Centrally ergodic one-parameter automorphism groups on semifinite injective von Neumann algebras

Yasuyuki Kawahigashi

Department of Mathematics, Faculty of Science

University of Tokyo, Hongo, Tokyo, 113, JAPAN

May, 1987

Abstract. We classify, up to stable conjugacy, centrally ergodic actions α of $\mathbf{R}$ on an injective semifinite von Neumann algebra with an invariant trace and with $\Gamma(\alpha) \neq \mathbf{R}$. We also classify actions of $\mathbf{R}$ on an injective semifinite von Neumann algebra with a non-trivial center, and which admit an invariant trace.

§0 Introduction

In this paper, we study one-parameter automorphism groups on injective semifinite factors up to stable conjugacy. Although there has been steady progress on the classification problem of discrete group actions on injective factors (cf. [Connes, 2], [Connes, 3], [Jones, 6], [Jones-Takesaki, 7], [Ocneanu, 11], and [Sutherland-Takesaki, 14]), the problem for continuous groups has not been studied. It is necessary to understand actions of $\mathbf{R}$ for the theory of general continuous groups, but little is known for actions of the real number group $\mathbf{R}$, beyond the Arveson-Connes spectrum theory and Takesaki duality. This paper is the starting point of our study of continuous group actions on injective factors. As a first step, we will study the easier case where the Connes spectrum $\Gamma(\alpha)$ is not equal to the entire group $\hat{\mathbf{R}} = \mathbf{R}$, and get the following main theorems.

Theorem 0.1. *Let $\mathcal{M}$ be a semifinite injective factor and α and β actions of $\mathbf{R}$ with $\Gamma(\alpha)$, $\Gamma(\beta) \neq \hat{\mathbf{R}}$ and $\alpha^{-1}(Int(\mathcal{M}))$, $\beta^{-1}(Int(\mathcal{M})) \neq \mathbf{R}$. Let*

$$\mathcal{M} \rtimes_\alpha \mathbf{R} \cong \mathcal{A}(\alpha)\bar{\otimes}\mathcal{N}(\alpha),$$

$$\mathcal{M} \rtimes_\beta \mathbf{R} \cong \mathcal{A}(\beta)\bar{\otimes}\mathcal{N}(\beta),$$

be the central decompositions, where $\mathcal{A}(\alpha)$ and $\mathcal{A}(\beta)$ are the centers and $\mathcal{N}(\alpha)$ and $\mathcal{N}(\beta)$ are factors. Then α and β are stably conjugate if and only if $\mathcal{N}(\alpha)$, $\mathcal{A}(\alpha)$ and $\mathcal{N}(\beta)$, $\mathcal{A}(\beta)$ are isomorphic respectively, and the flows given by $\hat{\alpha}$ and $\hat{\beta}$ on the centers $\mathcal{A}(\alpha)$ and $\mathcal{A}(\beta)$ are conjugate.

Note that if $\alpha^{-1}(\mathrm{Int}(\mathcal{M})) = \mathbf{R}$, then α is cocycle conjugate to the trivial action by a classical result ([Kallman, 8]).

Theorem 0.2. *In the context of Theorem 0.1, we have*

1. *If $\Gamma(\alpha) = 0$, then $\mathcal{N}(\alpha)$ is isomorphic to $\mathcal{L}(\mathcal{H})$ or the hyperfinite type II_∞ factor $\mathcal{R}_{0,1}$.*
2. *If $\Gamma(\alpha) \cong \mathbf{Z}$, then $\mathcal{N}(\alpha)$ is isomorphic to $\mathcal{R}_{0,1}$.*

Using Takesaki duality, we are also able to classify centrally ergodic actions of $\mathbf{R}$ on semifinite injective von Neumann algebras under the assumptions that the action admits an invariant trace, and the center is non-trivial.

Modular automorphism groups, a special type of actions of $\mathbf{R}$, have been well studied. The classification problem of injective type III factors was reduced to the classification of group actions on the hyperfinite type II factors. Our problem is related to the uniqueness problem of the injective type III_1 factor, which was solved recently by [Haagerup, 5]. Our result here is also related to Connes' classification of injective type III_0 and III_λ factors, $0 < \lambda < 1$, [Connes, 4].

The condition $\Gamma(\alpha) \neq \mathbf{R}$ in our result is a strong restriction on the action α. But the difficulty for the case $\Gamma(\alpha) = \mathbf{R}$ is very similar to that in the uniqueness problem of the injective type III_1 factor. In our context, Haagerup's deep result means that an action α of $\mathbf{R}$ on $\mathcal{R}_{0,1}$ with $\mathrm{tr} \circ \alpha_t = e^{-t}\mathrm{tr}$, $t \in \mathbf{R}$, is unique up to conjugacy, but the case of trace preserving actions with $\Gamma(\alpha) = \mathbf{R}$ is still open. The author hopes that this case will be settled in the near future.

In §1, we prove a version of Theorem 0.1 for semifinite injective algebras. In §2, we will show that the characteristic invariant is trivial and $\mathcal{N}(\alpha)$ is of type II_∞ if we have $\Gamma(\alpha) \cong \mathbf{Z}$ and $\mathcal{M}$ is a factor. In §3 we construct examples to show that all the possible cases in Theorem 0.2 can actually occur.

§1 General cases

Let $\mathcal{M}$ be a semifinite injective (separable) von Neumann algebra, and α a centrally ergodic action of $\mathbf{R}$ on $\mathcal{M}$ (i.e., $\mathcal{Z}(\mathcal{M})^\alpha = \mathbf{C}$) such that $\mathcal{M}$ has an invariant trace τ for α.

First, we deal with the case where $\Gamma(\alpha) = 0$. (See Définition 2.2.1 in [Connes, 1] for the definition of the Connes spectrum $\Gamma(\alpha)$.) We may exclude the case where all the α_t's are inner, because in this case the action is cocycle conjugate to the trivial one by a result in [Kallman, 8]. We will classify actions of this type up to stable conjugacy. (See page 216 in [Jones-Takesaki, 7] for the definition of stable conjugacy.) We note that by the result of [Katayama, 10], we know that the crossed product algebra by $\mathbf{R}$ is properly infinite unless the action is inner for every $t \in \mathbf{R}$. We construct an invariant for this action α as follows.

We denote by σ the action $\alpha \otimes \mathrm{Ad}\lambda_{\mathbf{R}}^{-1}$ on $\tilde{\mathcal{M}} = \mathcal{M}\bar{\otimes}\mathcal{L}(L^2(\mathbf{R}))$, which is the second dual action of α by Takesaki duality (see [Takesaki, 15]), where $\lambda_{\mathbf{R}}$ is the regular representation of $\mathbf{R}$. (Here we use the definition $\hat{\alpha}_s(u_t) = e^{-ist}u_t$ for the dual action.) In the following, we consider the system $\{\tilde{\mathcal{M}}, \sigma\}$ instead of the system $\{\mathcal{M}, \alpha\}$. We still have $\Gamma(\sigma) = 0$, so by the definition of the Connes spectrum, there exist a central projection e in the fixed point algebra $\tilde{\mathcal{M}}^\sigma$, which is isomorphic to the crossed product $\mathcal{M} \rtimes_\alpha \mathbf{R}$, and a positive ε such that

$$\mathrm{Sp}(\sigma^e) \cap ([-3\varepsilon, -\varepsilon] \cup [\varepsilon, 3\varepsilon]) = \emptyset.$$

Then by Lemme 5.2.3 in [Connes, 1], we have a positive non-singular self-adjoint h in $\tilde{\mathcal{M}}_e^{\sigma^e}$ such that

$$\mathrm{Sp}(t \mapsto (\mathrm{Ad}(h^{it}) \circ \sigma_t^e)) \cap [-\varepsilon, \varepsilon] = \{0\}.$$

We would like to get the equality $\mathrm{Sp}(\sigma) \cap [-\varepsilon, \varepsilon] = \{0\}$ by replacing σ within its stable conjugacy class. By considering $\sigma \otimes i$ on $\tilde{\mathcal{M}}\bar{\otimes}\mathcal{L}(L^2(\mathbf{R}))$ if necessary, we have a partial isometry u in $\tilde{\mathcal{M}}$ such that $e = uu^*$ and $1 = u^*u$ since e is properly infinite and the central ergodicity of σ implies that the central support of e in $\tilde{\mathcal{M}}$ is 1. We define

$$v_t = u^* h^{it} \sigma_t(u) \in \tilde{\mathcal{M}}, \qquad \text{for } t \in \mathbf{R},$$

so that we have

$$\begin{aligned}
v_t v_t^* &= u^* h^{it} \sigma_t(u) \sigma_t(u^*) h^{-it} u \\
&= u^* h^{it} e h^{-it} u = u^* e u = u^* u u^* u = 1, \\
v_t^* v_t &= \sigma_t(u^*) h^{-it} u u^* h^{it} \sigma_t(u) \\
&= \sigma_t(u^*) e \sigma_t(u) = \sigma_t(u^* u u^* u) = 1, \\
v_s \sigma_s(v_t) &= u^* h^{is} \sigma_s(u) \sigma_s(u^* h^{it} \sigma_t(u)) \\
&= u^* h^{is} \sigma_s(u) \sigma_s(u^*) h^{it} \sigma_{s+t}(u) \\
&= u^* h^{i(s+t)} \sigma_{s+t}(u) \\
&= v_{s+t}.
\end{aligned}$$

Thus $\{v_t\}$ is a unitary cocycle for σ, so we define a new action on $\tilde{\mathcal{M}}$ by $\mathrm{Ad}(v_t) \circ \sigma_t$, and denote it simply by σ again. Now we have $\mathrm{Sp}(\sigma) \cap [-\varepsilon, \varepsilon] = \{0\}$ as desired. By considering $\sigma \otimes i$ on $\tilde{\mathcal{M}} \bar{\otimes} \mathcal{L}(L^2(\mathbf{R}))$ if necessary, we may assume $\tilde{\mathcal{M}}^\sigma$ is properly infinite. Then by Lemme 5.3.4 in [Connes, 1], we have a unitary U in $\tilde{\mathcal{M}}^\sigma([\varepsilon, \infty))$ (see Définition 2.1.2 in [Connes, 1] for this notation) such that we have

$$\tilde{\mathcal{M}} \cong \tilde{\mathcal{M}}^\sigma \rtimes_\theta \mathbf{Z}, \qquad \theta = \mathrm{Ad}(U) \in \mathrm{Aut}(\tilde{\mathcal{M}}^\sigma).$$

Suppose $\mathcal{Z}(\tilde{\mathcal{M}}^\sigma) \cong L^\infty(Y, \nu)$. Then there exists a measurable real-valued function k on Y such that $\sigma_t(U) = e^{itk} U$. Because $\mathrm{Sp}_\sigma(U) \subset [\varepsilon, \infty)$ (see Définition 2.1.2 in [Connes, 1] for this notation), we may assume that $k(y) \geq \varepsilon$ for all $y \in Y$. Consider the crossed product $\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R}$. Then we have

$$\begin{aligned}
\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R} &\cong (\tilde{\mathcal{M}}^\sigma \rtimes_\theta \mathbf{Z}) \rtimes_\sigma \mathbf{R} \\
&\cong (\tilde{\mathcal{M}}^\sigma \rtimes_\sigma \mathbf{R}) \rtimes_{\bar{\theta}} \mathbf{Z},
\end{aligned}$$

where $\bar{\theta}$ is defined by

$$\bar{\theta}(\lambda(t)) = e^{itk}\lambda(t), \qquad \text{for } t \in \mathbf{R},$$

$$\bar{\theta}(x) = \theta(x) = UxU^*, \qquad \text{for } x \in \tilde{\mathcal{M}}^\sigma.$$

In the above expression, we also have $\tilde{\mathcal{M}}^\sigma \rtimes_\sigma \mathbf{R} \cong \tilde{\mathcal{M}}^\sigma \bar{\otimes} L^\infty(\mathbf{R})$, where we identify $\lambda(t)$ with the function, $s \in \mathbf{R} \mapsto e^{-ist} \in \mathbf{C}$, on $\mathbf{R}$. Then the action $\bar{\theta}$ on $L^\infty(Y,\nu)\bar{\otimes}L^\infty(\mathbf{R})$ is given by

$$(\bar{\theta}\varphi)(y,t) = \varphi(T^{-1}y, t-k(y)) \qquad \textit{for } \varphi \in L^\infty(Y,\nu)\bar{\otimes}L^\infty(\mathbf{R}),$$

where T is an automorphism on Y corresponding to the automorphism θ on $\mathcal{Z}(\tilde{\mathcal{M}}^\sigma) \cong L^\infty(Y,\nu)$. Thus we have the following lemma.

Lemma 1.1. *Under the above assumptions and notations, we have*

$$\mathcal{Z}(\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R}) = (L^\infty(Y,\nu)\bar{\otimes}L^\infty(\mathbf{R}))^{\bar{\theta}} \cong L^\infty(X,\mu),$$

where $X = \{\, (y,t) \mid y \in Y, 0 \le t < k(T^{-1}y) \,\}$ *and* $d\mu = d\nu \times$ *Lebesgue measure.*

Proof. Because the action $\bar{\theta}$ on $\mathcal{Z}(\tilde{\mathcal{M}}^\sigma \rtimes_\sigma \mathbf{R}) \cong L^\infty(Y,\nu)\bar{\otimes}L^\infty(\mathbf{R})$ is aperiodic, it can be regarded as a free action of $\mathbf{Z}$ on $\tilde{\mathcal{M}}^\sigma \rtimes_\sigma \mathbf{R}$. So by the relative commutant theorem for free actions of discrete groups (Lemma 7.11.10 in [Pedersen, 12]) we know that $\mathcal{Z}(\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R}) \subset \mathcal{Z}(\tilde{\mathcal{M}}^\sigma \rtimes_\sigma \mathbf{R})$. An element $x \in \mathcal{Z}(\tilde{\mathcal{M}}^\sigma \rtimes_\sigma \mathbf{R})$ is in $\mathcal{Z}(\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R})$ if and only if x commutes with the unitary u implementing $\bar{\theta}$ on $\tilde{\mathcal{M}}^\sigma \rtimes_\sigma \mathbf{R}$. But this condition is clearly equivalent to $\bar{\theta}(x) = x$.

For the set X as in the lemma, we know that

$$\bar{\theta}^n(X) \cap \bar{\theta}^m(X) = \emptyset, \qquad \text{for } n \neq m,$$

$$\bigcup_{n \in \mathbf{Z}} \bar{\theta}^n(X) = Y \times \mathbf{R},$$

so the fixed point algebra $(L^\infty(Y,\nu)\bar{\otimes}L^\infty(\mathbf{R}))^{\bar{\theta}}$ is isomorphic to its restriction $L^\infty(X,\mu)$. Q.E.D.

Note that the dual action $\hat{\sigma}$ on the "measure theoretic spectrum" $\{\,(y,t) \mid y \in Y, 0 \leq t < k(y)\,\}$ of this center is the flow under the ceiling function $k(T^{-1}y)$ for the transformation T^{-1} on Y because $\hat{\sigma}$ is just a translation for the $L^\infty(\mathbf{R})$ part in the above expression.

Because σ is centrally ergodic, θ is also centrally ergodic, thus $\tilde{\mathcal{M}}^\sigma$ is isomorphic to $\bar{\mathcal{M}}\bar{\otimes}L^\infty(Y,\nu)$ for an injective factor $\bar{\mathcal{M}}$.

We would like to express $\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R}$ in the form of $\mathcal{N}\bar{\otimes}L^\infty(X,\mu)$ and get the relation between $\hat{\sigma}$ and $\bar{\theta}$.

We consider $\bigoplus_{n=-\infty}^{\infty} \bar{\mathcal{M}}\bar{\otimes}L^\infty(X,\mu)$, $\bigoplus_{n=-\infty}^{\infty} \bar{\mathcal{M}}\bar{\otimes}L^\infty(Y,\nu)$, and $\bigoplus_{n=-\infty}^{\infty} \bar{\mathcal{M}}$ and denote the shift operator on these by S, that is, $(S(x))_n = (x_{n-1})$. Then we have

$$\begin{aligned}\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R} &\cong (\tilde{\mathcal{M}}^\sigma\bar{\otimes}L^\infty(\mathbf{R})) \times_{\bar{\theta}} \mathbf{Z} \\ &\cong (\bar{\mathcal{M}}\bar{\otimes}L^\infty(Y,\nu)\bar{\otimes}L^\infty(\mathbf{R})) \times_{\bar{\theta}} \mathbf{Z}.\end{aligned}$$

Here we have $L^\infty(Y,\nu)\bar{\otimes}L^\infty(\mathbf{R}) \cong \bigoplus_{n\in\mathbf{Z}} L^\infty(X,\mu)$ as in the proof of Lemma 1.1, and under this isomorphism we may identify $\bar{\theta}$ on $\bar{\mathcal{M}}\bar{\otimes}L^\infty(Y,\nu)\bar{\otimes}L^\infty(\mathbf{R})$ with S

on

$$\bigoplus_{n=-\infty}^{\infty}(\bar{\mathcal{M}}\bar{\otimes}L^{\infty}(X,\mu))).$$

Thus we have

$$\begin{aligned}\tilde{\mathcal{M}}\rtimes_{\sigma}\mathbf{R}&\cong(\bigoplus_{n=-\infty}^{\infty}(\bar{\mathcal{M}}\bar{\otimes}L^{\infty}(X,\mu)))\rtimes_{S}\mathbf{Z}\\&\cong((\bigoplus_{n=-\infty}^{\infty}\bar{\mathcal{M}})\rtimes_{S}\mathbf{Z})\bar{\otimes}L^{\infty}(X,\mu)\\&\cong\mathcal{N}\bar{\otimes}L^{\infty}(X,\mu),\end{aligned}$$

where we set $\mathcal{N}=(\bigoplus_{n=-\infty}^{\infty}\bar{\mathcal{M}})\rtimes_{S}\mathbf{Z}$, which is an injective factor isomorphic to $\bar{\mathcal{M}}$. We define the automorphism $R(\hat{\sigma},Y)$ on $(\bigoplus_{n=-\infty}^{\infty}\bar{\mathcal{M}}\bar{\otimes}L^{\infty}(Y,\nu))\rtimes_{S}\mathbf{Z}\cong\mathcal{N}\bar{\otimes}L^{\infty}(Y,\nu)$ by

$$\begin{aligned}R(\hat{\sigma},Y)&=S\circ\bigoplus_{n=-\infty}^{\infty}\theta\qquad\text{on }\bigoplus_{n=-\infty}^{\infty}\bar{\mathcal{M}}\bar{\otimes}L^{\infty}(Y,\nu)\\R(\hat{\sigma},Y)(V)&=V\qquad\text{for the unitary }V\text{ implementing }S.\end{aligned}$$

This notation is used because this automorphism is a "reduction" of $\hat{\sigma}$ as follows.

Because the action $\hat{\sigma}$ on the center $\mathcal{Z}(\tilde{\mathcal{M}}\rtimes_{\sigma}\mathbf{R})\cong L^{\infty}(X,\mu)$ is the flow built under the ceiling function $k(T^{-1}y)$ over the base (Y,T^{-1}) as seen in the proof of Lemma 1.1 and the remark after it, the action $\hat{\sigma}_t$ for $t\in\mathbf{R}$ gives us an isomorphism of $\mathcal{N}(y,s)\cong\mathcal{N}$, where $y\in Y$, $0\le s<k(T^{-1}y)$, onto $\mathcal{N}(y',s')\cong\mathcal{N}$, with y',s' as

follows.

$$(y', s') = \begin{cases} (T^{-n}y, s + t - k(T^{-1}y) + \cdots - k(T^{-n}y)), \\ \qquad \text{if } k(T^{-1}y) + k(T^{-2}y) + \cdots + k(T^{-n+1}y) \leq s \\ \qquad\qquad < k(T^{-1}y) + k(T^{-2}y) + \cdots + k(T^{-n}y), \qquad \text{for some } n > 0; \\ (T^n y, s + t - k(y) - \cdots - k(T^{n-1}y)) \\ \qquad \text{if } -k(y) - k(Ty) - \cdots - k(T^{n-1}y) \leq s \\ \qquad\qquad < -k(y) - k(Ty) - \cdots - k(T^{n-2}y), \qquad \text{for some } n > 0. \end{cases}$$

And if both s and s' are zero, then this automorphism coincides with the one given by $R(\hat{\sigma}, Y)$. Thus our notation $R(\hat{\sigma}, Y)$ is justified. Note that the modular automorphism group of the dual weight on the crossed product $\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R}$ is the identity, so the crossed product $\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R}$, and hence $\mathcal{N}$, is semifinite, and we have an invariant trace on the crossed product algebra $\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R}$.

We note that the crossed product algebra by $\mathbf{R}$ is properly infinite unless the action is inner for all $t \in \mathbf{R}$. But this case has been excluded by assumption, so the above factor $\mathcal{N}$ is isomorphic to the factor $\mathcal{N}_0$, which appears in the central decomposition of $\mathcal{M} \rtimes_\alpha \mathbf{R}$. Thus we may define the above factor $\mathcal{N}$ as the factor which appears in the central decomposition of $\mathcal{M} \rtimes_\alpha \mathbf{R}$, and this $\mathcal{N}$ is of type I_∞ or II_∞ by the above remark. We also note that $\hat{\sigma}$ on $\mathcal{Z}(\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R})$ is conjugate to $\hat{\alpha}$ on $\mathcal{Z}(\mathcal{M} \rtimes_\alpha \mathbf{R})$.

We regard the automorphism $R(\hat{\sigma}, Y)$ as a groupoid action of $Y \times \mathbf{Z}$ on the semifinite injective factor $\mathcal{N}$ as in §1 of [Sutherland-Takesaki, 14]. We will compare this action $R(\hat{\sigma}, Y)$ with the following *model* actions P and $\bar{P}$.

We define an action P of $Y \times \mathbf{Z}$ on $\mathcal{N}$ as follows. If $\mathcal{N}$ is of type II_∞, then take and fix an action φ of $\mathbf{R}$ on $\mathcal{N}$ such that we have $\mathrm{tr} \circ \varphi_t = e^t \mathrm{tr}$ where tr is the trace on $\mathcal{N}$ and $t \in \mathbf{R}$. If $\mathcal{N}$ is of type I_∞, then we just set $\varphi_t = \mathrm{Id} \in \mathrm{Aut}(\mathcal{N})$ for every $t \in \mathbf{R}$. Then we define P by

$$P(y,n) = \varphi_{-\log m(y,n)}, \qquad \text{for } y \in Y,\ n \in \mathbf{Z},$$

where $m(y,n)$ is the value of Radon-Nikodym derivative of T^{-n} at $y \in Y$. With this P, we also define an action $\bar{P}$ of a groupoid $X \times \mathbf{R}$ on $\mathcal{N}$ as follows. First we define a groupoid homomorphism p of $X \times \mathbf{R}$ to $Y \times \mathbf{Z}$ by

$$p(x,t) = \begin{cases} (p_Y(x), \mathrm{Card}\{s \mid 0 < s \leq t, F_s x \in Y \times 0\}), & \text{if } t \geq 0 \\ (p_Y(x), -\mathrm{Card}\{s \mid t < s \leq 0, F_s x \in Y \times 0\}), & \text{if } t < 0, \end{cases}$$

where the projection p_Y of X onto Y is defined by $p_Y(y,t) = y$. Then we define $\bar{P}$ by

$$\bar{P}(\gamma) = P(p(\gamma)), \qquad \text{where } \gamma \in X \times \mathbf{R}.$$

We can define the action $\tilde{P}$ of $\mathbf{R}$ on $\mathcal{N} \bar{\otimes} L^\infty(X,\mu)$ by

$$(P_t T)(tx) = P(x,t)(T(x)) \qquad \text{for } t \in \mathbf{R}, x \in X, \text{and } T = \int_X^\oplus T(x) d\mu(x).$$

If we need to express the dependence of P, $\bar{P}$ and $\tilde{P}$ on Y, ν, T, k, and $\mathcal{N}$, we use the notations $P(Y,\nu,T,\mathcal{N})$, $\bar{P}(Y,\nu,T,k,\mathcal{N})$, and $\tilde{P}(Y,\nu,T,\mathcal{N})$. With these definitions, we get the following key lemma.

Lemma 1.2. *With the above notation, $\hat{\sigma}$ and $\tilde{P}$ are cocycle conjugate as actions of* $\mathbf{R}$ *on* $\mathcal{M} \rtimes_\sigma \mathbf{R}$.

Proof. In the following proof, we make use of a combination of the usual method of decomposing actions into groupoid actions and integrating them, and a way of reducing a continuous groupoid to an orbitally discrete one.

We write just R for $R(\hat{\sigma}, Y)$. At first, we show two actions R and P of $Y \times \mathbf{Z}$ on $\mathcal{N}$ are cocycle conjugate.

Because we have an invariant trace on $L^\infty(Y, \nu)\bar{\otimes}\mathcal{N}$, the module of R (cf. §1 in [Sutherland-Takesaki, 14]) is the inverse of the Radon-Nikodym derivative for a measure on Y and the transformation T. So this module is equivalent to the module of P modulo coboundaries. Thus by conjugating by an automorphism on $L^\infty(Y, \nu)\bar{\otimes}\mathcal{N}$ if necessary, we may assume that these two modules coincide.

Regard R and P as Borel homomorphisms of our AF measured groupoid $Y \times \mathbf{Z}$ into the Polish group $\mathrm{Aut}(\mathcal{N})$. Then by the above, we have

$$R(\gamma) = P(\gamma) \mod \overline{\mathrm{Int}}(\mathcal{N}), \qquad \text{for } \gamma \in Y \times \mathbf{Z}.$$

So we can apply the cohomology theorem of Bures-Connes-Krieger-Sutherland [Sutherland, 13]. (Also see [Sutherland-Takesaki, 14, appendix].) Then we get Borel maps Q of $Y \times \mathbf{Z}$ to $\mathrm{Int}(\mathcal{N})$ and f of Y to $\overline{\mathrm{Int}}(\mathcal{N})$ such that we have

$$R(\gamma) = Q(\gamma) f(r(\gamma)) P(\gamma) f(s(\gamma))^{-1}, \qquad \gamma \in Y \times \mathbf{Z}.$$

Because the map f gives us an automorphism on $L^\infty(Y,\nu)\bar{\otimes}\mathcal{N}$ and the map Q gives us an unitary operator in $L^\infty(Y,\nu)\bar{\otimes}\mathcal{N}$, we know that the actions R and P of $Y\times\mathbf{Z}$ on $\mathcal{N}$ are cocycle conjugate.

We write this in the following way. We have a Borel function τ of Y to $\mathrm{Aut}(\mathcal{N})$, and a unitary cocycle u for $R(\hat{\sigma},Y)$ such that we have

$$\mathrm{Ad}(u(y,n))R(y,n) = \tau(T^{-n}(y))^{-1}P(y,n)\tau(y).$$

We use the map p_Y of X to Y defined by $p_Y(y,t) = y$ as above. We also use the groupoid homomorphism p of $X\times\mathbf{R}$ to $Y\times\mathbf{Z}$ as above, and a map q of $X\times\mathbf{R}$ to $X\times\mathbf{R}$ defined by the following:

$$q(\gamma\colon x_0 \to x_1) = (\gamma'\colon x_0 \to p_Y(x_1)).$$

We have $s(p(\gamma)) = p_Y(s(\gamma))$ and $r(p(\gamma)) = p_Y(r(\gamma))$. Next we want to get a suitable unitary cocycle for $\hat{\sigma}$, which is now regarded as an action of $X\times\mathbf{R}$ as above. We can make many choices of a unitary cocycle, but we use the simplest one. Taking the action $\hat{\sigma}$ into account, we define a map v of $X\times\mathbf{R}$ to $\mathcal{U}(\mathcal{N})$ by

$$v(\gamma) = \hat{\sigma}(\gamma q(\gamma)^{-1})(u(p(\gamma))).$$

Note v is a unitary cocycle for $\hat{\sigma}$, since for elements γ_1,γ_2 with product $\gamma_1\gamma_2$ defined in the groupoid $X \times \mathbf{R}$, we have $\gamma_1 q(\gamma_1)^{-1} = \gamma_1\gamma_2 q(\gamma_1\gamma_2)^{-1}$, so

$$\begin{aligned}
&v(\gamma_1\gamma_2)\\
&= \hat{\sigma}(\gamma_1\gamma_2 q(\gamma_1\gamma_2)^{-1})(u(p(\gamma_1)p(\gamma_2)))\\
&= \hat{\sigma}(\gamma_1 q(\gamma_1)^{-1})(u(p(\gamma_1))\hat{\sigma}(p(\gamma_1))(u(p(\gamma_2))))\\
&= \hat{\sigma}(\gamma_1 q(\gamma_1)^{-1})(u(p(\gamma_1)))\hat{\sigma}(\gamma_1 q(\gamma_1)^{-1} p(\gamma_1))(u(p(\gamma_2)))\\
&= \hat{\sigma}(\gamma_1 q(\gamma_1)^{-1})(u(p(\gamma_1)))\hat{\sigma}(\gamma_1\gamma_2 q(\gamma_2)^{-1})(u(p(\gamma_2)))\\
&= v(\gamma_1)\hat{\sigma}(\gamma_1)(v(\gamma_2)).
\end{aligned}$$

We define the Borel map Q of X to $\mathrm{Aut}(\mathcal{N})$ by

$$Q(x) = \bar{P}(\gamma)\tau(s(\gamma))\hat{\sigma}(\gamma)^{-1}\mathrm{Ad}(v(\gamma)^*),$$

where $\gamma \in X \times \mathbf{R}$ is given by $\gamma\colon (p_Y(x), 0) \to x$. We claim

$$\mathrm{Ad}(v(\gamma))\hat{\sigma}(\gamma) = Q(r(\gamma))^{-1}\bar{P}(\gamma)Q(s(\gamma)).$$

To see this, for an arbitrary $\gamma \in X \times \mathbf{R}$, we define $\gamma_1 = \gamma q(\gamma)^{-1}$, and $\gamma_0 =$

$q(\gamma)^{-1}p(\gamma)$, then we have $\gamma = \gamma_1 p(\gamma)\gamma_0^{-1}$ so that

$$\begin{aligned}
&\mathrm{Ad}(v(\gamma))\hat{\sigma}(\gamma) \\
&= \mathrm{Ad}(v(\gamma_1))\mathrm{Ad}(\hat{\sigma}(\gamma_1)(v(p(\gamma))))\mathrm{Ad}(\hat{\sigma}(\gamma_1 p(\gamma))(v(\gamma_0^{-1})))\hat{\sigma}(\gamma_1)\hat{\sigma}(p(\gamma))\hat{\sigma}(\gamma_0)^{-1} \\
&= \mathrm{Ad}(v(\gamma_1))\hat{\sigma}(\gamma_1)\mathrm{Ad}(v(p(\gamma)))\hat{\sigma}(p(\gamma))\hat{\sigma}(\gamma_0)^{-1}\mathrm{Ad}(v(\gamma_0)^*) \\
&= \mathrm{Ad}(v(\gamma_1))\hat{\sigma}(\gamma_1)\tau(r(p(\gamma)))^{-1}\bar{P}(p(\gamma))\tau(s(p(\gamma)))\hat{\sigma}(\gamma_0)^{-1}\mathrm{Ad}(v(\gamma_0)^*) \\
&= \mathrm{Ad}(v(\gamma_1))\hat{\sigma}(\gamma_1)\tau(s(\gamma_1))^{-1}\bar{P}(\gamma_1)^{-1}\bar{P}(\gamma)\bar{P}(\gamma_0)\tau(s(\gamma_0))\hat{\sigma}(\gamma_0)^{-1}\mathrm{Ad}(v(\gamma_0)^*) \\
&= Q(r(\gamma))^{-1}\bar{P}(\gamma)Q(s(\gamma)).
\end{aligned}$$

So $\hat{\sigma}$ and $\bar{P}$ are cocycle conjugate as actions of $X \times \mathbf{R}$. We define an automorphism $\bar{Q}$ on $\int_X^{\oplus} \mathcal{N}(x)\, d\mu(x) = L^\infty(X,\mu)\bar{\otimes}\mathcal{N} = \tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R}$ by

$$\bar{Q} = \int_X^{\oplus} Q(x)\, d\mu(x).$$

We also define a map $\bar{v}$ of $\mathbf{R}$ to $\mathcal{U}(\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R})$ by

$$\bar{v}(t)\left(\int_X^{\oplus} \xi(x)\, d\mu(x)\right) = \int_X^{\oplus} v(t, F_{-t}x)\xi(x)\, d\mu(x).$$

Then $\bar{v}$ is strongly continuous in t. Indeed, for any t and positive ε, if we define

$$\begin{aligned}
Y_{t,\varepsilon} &= \bigcup_{|s|\le\varepsilon} F_{t+s}(Y \times 0), \\
Y_t &= F_t(Y \times 0),
\end{aligned}$$

then for a fixed t, we have

$$\begin{aligned}
&\sup_{|t-s|<\varepsilon} \left\| (\bar{v}(t) - \bar{v}(s)) \left(\int_X^{\oplus} \xi(x)\, d\mu(x) \right) \right\| \\
&\leq 2 \left(\int_{Y_{t,\varepsilon}}^{\oplus} \|\xi(x)\|^2 \, d\mu(x) \right)^{1/2} \\
&\to 2 \left(\int_{Y_t}^{\oplus} \|\xi(x)\|^2 \, d\mu(x) \right)^{1/2} = 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

Thus $\bar{v}$ is a unitary cocycle for $\hat{\sigma}$ as an action of $\mathbf{R}$, and we have $\mathrm{Ad}(\bar{v}(t))\hat{\sigma}_t = Q^{-1}\tilde{P}_t Q$, so that $\hat{\sigma}$ and $\tilde{P}$ are cocycle conjugate as actions of $\mathbf{R}$ on $L^\infty(Y,\nu)\bar{\otimes}\mathcal{N} \cong \tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R}$. Q.E.D.

The following lemma shows that the model action is canonical up to conjugacy if the flow and $\mathcal{N}$ are given.

Lemma 1.3. *Let β be another action of $\mathbf{R}$ on $\mathcal{M}$ with the same properties as α. We construct ρ for β as we constructed σ for α. We use the notations $X(\sigma)$, $F(\sigma)$, $Y(\sigma)$, $\nu(\sigma)$, $T(\sigma)$, $k(\sigma)$, $\mathcal{N}(\sigma)$ and $X(\rho)$, $F(\rho)$, $Y(\rho)$, $\nu(\rho)$, $T(\rho)$, $k(\rho)$, $\mathcal{N}(\rho)$ for distinguishing these for σ and ρ. If $N(\sigma) \cong N(\rho)$ and the flows $F(\sigma)$ on $\mathcal{Z}(\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R})$ and $F(\rho)$ on $\mathcal{Z}(\tilde{\mathcal{M}} \rtimes_\rho \mathbf{R})$ are conjugate, then we know the actions $\tilde{P}(Y(\sigma),\nu(\sigma),T(\sigma),k(\sigma),\mathcal{N}(\sigma))$ and $\tilde{P}(Y(\rho),\nu(\rho),T(\rho),k(\rho),\mathcal{N}(\rho))$ are conjugate as actions of $\mathbf{R}$ on the crossed product algebra $\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R} \cong \tilde{\mathcal{M}} \rtimes_\rho \mathbf{R}$.*

Proof. By assumption, the flows built from the base $(Y(\sigma),T(\sigma))$ under $k(\sigma)$ and on $(Y(\rho),T(\rho))$ under $k(\rho)$ are conjugate. By Proposition 2.4 in [Katok, 9] , there are thus subsets $Y'(\sigma)$ of $Y(\sigma)$, $Y'(\rho)$ of $Y(\rho)$ and an isomorphism $\varphi\colon (Y'(\sigma),\nu(\sigma)) \to (Y'(\rho),\nu(\rho))$ such that $\varphi \circ T'(\sigma) = T'(\rho) \circ \varphi$ and $k'(\rho) \circ \varphi = k'(\sigma)$, where $T'(\sigma)$

is the transformation induced on $Y'(\sigma)$, $k'(\sigma)(y) = \sum_{j=0}^{n-1} k(\sigma)(T(\sigma)^j(y))$ where $n = \sup\{m > 0 \mid T(\sigma)^j(y) \notin Y'(\sigma)\ for\ 0 \leq j \leq m\}$, and $T'(\rho)$, $k'(\rho)$ are defined similarly.

Then by definition, $P(Y'(\sigma), \nu(\sigma), T'(\sigma), \mathcal{N}(\sigma))$ and $P(Y'(\rho), \nu(\rho), T'(\rho), \mathcal{N}(\rho))$ are conjugate. Because these can be regarded as reductions of $\bar{P}(Y(\sigma), \nu(\sigma), T(\sigma), k(\sigma), N(\sigma))$ and $\bar{P}(Y(\rho), \nu(\rho), T(\rho), k(\rho), N(\rho))$ respectively, we conclude $\tilde{P}(Y(\sigma), \nu(\sigma), T(\sigma), k(\sigma), N(\sigma))$ and $\tilde{P}(Y(\rho), \nu(\rho), T(\rho), k(\rho), N(\rho))$ are conjugate by a similar argument to the latter part of the proof of Lemma 1.2. Q.E.D.

Theorem 1.4. *Let $\mathcal{M}$ be a semifinite injective von Neumann algebra, and α and β centrally ergodic actions of $\mathbf{R}$ on $\mathcal{M}$ with $\Gamma(\alpha), \Gamma(\beta) = 0$. We assume $\alpha^{-1}(Int(\mathcal{M}))$, $\beta^{-1}(Int(\mathcal{M})) \neq \mathbf{R}$. We also suppose that each of α and β admits an invariant trace. Then α and β are stably conjugate if and only if $\mathcal{N}(\alpha) \cong \mathcal{N}(\beta)$ and the flows $\hat{\alpha}$ on $\mathcal{Z}(\mathcal{M} \rtimes_\alpha \mathbf{R})$ and $\hat{\beta}$ on $\mathcal{Z}(\mathcal{M} \rtimes_\beta \mathbf{R})$ are conjugate.*

Proof. Suppose α and β are stably conjugate. Then we clearly get the above two conditions.

Conversely assume the two conditions are satisfied. Considering σ and ρ for α and β as above, we know that $\mathcal{N}(\sigma) \cong \mathcal{N}(\rho)$ and that the flows $\hat{\sigma}$ on $\mathcal{Z}(\tilde{\mathcal{M}} \rtimes_\sigma \mathbf{R})$ and $\hat{\rho}$ on $\mathcal{Z}(\tilde{\mathcal{M}} \rtimes_\rho \mathbf{R})$ are conjugate. Then by Lemma 1.2 and Lemma 1.3, we know that $\hat{\sigma}$ and $\hat{\rho}$ are cocycle conjugate. Hence, σ and ρ are stably conjugate, and thus they are cocycle conjugate because $\tilde{\mathcal{M}}^\sigma$ are $\tilde{\mathcal{M}}^\rho$ are properly infinite by our definition of $\tilde{\mathcal{M}}$. (See Theorem 1 in [Jones-Takesaki, 7]). Then we have that the second dual actions of α and β are cocycle conjugate by our definition of σ and ρ, so that α and β are now stably conjugate. Q.E.D.

In general, $\Gamma(\alpha)$ must be equal to 0 or isomorphic to $\mathbf{Z}$ because it is a closed subgroup of $\mathbf{R}$ and we assume $\Gamma(\alpha) \neq \mathbf{R}$. Note that if α and β are stably conjugate, then $\Gamma(\alpha)$ and $\Gamma(\beta)$ are equal. Next, we study the case where the Connes spectrum is isomorphic to $\mathbf{Z}$.

Let α and $\mathcal{M}$ be as above. By changing the scale if necessary, we may assume that $\Gamma(\alpha)$ is equal to $\mathbf{Z}$. Because $\hat{\mathbf{R}}/\Gamma(\alpha) \cong \mathbf{T}$ is compact, by Corollaire 2.3.13 in [Connes, 1], we have $\alpha|_{\mathbf{Z}} = 1$ by changing α within its cocycle conjugacy class if necessary. Then we have

$$\mathcal{M} \rtimes_\alpha \mathbf{R} \cong \mathcal{M} \rtimes_{\alpha^0} \mathbf{T} \rtimes_i \mathbf{Z} \cong (\mathcal{M} \rtimes_{\alpha^0} \mathbf{T}) \bar{\otimes} L^\infty(\mathbf{T}, \mu),$$

where we use the notation α^0 for the action of $\mathbf{T}$ induced from α, i is the trivial action of $\mathbf{Z}$, and μ is Lebesgue measure on $\mathbf{T}$. It is easily seen that the Connes spectrum $\Gamma(\alpha^0)$ is also equal to $\mathbf{Z} = \hat{\mathbf{T}}$. Thus the crossed product $\mathcal{M} \rtimes_{\alpha^0} \mathbf{T}$ in the above formula is a (semifinite injective) factor. And we note that the translation on $\mathbf{T}$ corresponds to the flow on X in the above case, and this factor corresponds to the factor $\mathcal{N}(\alpha)$ in the discussion of the case $\Gamma(\alpha) = 0$. The action which corresponds to $R(\hat{\sigma}, Y)$ in this case is just $\hat{\alpha}^0$ on $\mathcal{M} \rtimes_{\alpha^0} \mathbf{T}$. If this $\hat{\alpha}^0$ does not preserve the trace, the second crossed product $\mathcal{M} \rtimes_{\alpha^0} \mathbf{T} \rtimes_{\hat{\alpha}^0} \mathbf{Z}$ must be a type III von Neumann algebra, which is a contradiction. So $\hat{\alpha}^0$ preserves the trace. Thus we get the following result. Note that $\mathcal{N}(\alpha)$ here is defined as the factor which appears in the central decomposition of $\mathcal{M} \rtimes_\alpha \mathbf{R}$.

Theorem 1.5. *Let $\mathcal{M}$, α and β as above, and assume $\Gamma(\alpha)$ and $\Gamma(\beta)$ are isomorphic to $\mathbf{Z}$. Then α and β are stably conjugate if and only if $\Gamma(\alpha)$ and $\Gamma(\beta)$ are equal, $\mathcal{N}(\alpha)$ and $\mathcal{N}(\beta)$ are isomorphic, and the characteristic invariants $\chi(\hat{\alpha}^0)$ and $\chi(\hat{\beta}^0)$ are equal.*

Proof. By the above argument, these three conditions are clearly necessary.

If these three conditions are satisfied, we know that $\hat{\alpha}$ and $\hat{\beta}$ are cocycle conjugate, so we get the stable conjugacy of α and β by Theorem 1.(a) in [Jones-Takesaki, 7]. Q.E.D.

By duality, we also get the following corollary.

Corollary 1.6. *Let $\mathcal{M}$ be a semifinite injective von Neumann algebra with a non-trivial center, and α and β centrally ergodic actions of $\mathbf{R}$ on $\mathcal{M}$. We suppose that α and β both admit an invariant trace, $\hat{\alpha}^{-1}(Int(\mathcal{M}\rtimes_\alpha\mathbf{R}))$, $\hat{\beta}^{-1}(Int(\mathcal{M}\rtimes_\beta\mathbf{R}))\neq\hat{\mathbf{R}}$, and the flows α and β on $\mathcal{Z}(\mathcal{M})$ are aperiodic. Then α and β are stably conjugate if and only if the flows $(\alpha,\mathcal{Z}(\mathcal{M}))$ and $(\beta,\mathcal{Z}(\mathcal{M}))$ are conjugate and $\mathcal{M}\rtimes_\alpha\mathbf{R}\cong\mathcal{M}\rtimes_\beta\mathbf{R}$.*

Proof. The conditions are clearly necessary. Assume the conditions are satisfied. By Takesaki duality, we have $\mathcal{N}(\hat{\alpha})\cong\mathcal{N}(\hat{\beta})$, and the flows of the second duals of α and β on the center are conjugate by assumption. So we know $\hat{\alpha}$ and $\hat{\beta}$ are stably conjugate because $\Gamma(\hat{\alpha})$, $\Gamma(\hat{\beta})\neq\mathbf{R}$ by the assumption $\mathcal{Z}(\mathcal{M})\neq\mathbf{C}$. Moreover, $\alpha^{-1}(\mathrm{Int}(\mathcal{M}))\neq\mathbf{R}$, so the crossed product $\mathcal{M}\rtimes_\alpha\mathbf{R}$ is properly infinite. Thus $\hat{\alpha}$ and $\hat{\beta}$ are cocycle conjugate, and hence α and β are stably conjugate again by Theorem 1.(a) in [Jones-Takesaki, 7]. Q.E.D.

§2 Factor cases

Next we make further study for factor cases in this section, and prove a limitation on the type of $\mathcal{N}$ when we have $\Gamma(\alpha) \cong \mathbf{Z}$.

First, we show an assumption in Theorem 1.4 can be dropped for factors.

Proposition 2.1. *Let $\mathcal{M}$ be a semifinite injective factor and α an action of $\mathbf{R}$ on $\mathcal{M}$ with $\Gamma(\alpha) \neq \hat{\mathbf{R}}$. Then any trace on $\mathcal{M}$ is invariant under α.*

Proof. If $\mathcal{M}$ is finite, the theorem is trivial because there exists only one normalized trace. If $\mathcal{M}$ is a type I_∞ factor, then all the automorphisms on $\mathcal{M}$ are inner, and a trace is invariant under inner automorphisms. Finally assume that $\mathcal{M}$ is isomorphic to $\mathcal{R}_{0,1}$, and take any trace τ on $\mathcal{R}_{0,1}$. Then by the uniqueness of the trace up to constant, we have a continuous function f on $\mathbf{R}$ with $\tau \circ \alpha_s = f(s)\tau$. Because we have $f(s+t) = f(s)f(t)$, there exists a constant $k \in \mathbf{R}$ such that $f(s) = e^{ks}$. If this constant k is not equal to zero, then by [Takesaki, 15] we have that $\mathcal{R}_{0,1} \rtimes_\alpha \mathbf{R}$ is a factor of type III_1, which is impossible because we assume $\Gamma(\alpha) \neq \hat{\mathbf{R}}$. Thus we have $k = 0$, and τ is thus invariant under α for this case, too. Q.E.D.

Because actions on factors are trivially centrally ergodic, we can apply Theorem 1.4 to this case. Moreover, the characteristic invariants of the dual actions in this case are trivial in fact. So we get the following simplification.

Theorem 2.2. *Let $\mathcal{M}$ be a semifinite injective factor, and α, β actions of $\mathbf{R}$ on $\mathcal{M}$ with $\Gamma(\alpha), \Gamma(\beta) \cong \mathbf{Z}$. Then α and β are stably conjugate if and only if we have $\Gamma(\alpha) = \Gamma(\beta)$ and $\mathcal{N}(\alpha) \cong \mathcal{N}(\beta)$.*

Proof. Because of Theorem 1.5, we have to show that the characteristic invariant of $R(\hat{\alpha}^0)$ is trivial. By the remark preceding Theorem 1.5, we may assume that

α has the period 1, and $\mathcal{M} \rtimes_{\alpha^0} \mathbf{T}$ is a factor. We consider the dual action $\hat{\alpha}^0$ on this factor. Suppose that there exists a non-zero integer p such that $(\hat{\alpha}^0)^p$ is inner. Take a unitary element u in $\mathcal{M} \rtimes_{\alpha^0} \mathbf{T}$ with $\mathrm{Ad}(u) = (\hat{\alpha}^0)^p$. We have a complex number γ with $(\hat{\alpha}^0)^p(u) = \gamma u$ and $\gamma^p = 1$. Then we have $(\hat{\alpha}^0)^{p^2} = \mathrm{Ad}(u^p)$, and $\hat{\alpha}^0(u^p) = \gamma^p u^p = u^p$. That is, we have $u^p \in (\mathcal{M} \rtimes_{\alpha^0} \mathbf{T})^{\hat{\alpha}^0}$. Thus we can take a unitary element v in $(\mathcal{M} \rtimes_{\alpha^0} \mathbf{T})^{\hat{\alpha}^0}$ with $v^{p^2} = u^{*p}$. If we let $\theta = \mathrm{Ad}(v)\hat{\alpha}^0$, we have

$$\mathcal{M} \rtimes_{\alpha^0} \mathbf{T} \rtimes_{\hat{\alpha}^0} \mathbf{Z} \cong \mathcal{M} \rtimes_{\alpha^0} \mathbf{T} \rtimes_{\theta} \mathbf{Z}.$$

Here the left hand side is isomorphic to the factor $\mathcal{M}\bar{\otimes}\mathcal{L}(\mathcal{H})$ by Takesaki duality, but the right hand side is not a factor because we have $\theta^{p^2} = \mathrm{Ad}(u^{*p})(\hat{\alpha}^0)^{p^2} = 1$ while $p^2 > 0$. Thus we have a contradiction. Q.E.D.

We now decide what type of $\mathcal{N}$ is possible for a given $\mathcal{M}$ of type II. In this paper after this point, $\mathcal{M}$ is a semifinite injective factor, α is an action of $\mathbf{R}$ on $\mathcal{M}$ with $\Gamma(\alpha) \neq \mathbf{R}$, and $\mathcal{N}$ is a semifinite injective factor given by the decomposition $\mathcal{M} \rtimes_\alpha \mathbf{R} \cong \mathcal{N}\bar{\otimes}L^\infty(X, \mu)$. By Takesaki duality, we get further restrictions.

Proposition 2.3. *If $\mathcal{M}$ is of type II and $\Gamma(\alpha)$ is isomorphic to $\mathbf{Z}$, then $\mathcal{N}$ cannot be of type I.*

Proof. Suppose that $\mathcal{N}$ is of type I. Then by the remark preceding Theorem 1.5, we may assume that there exists an action α of $\mathbf{T}$ on $\mathcal{M}$ such that $\mathcal{M} \rtimes_\alpha \mathbf{T}$ is a type I factor. Then by Takesaki duality, we have

$$\mathcal{M} \rtimes_\alpha \mathbf{T} \rtimes_{\hat{\alpha}} \mathbf{Z} \cong \mathcal{M}\bar{\otimes}\mathcal{L}(\mathcal{H}) \cong \mathcal{R}_{0,1}.$$

But all the automorphisms on a type I factor $\mathcal{M} \rtimes_\alpha \mathbf{T}$ is inner, hence we have $\mathcal{M} \rtimes_\alpha \mathbf{T} \rtimes_{\hat{\alpha}} \mathbf{Z}$ isomorphic to $(\mathcal{M} \rtimes_\alpha \mathbf{T})\bar{\otimes}L^\infty(\mathbf{T})$, which is also of type I. Thus $\mathcal{N}$ cannot be of type I. Q.E.D.

As we will see in the next section all possible cases except the above excluded case actually occur.

§3 Examples

It is not easy in general to construct examples of actions of $\mathbf{R}$, so in this section, we construct examples which show that the combinations $\mathcal{M}$ and $\mathcal{N}$ which are not excluded in section 2 actually happen. We make frequent use of representation of type II factors as crossed products and dual actions on them.

The following makes use of the fact that the factor $\mathcal{R}$ can be expressed as the "irrational rotation algebra."

Examples 3.1. Take an irrational number θ and we define an automorphism σ on $L^\infty(\mathbf{T})$ by

$$\sigma(f)(t) = f(e^{-2\pi i\theta}t), \qquad t \in \mathbf{T}, f \in L^\infty(\mathbf{T}).$$

Then it is well known that we have $L^\infty(\mathbf{T}) \rtimes_\sigma \mathbf{Z} \cong \mathcal{R}$. Then the dual action $\hat{\sigma}$ is an action of $\mathbf{T}$. Define $\alpha_t = \hat{\sigma}_{exp(2\pi it)}\ for\ t \in \mathbf{R}$, and note that

$$\mathcal{R} \rtimes_\alpha \mathbf{R} \cong L^\infty(\mathbf{T})\bar{\otimes}\mathcal{L}(l^2(\mathbf{Z}))\bar{\otimes}L^\infty(\mathbf{T}).$$

Thus $\mathcal{N}$ is $\mathcal{L}(l^2(\mathbf{Z}))$, which is of type I. Because the right hand side of the above formula is not a factor, $\Gamma(\alpha)$ is not equal to $\mathbf{R}$. By Proposition 2.3, the Connes spectrum $\Gamma(\alpha)$ is not isomorphic to $\mathbf{Z}$, either. Thus we have $\Gamma(\alpha) = 0$. By considering $\alpha \otimes i$ on $\mathcal{R}\bar{\otimes}\mathcal{L}(\mathcal{H}) \cong \mathcal{R}_{0,1}$, we also get an example for $\mathcal{R}_{0,1}$ instead of $\mathcal{R}$.

The next is similar to Example 3.1.

Example 3.2. Let σ be as in Example 3.1, and let i be the trivial action of $\mathbf{Z}$ on $\mathcal{R}$. Define an action τ of $\mathbf{Z}$ on $L^\infty(\mathbf{T})\bar{\otimes}\mathcal{R}$ by $\tau = \sigma \otimes i$. Because τ is centrally ergodic and outer, the crossed product is a (semifinite injective) factor. Because $\mathcal{R}$ is finite and $\mathbf{Z}$ is discrete, this crossed product is finite, and being infinite dimensional, is isomorphic to $\mathcal{R}$. Let α be an extension to $\mathbf{R}$ of the dual action $\hat{\tau}$ as in Example 2.1. Then we have

$$\mathcal{R} \rtimes_\alpha \mathbf{R} \cong L^\infty(\mathbf{T})\bar{\otimes}\mathcal{R}\bar{\otimes}\mathcal{L}(l^2(\mathbf{Z}))\bar{\otimes}L^\infty(\mathbf{T}).$$

Thus $\mathcal{N}$ in this case is $\mathcal{R}\bar{\otimes}\mathcal{L}(l^2(\mathbf{Z})) \cong \mathcal{R}_{0,1}$. For $t \in \mathbf{R}$, the flow given by $\hat{\alpha}$ on the second copy of $L^\infty(\mathbf{T})$ in the above formula is just translation by $\mathbf{R}$ on $\mathbf{T} \cong \mathbf{R}/\mathbf{Z}$. Thus we have $\Gamma(\alpha) \subset \mathbf{Z}$. For an integer t, the action $\hat{\alpha}_t$ on the first $L^\infty(\mathbf{T})$ is the rotation by $t\theta$. Thus we have $\Gamma(\alpha) = \{0\}$.

In the next example, we express the factor $\mathcal{R}$ as a crossed product of $\mathcal{R}$ by $\mathbf{Z}$.

Example 3.3. Take a free action σ of $\mathbf{Z}$ on the hyperfinite II$_1$ factor $\mathcal{R}$. (For instance, regard $\mathcal{R}$ as the infinite tensor product of $M_2(\mathbf{C})$ and define σ by the infinite tensor product of $\mathrm{Ad}\begin{pmatrix} e^{\sqrt{2}\pi i} & 0 \\ 0 & e^{-\sqrt{2}\pi i}\end{pmatrix}$.) Then the crossed product

$\mathcal{R} \rtimes_\sigma \mathbf{Z}$ is a factor because σ is free; it is also finite because $\mathcal{R}$ is finite and $\mathbf{Z}$ is discrete, and being infinite dimensional, is isomorphic to $\mathcal{R}$. By Takesaki duality, the crossed product $\mathcal{R} \rtimes_{\hat{\sigma}} \mathbf{T}$ is isomorphic to $\mathcal{R}\bar{\otimes}\mathcal{L}(l^2(\mathbf{Z})) \cong \mathcal{R}_{0,1}$. We define α to be the extension of $\hat{\sigma}$ to $\mathbf{R}$ as in Example 2.1. Then we have $\mathcal{R} \rtimes_\alpha \mathbf{R} \cong \mathcal{R}_{0,1}\bar{\otimes}L^\infty(\mathbf{T})$. For $t \in \mathbf{R}$, the flow given by $\hat{\alpha}$ on $L^\infty(\mathbf{T})$ is just translation by $\mathbf{R}$ on $\mathbf{T} \cong \mathbf{R}/\mathbf{Z}$. Thus we have $\Gamma(\alpha) = \mathbf{Z}$ in this case. For $\mathcal{M} = \mathcal{R}\bar{\otimes}\mathcal{L}(\mathcal{H}) \cong \mathcal{R}_{0,1}$, we can use $\alpha \otimes i$ as a new α. Then $\mathcal{N}$ in this case is $\mathcal{R}_{0,1}$ again, and $\Gamma(\alpha)$ is equal to $\mathbf{Z}$.

In the next example, we express the factor $\mathcal{R}_{0,1}$ using the group measure space construction with the group $\mathbf{R}$.

Example 3.4. We choose $L^\infty(X,\mu)$ and an action σ of $\mathbf{R}$ on $L^\infty(X,\mu)$ such that μ is invariant under the action and σ is ergodic and σ_t is not the identity for non-zero t. (For instance, take $X = \mathbf{T} \times [0,1)$ and μ to be the product of Lebesgue measures. Taking an irrational θ, define the flow F_t on X by

$$F_t(x,y) = (e^{in\theta}x, y+t-n), \qquad \text{for } t \in \mathbf{R}, x \in \mathbf{T}, y \in [0,1)\,,$$

where we set $n = [y+t]$. Define σ by this flow F_t.) Because this is a measure preserving action of a continuous group, we have that the crossed product $L^\infty(X,\mu) \rtimes_\sigma \mathbf{R}$ is isomorphic to $\mathcal{R}_{0,1}$. We define $\alpha = \hat{\sigma}$. Then by Takesaki duality we have

$$\mathcal{R}_{0,1} \rtimes_\alpha \mathbf{R} \cong L^\infty(X,\mu)\bar{\otimes}\mathcal{L}(L^2(\mathbf{R})).$$

The flow given by $\hat{\alpha}$ on $L^\infty(X,\mu)$ in the above formula is just the original F_t. Thus we have $\Gamma(\alpha) = 0$, and $\mathcal{N}$ in this case is $\mathcal{L}(L^2(\mathbf{R}))$ which is of type I_∞.

Summing up, we can determine which combinations of $\mathcal{M}$ and $\mathcal{N}$ are possible as follows.

Theorem 3.5. *In the context of Theorem 1.4 and Theorem 2.2, we have the following.*

(1) *If* $\mathcal{M} \cong \mathcal{R}$ *and* $\Gamma(\alpha) = 0$, *then* $\mathcal{N}$ *is isomorphic to* $\mathcal{L}(\mathcal{H})$ *or* $\mathcal{R}_{0,1}$.

(2) *If* $\mathcal{M} \cong \mathcal{R}$ *and* $\Gamma(\alpha) \cong \mathbf{Z}$, *then* $\mathcal{N}$ *is isomorphic to* $\mathcal{R}_{0,1}$.

(3) *If* $\mathcal{M} \cong \mathcal{R}_{0,1}$ *and* $\Gamma(\alpha) = 0$, *then* $\mathcal{N}$ *is isomorphic to* $\mathcal{L}(\mathcal{H})$ *or* $\mathcal{R}_{0,1}$.

(4) *If* $\mathcal{M} \cong \mathcal{R}_{0,1}$ *and* $\Gamma(\alpha) \cong \mathbf{Z}$, *then* $\mathcal{N}$ *is isomorphic to* $\mathcal{R}_{0,1}$.

Proof. The theorem is just a combination of the above results as follows. In general, $\mathcal{N}$ cannot be finite by [Katayama, 10] because we assume $\alpha^{-1}(\mathrm{Int}(\mathcal{M})) \neq \mathbf{R}$. For case (1), the factor $\mathcal{N}$ can be of I_∞, and II_∞ by Examples 3.1 and 3.2 respectively. For case (2), $\mathcal{N}$ cannot be of type I_∞ by Proposition 2.3. The factor $\mathcal{N}$ can be of type II_∞ by Example 3.3. For case (3), $\mathcal{N}$ can be of type I_∞ and II_∞ by Examples 3.4 and 3.1 respectively. Finally for case (4), $\mathcal{N}$ cannot be of type I_∞ by Proposition 2.3. The factor $\mathcal{N}$ can be of type II_∞ by Example 3.3. Q.E.D.

Acknowledgement

The author expresses his sincere thanks to Prof. M. Takesaki for numerous suggestions and successive encouragement, to Prof. C. Sutherland for pointing out a gap in a proof in the first draft, and to the referee for a great deal of remarks for improving explanation in this paper. He also expresses thanks for the support of travel fellowship of the Educational Project for Japanese Mathematical Scientists.

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