\mathbb{Z}$-graded vertex algebras

subject to the conditions

- $Y(1,x) = \mathrm{id}_V$, $\quad Y(u,x)1 = u + O(x)$
- $[D, Y(u,x)] = \frac{d}{dx} Y(u,x)$
- For $u \in V_k$, $u_m \in \operatorname{End} V$ defined as

$$Y(u,x) = \sum_{m \in \mathbb{Z}} u_m x^{-m-k},$$

  has degree $-m$.
- For all $u, v \in V$:

  $Y(u,x)v$ has only finitly many neg. powers of x

  $\exists M > 0 : (x-y)^M [Y(u,x), Y(v,y)] = 0$

... $\mathbb{Z}$-graded vertex algebras

$$(x-y)^M [Y(u,x), Y(v,y)] = 0$$

Fun with formal series

$$\delta(x/y) := \sum_{m\in\mathbb{Z}} x^m y^{-m} \in \mathbb{C}[[x^{\pm1}, y^{\pm1}]]$$

$$(x-y)\,\delta(x/y) = \underbrace{\sum_{m\in\mathbb{Z}} x^{m+1} y^{-m}}_{= \sum_{n\in\mathbb{Z}} x^n y^{-n+1}} - \sum_{m\in\mathbb{Z}} x^m y^{-m+1}$$

$$= 0 .$$

Thus $(x-y)\cdot - \circlearrowright \mathbb{C}[[x^{\pm1}, y^{\pm1}]]$ is not injective.

Relation between the two

$F, 1, \mu_n$

holomorphic integral scale cov. field th.

$V, 1, Y, D$

$\mathbb{Z}$-graded vertex algebra

$$F = V$$
$$1 = 1$$

$m(z, \phi \otimes \psi)$ = " $Y(x,\phi)\psi$ with formal x replaced by $z \in \mathbb{C}^*$ "

<u>Theorem.</u> The assignment defines a one-to-one correspondence.

Example : Heisenberg vertex algebra (aka $U(1)$-current algebra)

Lie algebra H with generators

$$a_m \quad (m \in \mathbb{Z}) \qquad , \qquad K$$

and bracket

$$[a_m, a_n] = m \cdot \delta_{m+n} K \qquad \text{and } K \text{ central}$$

Let $H_{\geq 0} = \operatorname{span}\{ K, a_m \mid m \geq 0 \}$

$\mathbb{C}_0$: $H_{\geq 0}$-module s.th.

K acts as 1 , a_m acts as 0 $(m \geq 0)$

Set

$$V_H := U(H) \underset{H_{\geq 0}}{\otimes} \mathbb{C}_0 \qquad \text{(induced module)}$$

... Heisenberg vertex algebra

$$V_H := U(H) \underset{H_{\geq 0}}{\otimes} \mathbb{C}_0$$

V_H has basis (set $\mathbb{1} := 1 \otimes 1 \in V_H$)

$$a_{-m_1} \cdots a_{-m_k} \mathbb{1} \qquad m_1 \geq m_2 \geq \cdots \geq m_k > 0$$

and is $\mathbb{Z}$-graded by $-$(total degree) of .

Set
$$Y(a_{-1}\mathbb{1}, x) := \sum_{m \in \mathbb{Z}} a_m x^{-m-1} \quad \in \operatorname{End}(V_H)[[x^{\pm 1}]]$$

<u>Theorem.</u> The data $V_H, \mathbb{1}, Y$ extends uniquely to a $\mathbb{Z}$-graded vertex algebra on V_H.

... Heisenberg vertex algebra

$$(x-y)^M \left[Y(u,x), Y(v,y) \right] = 0$$

Abbrev. $J(x) := Y(a_{-1}1, x) = \sum_{m\in\mathbb{Z}} a_m x^{-m-1}$

<u>Lemma</u> $(x-y)^2 [J(x), J(y)] = 0$

Pf.: $[J(x), J(y)] = \sum_{m,n\in\mathbb{Z}} \underbrace{[a_m, a_n]}_{= m\cdot\delta_{m+n,0} K} x^{-m-1} y^{-n-1}$ (K acts as id)

$$= \sum_{m\in\mathbb{Z}} m\, x^{-m-1} y^{m-1} = x^{-1} \frac{\partial}{\partial y} \sum_{m\in\mathbb{Z}} x^{-m} y^m$$

$$= x^{-1} \frac{\partial}{\partial y} \delta(y/x)$$

and

$$0 = \frac{\partial}{\partial y}\left((x-y)^2 \delta(y/x) \right) = (x-y)^2 \frac{\partial}{\partial y} \delta(y/x) . \qquad \square$$

Conformal field theory

<u>Definition</u> A <u>conformal field theory</u> is a scale cov. field theory $\mathcal{F}, 1, \mu_n$ together with

$$T, \bar{T} \in \mathcal{F}_2$$

(the holom. and antihol. components of the stress tensor)

s.th.

- $\bar{\partial} T = 0 = \partial \bar{T}$
- For appropriate $c, \bar{c} \in \mathbb{C}$ (hol./antihol. central charge):

$$m(z, T \otimes T) = \frac{c}{2} 1 \cdot z^{-4} + 2T \cdot z^{-2} + \partial T \cdot z^{-1} + \text{reg}$$

$$m(z, \bar{T} \otimes \bar{T}) = \frac{\bar{c}}{2} 1 \cdot \bar{z}^{-4} + 2\bar{T} \cdot \bar{z}^{-2} + \bar{\partial} \bar{T} \cdot \bar{z}^{-1} + \text{reg}$$

... conformal field theory

Notation: Define $L_m, \bar{L}_m : \mathcal{F} \to \mathcal{F}$ via Laurent expansion

$$m(z, T \otimes \phi) = \sum_{m \in \mathbb{Z}} L_m(\phi)\, z^{-m-2}$$

$$m(z, \bar{T} \otimes \phi) = \sum_{m \in \mathbb{Z}} \bar{L}_m(\phi)\, \bar{z}^{-m-2}$$

Lemma $L_m, \bar{L}_m : \mathcal{F}_\delta \longrightarrow \mathcal{F}_{\delta - m}$

Pf via scale covariance

- $\partial = L_{-1}$, $\bar{\partial} = \bar{L}_{-1}$, $\Delta = L_0 + \bar{L}_0$

... conformal field theory

<ins>Definition</ins> The <ins>Virasoro algebra</ins> is the Lie algebra with basis $\{ c, L_m \ (m \in \mathbb{Z}) \}$ and bracket

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{1}{12}(m^3 - m)\,\delta_{m+n,0}\, c$$

with c central.

... conformal field theory

Proposition $\mathcal{F}$ carries a rep$^{\underline{n}}$ of $\mathrm{Vir} \oplus \mathrm{Vir}$ via

$$(c_1, 0) \mapsto c \cdot \mathrm{id} \qquad (0, c_1) \mapsto \bar{c} \cdot \mathrm{id}$$
$$(L_m, 0) \mapsto L_m \qquad (0, L_m) \mapsto \bar{L}_m$$

Sketch of proof:

- Write $L_m \phi = \int_{\odot} z^{m+1} \, m(z, T \otimes \phi) \frac{dz}{2\pi i}$
  and check $L_m L_n \phi - L_n L_m \phi = \dots$
  using $m(z, T \otimes T) = \frac{c}{2} z^{-4} + \dots$
- Dito $\bar{L}_m \phi$
- **Lemma**. $m(z, T \otimes \bar{T})$ is regular in z
  Conclude that $[L_m, \bar{L}_n] = 0$.

... conformal field theory

Examples from scale cov. field theories

- boring example : $\mathcal{F} = \mathcal{F}_0 = A$ (f.dim. comm. alg.)

$$T = \bar{T} = 0$$

- slightly less boring example :

$\mathcal{F} = \mathbb{C}[x^{-1}]$, $D = \frac{d}{dx}$, $\Delta = -\deg$

cannot be made into a CFT

Pf.: $\mathcal{F}_2 = \mathbb{C}x^{-2}$, so $T = \alpha x^{-2}$ for some $\alpha \in \mathbb{C}$. But

$$m(z, x^{-1} \otimes x^{-1}) = \left(e^{z\frac{d}{dx}} x^{-1}\right) x^{-1} = x^{-2} \sum_{m=0}^{\infty} (-1)^m z^m x^{-m}$$

contains no negative powers of z. Hence $T = 0$, ↯ to $D = L_{-1}$

Local conformal symmetry

__Proposition.__ For $m \geq -1$, $\vec{z} = (z_1, \dots, z_k) \in \mathcal{M}_k$,

$$
\begin{aligned}
L_m\, \mu_k(\vec{z}, \phi_1 \otimes \cdots \otimes \phi_k)
&= \sum_{i=1}^{k} z_i^{m+1} \frac{\partial}{\partial z_i} \mu_n(\vec{z}, \phi_1 \otimes \cdots) \\
&\quad + \sum_{i=1}^{k} \underbrace{\sum_{\ell=0}^{\infty}} \binom{m+1}{\ell+1} z_i^{m-\ell}\, \mu_n(\vec{z}, \phi_1 \otimes \cdots \underbrace{L_\ell \phi_i} \cdots \otimes \phi_k)
\end{aligned}
$$

sum is finite as — will vanish for $\ell \gg 0$ as spectrum bounded below

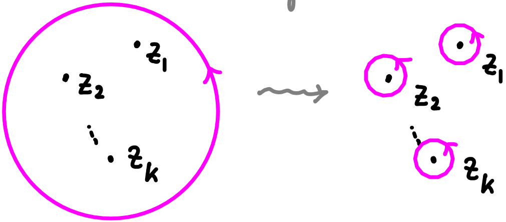
and similar for $\bar{L}_m$.

__Note__: $z^{m+1} \frac{\partial}{\partial z}$ $(m \geq -1)$ are globally defined holom. vector fields

... local conformal symmetry

Sketch of proof:

- Write L_n action as contour integral and deform contour



- Use Laurent expansion of T around z_i and the axiom that $L_{-1}\phi_i = \partial\phi_i \rightsquigarrow \frac{\partial}{\partial z_i}\mu_n(\cdots)$.

Vertex operator algebras

<u>Definition</u> A <u>vertex operator algebra</u> is a $\mathbb{Z}$-graded vertex algebra $(V, 1, Y)$ together with

$T \in V_2$ (Virasoro element)

s.th.

- for $Y(T,x) = \sum_{m \in \mathbb{Z}} L_m x^{-m-2}$ and an appropriate $c \in \mathbb{C}$:

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}$$

- $L_0|_{V_m} = m \cdot \mathrm{id}_{V_m}$ and $L_{-1} = D$

Relation between the two

$F, 1, \mu_n, T$ holomorphic integral conformal field th.

$V, 1, Y, T$ vertex operator algebra

$$F = V$$
$$1 = 1$$
$$T = T$$
$$m(z, \phi \otimes \psi) = \text{“ } Y(x,\phi)\psi \text{ with formal } x \text{ replaced by } z \in \mathbb{C}^* \text{ ”}$$

<u>Theorem.</u> The assignment $\uparrow$ defines a one-to-one correspondence.

Example: Heisenberg vertex operator algebra

Lie alg H : $[a_m, a_n] = m \cdot \delta_{m+n} K$

Basis of V_H : $a_{-m_1} \cdots a_{-m_k} 1$, $m_1 \geq m_2 \geq \cdots \geq m_k > 0$

$$(V_H)_2 = \mathbb{C}\, a_{-1} a_{-1} 1 \oplus \mathbb{C}\, a_{-2} 1$$

Prop For each $\beta \in \mathbb{C}$, $T = \frac{1}{2} a_{-1} a_{-1} 1 + \beta\, a_{-2} 1$ defines VOA structure on $\mathbb{Z}$-gr. vertex algebra V_H of central charge

$$c = 1 - 12\beta^2$$

These are all Virasoro elements for V_H.

Part III

I Some motivation from lattice models

conformal field theory

II

vertex operator algebra

III

braided tensor cat. of VOA-modules

& commutative algebra in (two copies of) cat. of VOA-modules

IV

Modules over a CFT

CFT : n-fold position dependent products

$$\mu_n : \mathcal{M}_n \times \mathcal{F}^{\otimes n} \longrightarrow \overline{\mathcal{F}}$$

CFT-module : n-fold position dependent action

$$\rho_n : \mathcal{M}_{n+1} \times \mathcal{F}^{\otimes n} \otimes M \longrightarrow \overline{M}$$

and "obvious axioms" s.th. $\mathcal{F}$ is a module over itself, or, indeed, over any sub-CFT of $\mathcal{F}$

In more detail ...

... modules over a CFT

Data : $M = \bigoplus_{\delta \in \mathbb{R}} M_\delta$ $\mathbb{R}$-graded $\mathbb{C}$-v.sp.

$\Delta : M \to M$, $\Delta|_{M_\delta} = \delta \cdot \mathrm{id}_{M_\delta}$

$\rho_n : \mathcal{M}_{n+1} \times V^{\otimes n} \otimes M \longrightarrow \overline{M}$

smooth in , linear in

Properties :

- $\dim M_\delta < \infty$, $M_\delta = 0$ for $\delta \ll 0$,
  there is a minimal gap ($\exists \varepsilon > 0$: $\delta \neq \delta'$ and $M_\delta \neq 0$, $M_{\delta'} \neq 0$ $\Rightarrow |\delta - \delta'| \geq \varepsilon$)

... modules over a CFT

- (permutation invariance) for all $\sigma \in S_n$

$$\rho_n\big((z_1,\dots,z_n,w),\ \phi_1\otimes\cdots\otimes\phi_n\otimes\psi\big)$$
$$= \rho_n\big((z_{\sigma 1},\dots,z_{\sigma n},w),\ \phi_{\sigma 1}\otimes\cdots\otimes\phi_{\sigma n}\otimes\psi\big)$$

- (unit)

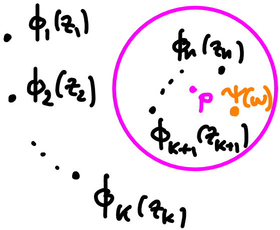
$$\rho_{n+1}\big((z_1,\dots,z_{n+1},w),\ \cdots\otimes\phi_n\otimes 1\otimes\psi\big)$$
$$= \rho_n\big((z_1,\dots,z_n,w),\ \cdots\phi_n\otimes\psi\big)$$

$$\rho_0(0,\psi) = \psi$$

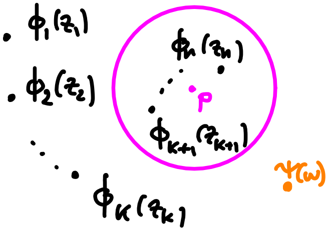
... modules over a CFT

- (factorisation)

A)



B)



$$\rho_n\big((\vec{z}, w), \phi_1 \otimes \cdots \otimes \psi\big)$$

$$= \begin{cases} A: \lim\limits_{d \to \infty} \rho_k\big((z_1, \dots, z_k, p), \phi_1 \otimes \cdots \otimes \phi_k \otimes P_{\le d}\, \xi_A\big) \\ \qquad \xi_A = \rho_{n-k}\big((z_{k+1} - p, \dots, w - p), \phi_{k+1} \otimes \cdots \otimes \phi_n \otimes \psi\big) \\ B: \lim\limits_{d \to \infty} \rho_{k+1}\big((z_1, \dots, z_k, p, w), \phi_1 \otimes \cdots \otimes \phi_k \otimes P_{\le d}\, \xi_B \otimes \psi\big) \\ \qquad \xi_B = \mu_{n-k}\big((z_{k+1} - p, \dots, z_n - p), \phi_{k+1} \otimes \cdots \otimes \phi_n\big) \end{cases}$$

... modules over a CFT

- (scale covariance) For $\lambda > 0$

$$\lambda^{\Delta} \rho_n\big((z_1, \dots, w), \phi_1 \otimes \cdots \otimes \psi \big)$$
$$= \rho_n\big((\lambda z_1, \dots, \lambda w), \lambda^{\Delta}\phi_1 \otimes \cdots \otimes \lambda^{\Delta}\psi \big)$$

- Axioms so far imply (see below): $\mathrm{Vir} \oplus \mathrm{Vir}$ acts on M.
Demand:

$$\Delta = L_0 + \bar{L}_0$$

$$\frac{\partial}{\partial w} \rho_n\big((\vec{z}, w), \phi_1 \otimes \cdots \otimes \phi_n \otimes \psi \big)$$
$$= \rho_n\big((\vec{z}, w), \phi_1 \otimes \cdots \otimes \phi_n \otimes L_{-1}\psi \big)$$

$$\frac{\partial}{\partial \bar{w}} \cdots = \cdots \bar{L}_{-1} \cdots$$

... modules over a CFT

Let $\rho : \mathbb{C}^* \times \mathcal{F} \otimes M \longrightarrow M$

$$(z, \phi \otimes \psi) \longmapsto \rho_2((z,0), \phi \otimes \psi)$$

<u>Lemma</u> $\rho(z, T \otimes \psi)$ is analytic on $\mathbb{C}^*$ and the L_m in

$$\rho(z, T \otimes \psi) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}$$

are degree $-m$ operators on M and define a rep^n of Vir of central charge c (same as for $\mathcal{F}$)

... modules over a CFT

For boring example ($\mathcal{F} = \mathcal{F}_0 = A$ f.dim. comm. alg.
$T = 0 = \bar{T}$):

$\mathcal{F}$-module $\hat{=}$ f.dim. A-module

Let $E \subset \mathcal{F}$ be a sub-CFT. Then $\mathcal{F}$ is an E-module.

In particular, $E = \langle V_{\text{hol,int.}} , V_{\text{antihol,int.}} \rangle$

... modules over a CFT

M, N : $\mathcal{F}$-modules

$$\mathrm{Hom}_{\mathcal{F}}(M,N) = \{ \text{grading pres. lin. maps } M \xrightarrow{f} N, \; f \circ \rho_M(\cdots \otimes \psi) = \rho_N(\cdots \otimes f\psi) \}$$

$\rightsquigarrow$ category $\mathcal{F}$-mod

$\mathcal{F}$-mod is a $\mathbb{C}$-linear abelian category.

Modules over a VOA

Definition (with omissions). V : VOA

A V-module is a $\mathbb{C}$-graded v.sp. M (fin. dim. components, grading bounded below) and a lin. map

$$\begin{aligned} Y_M : V &\longrightarrow (\operatorname{End} M)[[x^{\pm 1}]] \\ v &\longmapsto Y_M(v,x) \end{aligned}$$

s.th.

"truncation, vacuum, grading by L_0, Jacobi-identity"

V-modules form a $\mathbb{C}$-linear abelian category $V\text{-mod}$.

... modules over a VOA

Relation:

V : hol. integral CFT $\xleftrightarrow{1:1}$ V : VOA

M : CFT-module for V $\xleftrightarrow{1:1}$ M : $\mathbb{R}$-graded VOA-module for V s.th. spectrum of weights has a minimal gap

$$V\text{-mod}_{(CFT)} \quad \simeq \quad V\text{-mod}'_{(VOA)}$$

subcat. satisfying

- restriction to $\mathbb{R}$-grading: lazyness
- minimal gap: automatic for irreducible VOA-modules (and thus also for VOA-modules with finite composition series)

Example : Heisenberg VOA

Lie alg H : $[a_m, a_n] = m \cdot \delta_{m+n} K$

Basis of V_H : $a_{-m_1} \cdots a_{-m_k} 1$, $m_1 \geq m_2 \geq \dots \geq m_k > 0$

$$V_H = U(H) \underset{H_{\geq 0}}{\otimes} \mathbb{C}_0$$

$$T = \frac{1}{2} a_{-1} a_{-1} 1 + \beta a_{-2} 1 \quad (\beta \in \mathbb{C})$$

Let $q \in \mathbb{C}$ and let

$\mathbb{C}_q$: 1-dim. $H_{\geq 0}$-module s.th.
- K acts as id
- a_m as 0 $(m > 0)$
- a_0 as $q \cdot \mathrm{id}$

Define

$$M_q := U(H) \underset{H_{\geq 0}}{\otimes} \mathbb{C}_q$$

... example: Heisenberg VOA

$$M_q := U(H) \otimes_{H_{\geq 0}} \mathbb{C}_q$$

- $V_H = M_0$ as H-module
- Write $1_q := 1 \otimes 1 \in M_q$. M_q has basis $a_{-m_1} \cdots a_{-m_k} 1$, $m_1 \geq m_2 \geq \dots \geq m_k > 0$
- a_0 acts on M_q as $q \cdot \mathrm{id}_{M_q}$

Proposition There is a unique V_H-module structure on M_q s.th. for $\psi \in M_q$:

$$Y_{M_q}(a_{-1} 1)\psi = \sum_{m \in \mathbb{Z}} x^{-m-1} a_m \psi$$

... example : Heisenberg VOA

- L_m acts on M_q as $\left(T = \frac{1}{2} a_{-1} a_{-1} \mathbb{1} + \beta a_{-2} \mathbb{1} \right)$

$$L_m = \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_k a_{m-k} : \; - \beta (m+1) a_m$$

normal ordering $\quad : a_k a_\ell : \; := \begin{cases} a_k a_\ell & k \leq \ell \\ a_\ell a_k & k > \ell \end{cases}$

$$L_0 \mathbb{1}_q = \left(\frac{1}{2} a_0^2 - \beta a_0 \right) \mathbb{1}_q = \frac{1}{2} q (q - 2\beta) \mathbb{1}_q$$

- Thus grading on M_q is $\mathbb{Z}_{\geq 0} + \frac{1}{2} q (q - 2\beta)$

... example: Heisenberg VOA

For a $\mathbb{C}$-graded v.sp. W, let $H_{\geq 0}$ act as

K as id_W,
a_m $(m>0)$ as 0
a_0 as grading

define

$$\widehat{W} := U(H) \otimes_{H_{\geq 0}} W$$

Check: $\hat{W}$ has f.-dim. homog. comp. & grading bnd. below

$\Leftrightarrow$

W has f.-dim. components and $\mathrm{Re}(q+\beta)^2$ bnd. below for all q s.th. $W_q \neq 0$.

Call these $\uparrow$ $\mathrm{Vect}^{\mathrm{bnd}}_{\mathbb{C}\text{-gr}}$

... example : Heisenberg VOA

Get a fully faithful functor

$$\widehat{(\ )} : \mathrm{Vect}^{\mathrm{bnd}}_{\mathbb{C}\text{-gr}} \longrightarrow V_H\text{-mod}$$

of $\mathbb{C}$-linear categories.

(Its image consists of all V_H-modules on which a_0 acts diagonalisably.)

Conformal blocks

<u>So far</u> : CFT : μ_n

$\phi_1(z_1)$ · $\phi_2(z_2)$ · $\phi_3(z_3)$ · $\phi_4(z_4)$

all insertions from F

CFT module : ρ_n

$\phi_1(z_1)$ · $\phi_2(z_2)$ · $\phi_3(z_3)$ · $\psi(w)$

one insertion from M

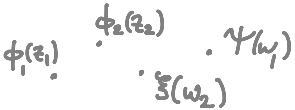
conformal block :

$\phi_1(z_1)$ · $\phi_2(z_2)$ · $\psi(w_1)$ · $\xi(w_2)$

insertions from several modules $M, N, \ldots$

… conformal blocks

conformal block :

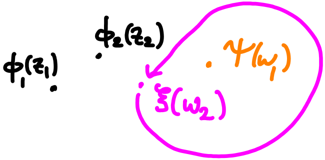


Complication

single valued in z_1, z_2

but

typically not single valued as fn of w_1, w_2



... conformal blocks

Replace configuration space $\mathcal{M}_n$.

$\mathcal{M}_{n,k}$ ~ "arguments for functions with n single valued variables and k multivalued variables"

$$\begin{array}{c} \widetilde{\mathcal{M}}_k \\ \pi \downarrow \\ \mathcal{M}_k \end{array} \quad : \quad \text{universal cover of } \mathcal{M}_k$$

$$\mathcal{M}_{n,k} := \left\{ (\vec{z}, \tilde{w}) \in \mathcal{M}_n \times \widetilde{\mathcal{M}}_k \mid (\vec{z}, \pi(w)) \in \mathcal{M}_{n+k} \right\}$$

e.g. $\mathcal{M}_{n,1} = \mathcal{M}_{n+1}$, but $\mathcal{M}_{n,2} \neq \mathcal{M}_{n+2}$

... conformal blocks

$\mathcal{F}$: CFT

$M_1, \dots, M_k, N$: $\mathcal{F}$-modules

A <u>conformal block for $\mathcal{F}$ with insertions $M_1, \dots, M_k$ taking values in $\overline{N}$</u> is a collection $(\gamma_{n,k})_{n \geq 0}$ of functions

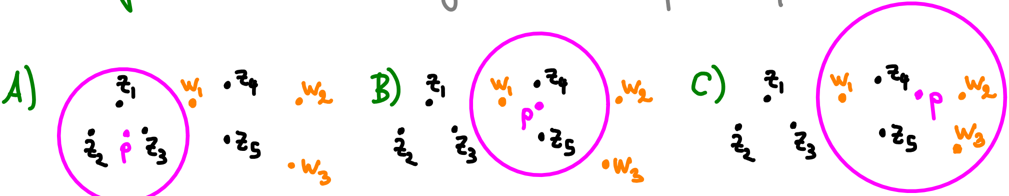
$$\gamma_{n,k} : \underbrace{\mathcal{M}_{n,k}}_{\text{smooth}} \times \underbrace{\mathcal{F}^{\otimes n} \otimes M_1 \otimes \cdots \otimes M_k}_{\text{linear}} \longrightarrow \overline{N}$$

s.th.

- permutation invariance <u>in first n arguments</u>
- unit condition <u>in " n arguments</u>

... conformal blocks

- scale covariance in all $n+k$ arguments
- factorisation: (only ≤ 1 or all points from $w_1, \dots, w_k$)



$$\gamma_{n,k} = \begin{cases} A: & \lim\limits_{d\to\infty} \gamma_{\ell,k}\left(\dots\dots, \ \dots\ P_{\leq d}\, \mu_{n-\ell} \dots \right) \\ B: & \lim\limits_{d\to\infty} \gamma_{\ell,k}\left(\dots\dots, \ \dots\ P_{\leq d}\, \rho_{n-\ell+1} \dots \right) \\ C: & \lim\limits_{d\to\infty} \rho_{\ell}\left(\dots\dots, \ \dots\dots\ P_{\leq d}\, \gamma_{n-\ell,k} \right) \end{cases}$$

... conformal blocks

Let

$$B(M_1, \dots, M_k ; N)$$

be the space of conformal blocks of this type

__Observations__

- $B(M_1, \dots, M_k ; N)$ is a $\mathbb{C}$-vector space via

$$\lambda (\gamma_{n,k})_{n \in \mathbb{Z}_{\geq 0}} + \mu (\gamma'_{n,k})_{n \in \mathbb{Z}_{\geq 0}}$$
$$= (\lambda \gamma_{n,k} + \mu \gamma'_{n,k})_{n \in \mathbb{Z}_{\geq 0}} \qquad (\lambda, \mu \in \mathbb{C})$$

as all conditions are linear in γ.

... conformal blocks

- $\mathcal{B}(\underbrace{-,\dots,-}_{k \text{ arguments}};-)$ defines a functor
$$\underbrace{\mathcal{F}\text{-mod}^{op} \times \dots \times \mathcal{F}\text{-mod}^{op}}_{k \text{ factors}} \times \mathcal{F}\text{-mod} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

- $\gamma_{n,k} \in \mathcal{B}(M_1,\dots,M_k;N)$ is uniquely determined by $\gamma_{0,k}$.

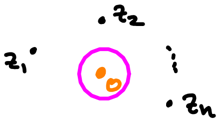
- $$\mathcal{B}(M;N) \longrightarrow \mathrm{Hom}_{\mathcal{F}}(M,N)$$
$$(\gamma_{n,1})_n \longmapsto \gamma_{0,1}(w=0,-)$$
is well-defined and a $\mathbb{C}$-linear isomorphism (natural in M,N)

… conformal blocks

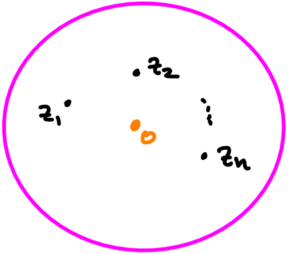
Sketch of proof of last point:

- use scale covariance to see $\psi \in M_\delta \Rightarrow Y_{0,1}(0,\psi) \in N_\delta$ (thus $Y_{0,1}(0,-)$ maps to N, not only $\overline{N}$)

- use factorisations



vs



to check that in $\mathrm{Hom}_{\mathcal{F}}$

- given f, check that $(f \circ p_n)_n \in \mathcal{B}(M,N)$

... conformal blocks

- The unit map provides a natural isomorphism

$$B(M_1, \dots, F, \dots, M_k, N) \xrightarrow{\sim} B(M_1, \dots, M_k, N)$$
$$(\gamma_{n,k+1})_n \longmapsto (\gamma_{n,k+1}(\dots\ 1\ \dots))_n$$

argument for F

- Relation to VOA-formalism:

  Given VOA-modules M, N, K, one can define <u>intertwining operators</u> from M, N to K.
  For $\gamma \in B(M,N;K)$, $\gamma_{0,2}((w,0), -\otimes-)$ is such an operator (upon replacing w by formal var.)

... conformal blocks

- Let $E \subset \mathcal{F}$ be a sub-CFT. Then

$$\boxed{(\mu_{n+k})_n \in \mathcal{B}_E(\mathcal{F}, \ldots, \mathcal{F}; \mathcal{F})}$$

via

$$\mathcal{M}_{n,k} \times E^{\otimes n} \otimes \mathcal{F}^{\otimes k} \longrightarrow \mathcal{F}$$

$$((\vec{z}, \tilde{w}), e_1 \otimes \cdots \otimes e_n \otimes \phi_1 \otimes \cdots \otimes \phi_k) \longmapsto \mu_{n+k}((\vec{z}, \pi(\tilde{w})), e_1 \otimes \cdots \otimes \phi_k)$$

Horizontal sections

Trivial bundle

$$E := \begin{array}{c} \mathcal{M}_{n,k} \times \mathrm{Lin}\left(F^{\otimes n} \otimes M_1 \otimes \cdots \otimes M_k , \bar{N} \right) \\ \downarrow \\ \mathcal{M}_{n,k} \end{array}$$

(so $\gamma_{n,k} \in \Gamma(E)$)

Connection $\nabla = d + A$ on E with, for $f \in \Gamma(E)$,

$$A(f) = -\sum_{i=1}^{n+k} \Big(f_{z,w}\left(- \otimes - \cdots \otimes L_{-1} - \otimes \cdots\right) dz_i + f_{z,w}\left(- \otimes \cdots \otimes \bar{L}_{-1} - \otimes \cdots\right) d\bar{z}_i \Big)$$

Flat on E : $[\nabla_\alpha, \nabla_\beta] = 0 \qquad \alpha, \beta = \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}$

... horizontal sections

<u>Lemma.</u> $\gamma \in B(M_1, \ldots, M_k, N)$

$$\frac{\partial}{\partial z_i} \gamma_{n,k}\left((z_1, \ldots, z_n), \widetilde{(w_1, \ldots, w_k)}, \cdots \otimes \phi_i \otimes \cdots \right)$$
$$= \gamma_{n,k}\left((z_1, \ldots, z_n), \widetilde{(w_1, \ldots, w_k)}, \cdots \otimes L_{-1} \phi_i \otimes \cdots \right)$$

and similarly for $\frac{\partial}{\partial \bar{z}_i}$, $\frac{\partial}{\partial w_j}$, $\frac{\partial}{\partial \bar{w}_j}$.

Sketch of proof:

Use factorisation and the fact that the corresp. properties hold for μ_n, ρ_n.

<u>Corollary</u> The $\gamma_{n,k}$ are horizontal sections of E.

Monodromies

Fix basepoint $p_0 := (1, 2, \dots, k) \in \mathcal{M}_k$

$\tilde{w} \in \tilde{\mathcal{M}}_k$ is a pair $\tilde{w} = (\vec{w}, [\beta])$,

$\in \mathcal{M}_k$

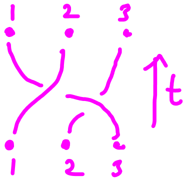
homotopy class of paths $p_0 \to \vec{w}$ in $\mathcal{M}_k$

$\sigma \in S_k$: a permutation

$\alpha : [0,1] \to \mathcal{M}_k$: a path $p_0 \to (\sigma 1, \dots \sigma k)$

Diffeom. $\alpha^* : \tilde{\mathcal{M}}_k \longrightarrow \tilde{\mathcal{M}}_k$

$$(w, [\beta]) \longmapsto (w, [\beta \circ \alpha])$$



... monodromies

For a conf. block $\mathcal{Y}_{n,k}$ write

$$(\alpha^* \mathcal{Y}_{n,k})\big((\vec{z}, \tilde{w}), \phi_1 \otimes \cdots \phi_n \otimes \psi_1 \cdots \psi_k \big)$$

$$:= \mathcal{Y}_{n,k}\big((\vec{z}, \alpha^* \tilde{w}), \phi_1 \otimes \cdots \phi_n \otimes \psi_{\sigma^{-1} 1} \cdots \psi_{\sigma^{-1} k} \big)$$

<u>Lemma</u> $\gamma \in B(M_1, \dots, M_k; N)$

$$\Rightarrow \alpha^* \gamma \in B(M_{\sigma 1}, \dots, M_{\sigma k}; N),$$

and $\gamma \mapsto \alpha^* \gamma$ is a linear isomorphism.

$\rightsquigarrow$ braid group action on $B(M, \dots, M; N)$

Example : Heisenberg VOA

Fix $q_1, \dots, q_k, p \in \mathbb{C}$, consider $B(M_{q_1}, \dots, M_{q_k}, M_p)$

$\phi_1, \dots, \phi_n \in V_H$, $\phi_{n+1} \in M_{q_1}, \dots, \phi_{n+k} \in M_{q_k}$

By contour integration:

$$a_0 \, \gamma_{n,k}\left((\vec{z}, \tilde{w}), \phi_1 \otimes \cdots \otimes \phi_{n+k} \right)$$

$$= \sum_{i=1}^{n+k} \gamma_{n,k}\left((\vec{z}, \tilde{w}), \phi_1 \cdots \otimes a_0 \phi_i \otimes \cdots \phi_{n+k} \right)$$

$\Rightarrow$ $\gamma_{n,k}$ non-zero only for $p = q_1 + \cdots + q_k$

... example : Heisenberg VOA

<u>Lemma</u> $\gamma_{n,k}((\vec{z},\tilde{w}), \phi_1 \otimes \cdots)$ uniquely determined by $1_p^* \, \gamma_{0,k}((\tilde{w}), 1_{q_1} \otimes \cdots 1_{q_k})$

Sketch of proof:

Use contour integration to remove negative modes in $\phi_i = a_{-m_1} \cdots a_{-m_\ell} 1_{q_i}$ for each i.

Thus :

$$\dim B(M_{q_1}, \dots, M_{q_k}, M_p) = \begin{cases} 0 \;; & q_1 + \cdots + q_k \neq p \\ \leq 1 \;; & \text{—''—} = p \end{cases}$$

actually $=1$ (w/o proof)

... example : Heisenberg VOA

<u>Proposition.</u> $f = 1_p^* \, \gamma_{n,K} \left((\vec{w}, [\alpha]), 1_{q_1} \otimes \cdots \otimes 1_{q_K} \right)$ solves

$$\left(\frac{\partial}{\partial w_i} + \sum_{j \neq i} \frac{q_i q_j}{w_j - w_i} \right) f = 0$$

(Knizhnik - Zamolodchikov equation)

Solution :

$$f = (\text{const}) \prod_{i<j} (w_i - w_j)^{q_i q_j}$$

... example : Heisenberg VOA

Sketch of proof: compute

$$L_{-1} 1_{q_i} = \frac{1}{2} \sum_{m \in \mathbb{Z}} a_m a_{-1-m} 1_{q_i} = a_{-1} a_0 1_{q_i} = q_i a_{-1} 1_{q_i}$$

$$\frac{\partial}{\partial w_i} 1_p^* \gamma_{0,K}\left((\vec{w}, [\alpha]), 1_{q_1} \otimes \cdots \right)$$

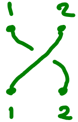
$$= q_i 1_p^* \gamma_{0K}\left((\vec{w}, [\alpha]), \cdots \otimes a_{-1} 1_{q_i} \otimes \cdots \right)$$

$$= q_i \int_{\bigcirc w_i} \frac{dz}{2\pi i} (z-w_i)^{-1} 1_p^* \gamma_{1,K}\left((\vec{w}, [\alpha]), z, a_{-1} 1_0 \otimes 1_{q_1} \cdots 1_{q_K} \right)$$

$$= -q_i \sum_{j \neq i} \int_{\bigcirc w_j} \frac{dz}{2\pi i} (\text{-- same --}) + q_i \int_{\bigcirc w_1 \cdots w_K} \frac{dz}{2\pi i} (\text{ same })$$

$$= -q_i \sum_{j \neq i} q_j (w_j - w_i)^{-1} 1_p^* \gamma_{0,K}\left((\vec{w}, [\alpha]), 1_{q_1} \cdots \right) + 0$$

... example : Heisenberg VOA

Compute action of  on $\mathcal{B}(M_p, M_q ; M_{p+q})$

$\nu^{p,q}$ basis el.

Abbrev.

$$f((z,w),[\tau]) = 1^*_{p+q} \circ \nu^{p,q}_{0,2}\big((z,w,[\tau]), 1_p \otimes 1_q\big)$$

Choose $\nu^{p,q}$ (uniquely) s.th.

$$f\left((1,2), \begin{bmatrix}\text{const}\\ \text{path}\end{bmatrix}\right) = 1$$

Had

$$f((z,w),[\tau]) = (\text{const})\,(z-w)^{pq}$$

determined by anal. cont. along τ

... example : Heisenberg VOA

$$f\big((z,w),[\tau]\big) = (\text{const})\,(z-w)^{pq}$$

Let $\alpha =$  . Using ↑ one checks

$$\underbrace{\alpha^* \nu^{p,q}}_{\in \mathcal{B}(M_q, M_p ; M_{p+q})} = e^{\pi i pq}\, \nu^{q,p}$$

Tensor product functor on $\mathcal{F}$-mod

⚠ Starting from here, we will need lots of extra assumptions. These hold e.g. for

- the Heisenberg VOA
- VOAs that are rational, C_2-cofinite, self-dual and have $V_0 = \mathbb{C}1$.

(cf tensor product theory by Huang, Lepowski)

... tensor product functor on $\mathcal{F}$-mod

Abbreviate $\vec{M} = M_1, \ldots, M_k$, $M_i \in \mathcal{F}$-mod

Assume:

The functors $B(\vec{M}, -)$ are representable for all $\vec{M}$:
There is $T(\vec{M}) \in \mathcal{F}$-mod s.th.

$$B(\vec{M}, N) \xrightarrow[\eta_N]{\sim} \mathrm{Hom}_{\mathcal{F}}(T(\vec{M}), N) ,$$

naturally in N.

... tensor product functor on F-mod

Since $B(\vec{M}, N)$ is natural also in $\vec{M}$ get functors

$$T : \underbrace{F\text{-mod} \times \dots \times F\text{-mod}}_{k \text{ times}} \longrightarrow F\text{-mod}$$

for each $k \geq 1$.

For $n=1$ (see above) : $T : F\text{-mod} \to F\text{-mod}$,

$T = \mathrm{id}$.

... tensor product functor on $\mathcal{F}$-mod

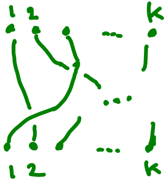
Unit isomorphisms

$$\begin{tikzcd}
\mathcal{B}(\vec{M} \mp \vec{N}; W) \arrow[r, "\sim", "\eta_W"'] \arrow[d, "\wr"] & \mathrm{Hom}_{\mp}(T(\vec{M} \mp \vec{N}), W) \arrow[d, dashed, "\wr"] \\
\mathcal{B}(\vec{M}\, \vec{N}; W) \arrow[r, "\sim"] & \mathrm{Hom}_{\mp}(T(\vec{M}\, \vec{N}), W)
\end{tikzcd}$$

Get natural iso's

$$T(\vec{M} \mp \vec{N}) \xrightarrow[u]{\sim} T(\vec{M}\, \vec{N})$$

Commutativity isomorphisms

Let $\alpha =$  be an element of the braid group B_k

$\sigma \in S_k$: corresponding permutation.

$$\begin{tikzcd}
B(M_1, \dots, M_k, N) \arrow[r, "\sim", "\eta_N"'] \arrow[d, "\alpha^*", "\wr"'] & \mathrm{Hom}_{\mathcal{I}}(T(M_1 \dots), N) \arrow[d, dashed, "\wr"'] \\
B(M_{\sigma 1}, \dots, M_{\sigma k}, N) \arrow[r, "\sim", "\eta_N"'] & \mathrm{Hom}_{\mathcal{I}}(T(M_{\sigma 1} \dots), N)
\end{tikzcd}$$

Get nat. iso's $\quad T(M_1, \dots, M_k) \xrightarrow[c]{\sim} T(M_{\sigma 1}, \dots, M_{\sigma k})$

Example: Heisenberg VOA

$$\mathrm{Vect}^{\mathrm{bnd}}_{\mathbb{C}\text{-gr}} \xrightarrow{\widehat{(\ )}} V_H\text{-mod} \qquad \text{fully faithful}$$

Tensor product (2 factors)

$$\mathcal{B}(M_p, M_q; W) \xrightarrow{\sim} \mathrm{Hom}_{V_H}(T(M_p, M_q), W)$$

$$\begin{array}{ccc} \mathcal{B}(\widehat{\mathbb{C}_p}, \widehat{\mathbb{C}_q}; \widehat{X}) & \xrightarrow{\sim} & \mathrm{Hom}_{V_H}(T(\widehat{\mathbb{C}_p}, \widehat{\mathbb{C}_q}), \widehat{X}) \\ \downarrow \wr & & \\ \mathrm{Hom}_{\mathrm{gr}}(\mathbb{C}_p \otimes \mathbb{C}_q, X) & & \end{array}$$

Thus: can choose

$$T(\widehat{\mathbb{C}_p}, \widehat{\mathbb{C}_q}) = \widehat{\mathbb{C}_p \otimes \mathbb{C}_q}$$

Hence:

$\widehat{(\ )}$ is a (non-coherent) tensor functor w.r.t. T with two arguments

... example: Heisenberg VOA

Commutativity isom. (2 factors), $\alpha =$ 

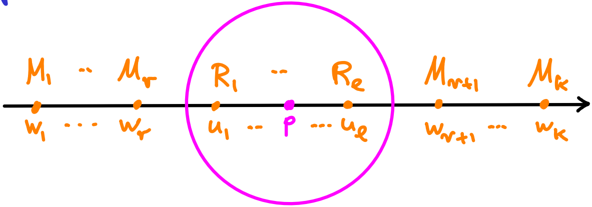
$$\begin{tikzcd}
B(\hat{\mathbb{C}}_p, \hat{\mathbb{C}}_q, \hat{\mathbb{C}}_{p+q}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{gr}(\mathbb{C}_p \otimes \mathbb{C}_q, \mathbb{C}_{p+q}) \arrow[d, dashed] \\
B(\hat{\mathbb{C}}_q, \hat{\mathbb{C}}_p, \hat{\mathbb{C}}_{p+q}) \arrow[r, "\sim"] & \mathrm{Hom}_{gr}(\mathbb{C}_q \otimes \mathbb{C}_p, \mathbb{C}_{p+q})
\end{tikzcd}$$

$$\nu^{pq} \mapsto e^{\pi i pq} \cdot \nu^{qp}$$

So induced isom. on $\mathrm{Vect}^{\mathrm{bd}}_{\mathbb{C}\text{-}gr}$ is

$$c : \mathbb{C}_p \otimes \mathbb{C}_q \longrightarrow \mathbb{C}_q \otimes \mathbb{C}_p$$
$$1 \otimes 1 \longmapsto e^{\pi i pq} \cdot 1 \otimes 1$$

Associativity isomorphisms and coherence



Assume:

Let $\rho \in B(\vec{M}\, W\, \vec{N}, X)$, $\sigma \in B(\vec{R}, W)$

Then $(k = |\vec{M}| + |\vec{N}|,\ \ell = |\vec{R}|)$

$$\lim_{d \to \infty} \rho_{n,k+1}\Big(\cdots, \phi_1 \otimes \cdots \otimes \underbrace{P_{\leq d}\, \sigma_{m,\ell}(\cdots)}_{k+1\text{'st entry}} \otimes \cdots \otimes \phi_k \Big)$$

converges to an element of

$B(\vec{M}\, \vec{R}\, \vec{N}, X)$

... associativity isomorphisms and coherence

Assumption gives

$$\begin{tikzcd} B(\vec{M}\, W\, \vec{N}, X) \otimes B(\vec{R}, W) \arrow[r] \arrow[d, "\wr"', "\eta_X \otimes \eta_W"] & B(\vec{M}\,\vec{R}\,\vec{N}, X) \arrow[d, "\wr"', "\eta_X"] \\ \mathrm{Hom}_{\mathcal{F}}(T(\vec{M}\, W\, \vec{N}), X) \otimes \mathrm{Hom}_{\mathcal{F}}(T(\vec{R}), W) \arrow[r, dashed] & \mathrm{Hom}_{\mathcal{F}}(T(\vec{M}\,\vec{R}\,\vec{N}), X) \end{tikzcd}$$

For $W = T(\vec{R})$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{F}}(T(\vec{M}\, T(\vec{R})\, \vec{N}), X) \arrow[d, "f \mapsto f \otimes \mathrm{id}"] \\ \mathrm{Hom}_{\mathcal{F}}(T(\vec{M}\, T(\vec{R})\, \vec{N}), X) \otimes \mathrm{Hom}_{\mathcal{F}}(T(\vec{R}), T(\vec{R})) \arrow[d, dashed] \\ \mathrm{Hom}_{\mathcal{F}}(T(\vec{M}\,\vec{R}\,\vec{N}), X) \end{tikzcd}$$

Get maps
(a priori not iso's)

$$T(\vec{M}\, T(\vec{R})\, \vec{N}) \xrightarrow{a} T(\vec{M}\,\vec{R}\,\vec{N})$$

... associativity isomorphisms and coherence

<u>Theorem</u>. The collection of maps

$$T(\vec{M}\ T(\vec{R})\ \vec{N}) \xrightarrow{a} T(\vec{M}\ \vec{R}\ \vec{N}) \quad \text{(not necc. isom.)}$$

$$T(\vec{M}\ \mp\ \vec{N}) \xrightarrow[u]{\sim} T(\vec{M}\ \vec{N})$$

$$T(M_1, \dots, M_k) \xrightarrow[c]{\sim} T(M_{\sigma 1}, \dots, M_{\sigma k})$$

is coherent, i.e. any two compositions of a, u, c with same source and target

$$T(\cdots T(\cdots T\cdots\ )\cdots T\cdots) \longrightarrow T(\cdots\ T\text{'s}\cdots\ )$$

and representing the same element in the braid group are equal

Idea of proof: different compositions $\hat{=}$ reordering of limits. Write limits as series and use absolute convergence to reorder.

Braided tensor structure on $\mathcal{F}$-mod

Assumption: The composition

$$\begin{array}{c} B(\vec{M}\, T(\vec{R})\, \vec{N},\ X) \\ \Big\downarrow\ f \mapsto f \otimes \iota \\ B(\vec{M}\, T(\vec{R})\, \vec{N},\ X) \otimes B(\vec{R}, T(\vec{R})) \\ \Big\downarrow\ \text{compose blocks} \\ B(\vec{M}\, \vec{R}\, \vec{N}, X) \end{array}$$

inverse image of id under $B(\vec{R}, T(\vec{R})) \xrightarrow[\eta]{\sim} \mathrm{Hom}_{\mathcal{F}}(T(\vec{R}), T(\vec{R}))$

is an isomorphism.

Consequence: $T(\vec{M}\, T(\vec{R})\, \vec{N}) \xrightarrow{a} T(\vec{M}\, \vec{R}\, \vec{N})$ is iso.

... braided tensor structure on $\mathcal{F}$-mod

Then T, a, u, c determined by

$$T : \mathcal{F}\text{-mod} \times \mathcal{F}\text{-mod} \longrightarrow \mathcal{F}\text{-mod}$$

$$\alpha_{K,L,M} : \quad T(T(K,L),M) \xrightarrow{\sim} T(K,T(L,M))$$

$$\begin{tikzcd}
T(T(K,L),M) \arrow[dr, "a"] & & T(K,T(L,M)) \arrow[dl, "a"'] \\
& T(K,L,M) &
\end{tikzcd}$$

$$\lambda_M : T(\mathcal{F},M) \xrightarrow[u]{\sim} M \quad , \quad \rho_M : T(M,\mathcal{F}) \xrightarrow[u]{\sim} M$$

$$c_{M,N} : \quad T(M,N) \xrightarrow{\sim} T(N,M)$$

Corollary (to coherence): $\mathcal{F}$-mod with $\uparrow$ is a braided tensor category.

Example: Heisenberg VOA

So far:

$$\mathrm{Vect}^{\mathrm{bnd}}_{\mathbb{C}\text{-gr}} \xrightarrow{\widehat{(\ )}} V_H\text{-mod}, \qquad \widehat{R \otimes S} \underset{\text{nat.}}{\cong} T(\hat{R}, \hat{S})$$

Transported braiding:

$$c : \mathbb{C}_p \otimes \mathbb{C}_q \longrightarrow \mathbb{C}_q \otimes \mathbb{C}_p$$
$$1 \otimes 1 \longmapsto e^{\pi i p q}\, 1 \otimes 1$$

Transported associator and unit iso's (w/o proof):

those of $\mathrm{Vect}^{\mathrm{bnd}}_{\mathbb{C}\text{-gr}}$

Part IV

I Some motivation from lattice models

conformal field theory

II

vertex operator algebra

III

braided tensor cat. of
VOA - modules

IV

& commutative algebra
in (two copies of)
cat. of VOA-modules

Algebras in tensor categories

$\mathcal{C} = (\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$: tensor category

$\alpha_{u,v,w} : (u \otimes v) \otimes w \xrightarrow{\sim} u \otimes (v \otimes w)$ associator

$\lambda_u : 1 \otimes u \xrightarrow{\sim} u$, $\rho_u : u \otimes 1 \to u$ unit isos

subject to triangle and pentagon identities.

<u>Definition.</u> An <u>algebra</u> (or <u>monoid</u>) in $\mathcal{C}$ is a triple (A, μ, η) s.th. $A \in \mathcal{C}$, $AA \xrightarrow{\mu} A$, $1 \xrightarrow{\eta} A$

$$\begin{tikzcd}
A1 \arrow[r, "1\eta"] \arrow[dr, "\sim"'] & AA \arrow[d, "\mu"] & 1A \arrow[l, "\eta 1"'] \arrow[dl, "\sim"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
(AA)A \arrow[r, "\mu 1"] \arrow[d, "\wr"] & AA \arrow[dr, "\mu"] & \\
& & A \\
A(AA) \arrow[r, "1\mu"] & AA \arrow[ur, "\mu"'] &
\end{tikzcd}$$

... algebras in tensor categories

E.g.:

- an algebra in Vect_k is a unital assoc. k-algebra
- an algebra in $\mathrm{Vect}_{G\text{-}gr}$: unit and mult. are degree zero maps

Definition. A left <u>A-module</u> in $\mathcal{C}$ is a pair (M,ρ) s.th. $M \in \mathcal{C}$, $AM \xrightarrow{\rho} M$,

$$\begin{tikzcd}
1M \arrow[r, "\eta 1"] \arrow[dr, "\wr"'] & AM \arrow[d, "\rho"] \\
& M
\end{tikzcd}
\qquad
\begin{tikzcd}
(AA)M \arrow[r, "\mu 1"] \arrow[d, "\wr"] & AM \arrow[dr, "\rho"] & \\
A(AM) \arrow[r, "1\rho"] & AM \arrow[r, "\rho"'] & A
\end{tikzcd}$$

... algebras in tensor categories

$\mathcal{C}$: braided tensor cat.,

$c_{U,V} : U\otimes V \xrightarrow{\sim} V\otimes U$ braiding (subject to two hexagon conditions)

<u>Definition.</u> A : algebra in $\mathcal{C}$

A is <u>commutative</u> if

$$\begin{tikzcd} AA \arrow[d, "c_{A,A}"'] \arrow[dr, "\mu"] & \\ AA \arrow[r, "\mu"'] & A \end{tikzcd}$$

M : A-module.

M is <u>local</u> if

$$\begin{tikzcd} AM \arrow[d, "c_{A,M}"'] \arrow[dr, "\rho"] & \\ MA \arrow[d, "c_{M,A}"'] & M \\ AM \arrow[ur, "\rho"'] & \end{tikzcd}$$

... algebras in tensor categories

Are there algebras "which cannot be made bigger"?

<u>Definition</u> $\mathcal{C}$: abelian braided tensor cat.

A commutative algebra A in $\mathcal{C}$ is called <u>Lagrangian</u> if every local A-module is isomorphic (as an A-module) to a direct sum of copies of ${}_A A$

Remarks:

- If A is Lagrangian and $\mathcal{C}$ does not contain infinite inclusions of proper subobjects $\cdots \hookrightarrow U \hookrightarrow U \hookrightarrow U \hookrightarrow U$, then ${}_A A$ is simple
  (since submodules of local modules are local)

... algebras in tensor categories

- B : comm. alg. in $\mathcal{C}$, $A \hookrightarrow B$ subalgebra, A Lagrangian
  Then $B \simeq \bigoplus A$ as A-module.

  If we want a "unique vacuum condition" in the form $\mathcal{C}(1, B) = \mathbb{C} \cdot \eta$ (for $\mathbb{C}$-linear $\mathcal{C}$), then we cannot extend A.

- In $\mathrm{Vect}(\mathbb{C})$, the only fin. dim. Lag. alg. is $\mathbb{C}$.
  (but there are inf. dim. ones, eg function fields)

- In $\mathrm{Vect}_{\mathbb{C}\text{-gr}}(\mathbb{C})$ (with symmetric braiding), there is no Lagrangian algebra with finite or countable basis.
  (since $A[\gamma]$ is an A-module for any shift $\gamma \in \mathbb{C}$)

Example: $\mathrm{Vect}_{\mathbb{C}\text{-gr}}$ with non-trivial braiding

Braiding $c_{\mathbb{C}_p, \mathbb{C}_q} = e^{\pi i p q}$

Abbreviate $\mathcal{C} := \mathrm{Vect}_{\mathbb{C}\text{-gr}}$ with as braiding

Family of algebras in $\mathcal{C}$ $(r \in \mathbb{C}^*)$

$$A_r := \bigoplus_{m \in \mathbb{Z}} \mathbb{C}_{r \cdot m} \qquad \mu(1_{mr} \otimes 1_{nr}) = 1_{(m+n)r}$$

$$\eta : \mathbb{C}_0 \hookrightarrow A$$

A-mod is semi-simple with simple modules $A_r[q]$

degree shift

... example : $\mathrm{Vect}_{\mathbb{C}\text{-gr}}$ with non-trivial braiding

- A_r is commutative iff $r^2 \in 2\mathbb{Z}$

  (since $\mu \circ c_{A,A}(1_{mr} \otimes 1_{nr}) = e^{\pi i r^2 mn}\, 1_{(m+n)r}$)

- Let $r^2 \in 2\mathbb{Z}$

  $A_r[q]$ is local iff $q \in \frac{1}{r}\mathbb{Z}$

  (since $\mu \circ c_{A[q],A} \circ c_{A,A[q]}(1_{mr} \otimes 1_{nr+q})$
  $= e^{2\pi i\, mr(nr+q)}\, 1_{(m+n)r+q}$)

Thus : A_r is never local.

(since $\frac{1}{r}\mathbb{Z} \not\subseteq r\mathbb{Z}$ for $r^2 \in 2\mathbb{Z}$)

... example : $\mathrm{Vect}_{\mathbb{C}\text{-gr}}$ with non-trivial braiding

Consider $\mathbb{C}^2$ - graded v. sp. with braiding

$$c_{\mathbb{C}_{\vec{p}},\mathbb{C}_{\vec{q}}} = e^{\pi i (p_1 q_1 - p_2 q_2)}$$

Lattice

$$\Lambda_r = \mathbb{Z}\begin{pmatrix}1\\1\end{pmatrix}\cdot r \oplus \mathbb{Z}\begin{pmatrix}1\\-1\end{pmatrix}\cdot \frac{1}{2r} \qquad r \in \mathbb{R}_{>0}$$

$$B_r := \bigoplus_{\vec{p}\in\Lambda_r} \mathbb{C}_{\vec{p}}$$

Write $\vec{p} = p_+ \begin{pmatrix}1\\1\end{pmatrix} r + p_- \begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{2r}$

$$\mu(1_{\vec{p}} \otimes 1_{\vec{q}}) = e^{\pi i p_+ q_-}\, 1_{\vec{p}+\vec{q}}$$

... example : $\mathrm{Vect}_{\mathbb{C}\text{-gr}}$ with non-trivial braiding

Lemma B_r is a commutative algebra

Pf

Assoc.: $\mu(\mu(1_{\vec p}\, 1_{\vec q})\, 1_{\vec r}) = (-1)^{p_+ q_- + (p_+ + q_+) r_-}\, 1_{\vec p + \vec q + \vec r}$

$\mu(1_{\vec p}\, \mu(1_{\vec q}\, 1_{\vec r})) = (-1)^{q_+ r_- + p_+(q_- + r_-)}\, 1_{\vec p + \vec q + \vec r}$ ✓

Comm.: $\mu \circ c_{A,A}(1_{\vec p}\, 1_{\vec q}) = e^{\pi i (p_+ q_- + p_- q_+)}\, e^{\pi i\, q_+ p_-}\, 1_{\vec p + \vec q}$

$= e^{\pi i\, p_+ q_-}\, 1_{\vec p + \vec q} = \mu(1_{\vec p}\, 1_{\vec q})$ ✓

$\square$

B_r-mod is semi simple, simple modules are $B_r[\vec q]$

... example : $\mathrm{Vect}_{\mathbb{C}\text{-gr}}$ with non-trivial braiding

Lemma. $B_r[\vec{q}]$ local $\Leftrightarrow \vec{q} \in \Lambda_r$

Pf

$$\rho \circ c_{A[q]A} \circ c_{A\,A[q]} \left(1_{\vec{s}}\, 1_{\vec{t}+\vec{q}}\right) \qquad (\vec{s},\vec{t} \in \Lambda)$$

$$= \underbrace{e^{2\pi i \left(s_+ (t_- + q_-) + s_- (t_+ + q_+)\right)}}_{= e^{2\pi i (s_+ q_- + s_- q_+)}} \rho\left(1_{\vec{s}}\, 1_{\vec{t}+\vec{q}}\right)$$

Thus need $q_\pm \in \mathbb{Z}$, i.e. $\vec{q} \in \Lambda$. $\square$

Altogether : B_r is local.

Aside : There are more Lag. alg. , e.g. $\bigoplus_{x \in \mathbb{R}} \mathbb{C}_{(x,x)}$ with product $\mu(1_{(x,x)}\, 1_{(y,y)}) = 1_{(x+y,x+y)}$.

CFT and commutative algebras

$(\mathcal{F}, \mu, 1)$: CFT , $E \subset \mathcal{F}$: sub-CFT

Thus $\mathcal{F} \in E\text{-mod}$

$$(\mu_{n+k})_n \in B_E(\mathcal{F}^{\otimes n}; \mathcal{F})$$

Assume : Assumptions in part III hold for E,
in particular, E-mod is braided tensor cat.

$$B_E(\mathcal{F}, \mathcal{F}; \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_E(T(\mathcal{F}, \mathcal{F}), \mathcal{F})$$

$$(\mu_{n+2})_n \longmapsto \tilde{\mu}$$

... CFT and commutative algebras

Proposition. F with mult. $\tilde{\mu}$ and unit $E \hookrightarrow F$ is a commutative algebra in E-mod.

Sketch of proof:

Assoc.: by factorisation the two compositions

$$B(F, F; F) \otimes B(F, F; F) \longrightarrow B(F, F, F; F)$$

map to the same element

Comm.: μ_{n+k} is defined on $\mathcal{M}_{n+k}$, not only on the cover $\mathcal{M}_{n,k}$, hence monodromies α^* act trivially on $\mu \in B(F^{\otimes k}, F)$

Conversely, let $(F, \tilde{\mu}, \tilde{\eta})$ be a commutative algebra in E-mod.

Define $\mu_n : \mathcal{M}_n \times F^{\otimes n} \longrightarrow F$ via

$$\begin{array}{ccc}
B(F, \dots, F; F) & \xrightarrow{\sim} & \mathrm{Hom}_E(T(F, \dots, F), F) \\
 & & \wr\wr \\
\in & & \mathrm{Hom}_E(T(F, T(F, \dots F)), F) \\
 & & \in \\
(\mu_{n+k})_n & \longmapsto & \tilde{\mu} \circ (id \otimes \tilde{\mu}) \circ \dots
\end{array}$$

Then comm. of $\tilde{\mu}$ $\Rightarrow$ μ_{n+k} decends to $\mathcal{M}_{n+k}$

assoc. of $\tilde{\mu}$ $\Rightarrow$ factorisability of μ_n

<u>Prop.</u> A comm. alg in E-mod defines a CFT with sub-CFT E.

... CFT and commutative algebras

One stategy to build CFTs :

1) fix a CFT E (e.g. a VOA V or $V \otimes \bar{V}$ for $\bar{V}$ an antihol. copy of V)

2) try to gain some control over E-mod

3) find commutative algebras in E-mod

Example : Heisenberg VOA

1) fix Heisenberg VOA $E = V_H$

2) braided tensor functor $\mathrm{Vect}^{\mathrm{bnd}}_{\mathbb{C}\text{-gr}} \xrightarrow{\widehat{(\ )}} V_H\text{-mod}$

3) Had found algebras A_r $(r^2 \in 2\mathbb{Z})$

$A_r \in \mathrm{Vect}^{\mathrm{bnd}}_{\mathbb{C}\text{-gr}}$ iff $r \in \mathbb{R}$.

$\rightsquigarrow$ hol. CFT $\widehat{A_r} = \bigoplus_{m \in \mathbb{Z}} \widehat{\mathbb{C}}_{mr}$ (rank 1 lattice VOA)

... example : Heisenberg VOA

1) fix $E = V_H \otimes \overline{V}_H$ antihol. copy

2) braided tensor functor $\mathrm{Vect}^{\mathrm{bnd}}_{\mathbb{C}^2\text{-gr}} \xrightarrow{\widehat{(\ )}} V_H \otimes \overline{V}_H\text{-mod}$

3) Had found alg. B_r $(r \in \mathbb{R}_{>0})$

$\rightsquigarrow$ (non-hol. CFT) $\widehat{B}_r = \bigoplus_{(p,q)\in\Lambda_r} \widehat{\mathbb{C}_p \otimes \mathbb{C}_q}$

One observes ;

1^{st} example: non-Lag. & partition fn (graded trace of F) not modular invariant

2^{nd} example: Lagrangian & partition fn mod. inv. (but not holom. in τ)

Outlook: the rational case

Let V be a rational (& extra conditions) VOA

$\rightsquigarrow$ V-mod is a modular tensor category

Let $\mathcal{C}$ be a modular tensor cat

Can show:

$\mathcal{C}$ with inverse braiding and twist

$$\{ \text{Lagrangian algebras in } \mathcal{C} \boxtimes \overline{\mathcal{C}} \text{ up to alg. iso.} \}$$

$$\updownarrow \wr$$

$$\{ \text{indecomposable } \mathcal{C}\text{-module categories up to equivalence} \}$$

So: Classification of "maximal" CFTs over $V \otimes \overline{V}$ becomes a question in categorified representation theory

I Some motivation from lattice models

conformal field theory

II

vertex operator algebra

III

braided tensor cat. of VOA-modules

& commutative algebra in (two copies of) cat. of VOA-modules

IV