Classification of symmetry protected topological phases in quantum spin systems

Yoshiko Ogata

Graduate School of Mathematical Sciences, The University of Tokyo

17/12/2020

Symmetry Protected Topological (SPT) phases

We consider the set of Hamiltonians satisfying the following.

(i) a unique gapped ground state,

(ii) satisfy a given symmetry β,

(iii) can be smoothly deformed into a fixed reference on-site Hamiltonian without a phase transition. (Excludes long- range entanglement.)

We would like to classify them with the criterion :

> Two Hamiltonians are equivalent if they can be smoothly deformed into each other without a phase transition, preserving the symmetry.

The resulting phases are the SPT-phases.

Symmetry Protected Topological (SPT) phases

This talk is about

classification of SPT-phases
in $\nu = 1$ and $\nu = 2$-dimensional quantum spin systems,
with on-site finite group symmetry.

Symmetry Protected Topological (SPT) phases

This talk is about

classification of SPT-phases
in $\nu = 1$ and $\nu = 2$-dimensional quantum spin systems,
with on-site finite group symmetry.

We consider this problem in the operator algebraic framework of quantum statistical mechanics. Namely, instead of considering finite systems and taking the thermodynamic limit, we start from infinite systems.
The reason for that is

the invariant of the classification is most naturally defined in infinite systems.

Cf. Fredholm index is 0 for any operators on a finite Hilbert space.

Cf. Fredholm index

We consider this problem in the operator algebraic framework of quantum statistical mechanics. Namely, instead of considering finite systems and taking the thermodynamic limit, we start from infinite systems. The reason for that is the invariant of the classification is most naturally defined in infinite systems.

Fredholm index of an operator T (with $\dim\ker T, \dim\ker T^* < \infty$ and $T\mathcal{H}$ closed) on a Hilbert space $\mathcal{H}$ is defined by

$$\mathrm{ind}\, T := \dim\ker T - \dim\ker T^*.$$

This is always 0 if $\mathcal{H}$ is of finite dimensional.
It is not the case if $\dim\mathcal{H} = \infty$.
For example, on $\mathcal{H} = l^2(\mathbb{N})$, the unilateral shift

$$S\left(\xi_1, \xi_2, \xi_3, \ldots\right) := \left(0, \xi_1, \xi_2, \xi_3, \ldots\right), \quad \left(\xi_1, \xi_2, \xi_3, \ldots\right) \in l^2(\mathbb{N}).$$

has $\ker S = \{0\}$, $\ker S^* = (1, 0, 0, 0, \ldots)$ and

$$\mathrm{ind}\, S = 0 - 1 = -1.$$

An analogous thing happens to our index.

Quantum spin system

Let $d \in \mathbb{N}$ be fixed.Let $\nu \in \mathbb{N}$ be a spacial dimension.
A ν-dimensional quantum spin system is the C^*-algebra

$$\mathcal{A} := \bigotimes_{\mathbb{Z}^\nu} \mathrm{M}_d, \quad \mathrm{M}_d : \text{ matrix algebra of size } d.$$

For each $\Gamma \subset \mathbb{Z}^\nu$, $\mathcal{A}_\Gamma := \bigotimes_\Gamma \mathrm{M}_d$ is naturally regarded as a subalgebra of $\mathcal{A}$. We use the notation

$$\mathcal{A}_{\mathrm{loc}} := \bigcup_{\Lambda \Subset \mathbb{Z}^\nu} \mathcal{A}_\Lambda.$$

Interaction

An interaction is a map

$$\Phi : \mathfrak{S}_{\mathbb{Z}^\nu} \to \mathcal{A}_{\mathrm{loc}}$$

where $\mathfrak{S}_{\mathbb{Z}^\nu}$ is the set of all finite subsets of $\mathbb{Z}^\nu$, satisfying

$$\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$$

for all $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$.
An interaction Φ is of finite range if there exists $m \in \mathbb{N}$ such that $\Phi(X) = 0$ for X with diameter larger than m. It is uniformly bounded if

$$\sup_{X \in \mathfrak{S}_{\mathbb{Z}^\nu}} \|\Phi(X)\| < \infty.$$

Interaction such that

$$\Phi(X) = 0, \quad \text{if} \quad |X| \neq 1,$$

is called on-site/ trivial interaction.

Dynamics

For an interaction Φ and a finite set $\Lambda \subset \mathbb{Z}^\nu$, we define the local Hamiltonian as

$$(H_\Phi)_\Lambda := \sum_{X \subset \Lambda} \Phi(X).$$

For a uniformly bounded finite range interaction Φ, the limit

$$\alpha_t^\Phi(A) = \lim_{\Lambda \to \mathbb{Z}^\nu} e^{it(H_\Phi)_\Lambda} A e^{-it(H_\Phi)_\Lambda}, \quad t \in \mathbb{R}, \quad A \in \mathcal{A}$$

exists and define a dynamics α^Φ on $\mathcal{A}$.

Ground states

Definition

Let δ_Φ be the generator of α^Φ. A state ω on $\mathcal{A}$ is called an α^Φ-ground state if the inequality

$$-i\omega\left(A^*\delta_\Phi\left(A\right)\right) \geq 0$$

holds for any element A in the domain $\mathcal{D}(\delta_\Phi)$ of δ_Φ.

Let us consider this condition for a finite system M_n with dynamics

$$\alpha_t(A) = e^{itH}Ae^{-itH}, \quad t \in \mathbb{R}, \quad A \in \mathfrak{A},$$

Let P be the spectral projection of H corresponding to the lowest eigenvalue. Then a state ω is an α-ground state if and only if the support $s(\omega)$ of ω satisfies $s(\omega) \leq P$.

Unique gapped ground states

Definition

Suppose that there is a unique α^{Φ}-ground state ω_{Φ}. Then we say Φ has a unique gapped ground state in the bulk if there exists a $\gamma > 0$ such that

$$-i\omega_{\Phi}\left(A^{*}\delta_{\Phi}\left(A\right)\right) \geq \gamma\omega_{\Phi}(A^{*}A), \quad \textit{for all } A \in \mathcal{D}(\delta_{\Phi}) \textit{ with } \omega_{\Phi}(A) = 0.$$

Let us consider this condition for a finite system M_n with dynamics

$$\alpha_t(A) = e^{itH} A e^{-itH}, \quad t \in \mathbb{R}, \quad A \in \mathfrak{A},$$

with a self-adjoint element H in M_n. Then the above condition means that "the lowest eigenvalue of H is non-degenerated and the difference between the lowest eigenvalue and the second lowest eigenvalue is at least γ".

On-site symmetry

Let G be a finite group and U a unitary representation of G on $\mathbb{C}^d$.
Let β be an action of G on $\mathcal{A}$ such that

$$\beta_g(A) := \left(\bigotimes_{x\in\Lambda} U(g)\right) A \left(\bigotimes_{x\in\Lambda} U(g)^*\right), g \in G, \Lambda \Subset \mathbb{Z}^\nu, A \in \mathcal{A}_\Lambda.$$

We say an interaction Φ is β-invariant if $\beta_g(\Phi(X)) = \Phi(X)$ for all $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$ and $g \in G$.

Classification of gapped Hamiltonians without symmetry

We would like to classify

$$\mathcal{P}_{U.G.} := \left\{ \Phi \;\middle|\; \begin{array}{c} \text{finite range uniformly bounded interactions} \\ \text{with unique gapped ground state} \end{array} \right\}$$

with respect to the following criterion.

Classification of gapped Hamiltonians without symmetry

We would like to classify

$$\mathcal{P}_{U.G.} := \left\{ \Phi \;\middle|\; \begin{array}{c} \text{finite range uniformly bounded interactions} \\ \text{with unique gapped ground state} \end{array} \right\}$$

with respect to the following criterion.

> Two interactions $\Phi_0, \Phi_1 \in \mathcal{P}_{U.G.}$ are equivalent ($\Phi_0 \sim \Phi_1$) if there is a smooth path in $\mathcal{P}_{U.G.}$ connecting them.

Smoothness means

- $[0,1] \ni s \mapsto \Phi_s(X) \in \mathcal{A}_X$ is smooth,
- the gap is uniformly bounded from below by some $\gamma > 0$ along the path,
- the path of expectation values $[0,1] \ni s \mapsto \omega_{\Phi(s)}(A) \in \mathbb{C}$ of sub-exponentially localized elements $A \in \mathcal{A}_{\mathbb{Z}^\nu}$ with respect to the ground state $\omega_{\Phi(s)}$ is regular with respect to $s \in [0,1]$.

Reference on-site interaction

Throughout this talk, we fix some on-site interaction Φ_0 with unique gapped ground state.
Note that its ground state ω_{Φ_0} is a product state.

Interaction such that

$$\Phi_0(X) = 0, \quad \text{if} \quad |X| \neq 1,$$

is called an on-site interaction.

SPT phases

Now we introduce the on-site symmetry β to the game.
The set we consider in this talk is

$$\mathcal{P}^0_{U.G.\beta} := \{\Phi \in \mathcal{P}_{U.G.} \mid \Phi \sim \Phi_0 \text{ and } \beta\text{-invariant}\}.$$

Two interactions $\Phi_0, \Phi_1 \in \mathcal{P}^0_{U.G.\beta}$ are equivalent ($\Phi_0 \sim_\beta \Phi_1$) if there is a smooth path in $\mathcal{P}^0_{U.G.\beta}$ connecting them.

$\mathcal{P}^0_{U.G.\beta}$ may split into multiple equivalence classes with respect to $\sim_\beta$.

Definition (Symmetry Protected Topological (SPT) phases)

Each equivalence class of $\mathcal{P}^0_{U.G.\beta}$ with respect to $\sim_\beta$ is called Symmetry Protected Topological (SPT) phases.
(Gu-Wen '09)

The question I would like to ask in this talk is the following:

Question

How can we see that $\Phi_1 \in \mathcal{P}^0_{U.G.\beta}$ and $\Phi_2 \in \mathcal{P}^0_{U.G.\beta}$ belong to different equivalence class?

The question I would like to ask in this talk is the following:

Question

How can we see that $\Phi_1 \in \mathcal{P}^0_{U.G.\beta}$ and $\Phi_2 \in \mathcal{P}^0_{U.G.\beta}$ belong to different equivalence class?

Natural approach:
Define some index on $\mathcal{P}^0_{U.G.\beta}$ and show that it is an invariant of $\sim_\beta$.

The question I would like to ask in this talk is the following:

Question

How can we see that $\Phi_1 \in \mathcal{P}^0_{U.G.\beta}$ and $\Phi_2 \in \mathcal{P}^0_{U.G.\beta}$ belong to different equivalence class?

Natural approach:
Define some index on $\mathcal{P}^0_{U.G.\beta}$ and show that it is an invariant of $\sim_\beta$.
Indeed, physicist took this approach, [Pollmann et.al. '10,'12 Chen-Gu-Wen '11 Schuch et.al. '11, Molnar '18 et.al.], and they conjectured that

for ν-dimensional quantum spin systems, there is a $H^{\nu+1}(G, U(1))$-valued invariant for the classification,

based on analysis of MPS/PEPS, and TQFT.

Indeed, physicist took this approach, [Pollmann et.al. '10,'12 Chen-Gu-Wen '11 Schuch et.al. '11, Molnar '18 et.al.], and they claimed that

for ν-dimensional quantum spin systems, there is a $H^{\nu+1}(G, U(1))$-valued invariant for the classification

based on analysis of MPS/PEPS, and TQFT.

i-cochain of G :

$$C^{i}\left(G, \mathrm{U}(1)\right) = \left\{ f \mid G^{\times i} \to \mathrm{U}(1) \right\}.$$

The i-th differential $d^{i} : C^{i}\left(G, \mathrm{U}(1)\right) \to C^{i+1}\left(G, \mathrm{U}(1)\right)$ is given by

$$\left(d^{i}\sigma\right)\left(g_0, \ldots, g_i\right) = \sigma\left(g_1, \ldots, g_i\right) \left(\prod_{k=1}^{i} \sigma(g_0, \ldots, g_{k-1}g_k, \ldots, g_i)^{(-1)^k} \right) \sigma\left(g_0, \ldots, g_{i-1}\right)^{(-1)^{i+1}}.$$

$$Z^{i}\left(G, \mathrm{U}(1)\right) := \ker d^{i}, \quad B^{i}\left(G, \mathrm{U}(1)\right) := \mathrm{Im}\, d^{i-1}$$
$$H^{i}\left(G, \mathrm{U}(1)\right) := Z^{i}\left(G, \mathrm{U}(1)\right) / B^{i}\left(G, \mathrm{U}(1)\right).$$

Main result

Theorem (Main Theorem O'20, O'20+)

There is an $H^{\nu+1}(G, \mathrm{U}(1))$*-valued invariant if* $\nu = 1, 2$*.*

$\nu = 1$ case

The invariant should be $H^2(G, \mathrm{U}(1))$-valued.

$\nu = 1$ case

The invariant should be $H^2(G, \mathrm{U}(1))$-valued.
It is well known that $H^2(G, \mathrm{U}(1))$ shows up naturally from projective representation

$$u_g u_h = \sigma(g, h) u_{gh}, \quad g, h \in G.$$

$\nu = 1$ case

The invariant should be $H^2(G, \mathrm{U}(1))$-valued.
It is well known that $H^2(G, \mathrm{U}(1))$ shows up naturally from projective representation

$$u_g u_h = \sigma(g, h) u_{gh}, \quad g, h \in G.$$

A projective representation was already there!

(i) **Thm** [T. Matsui '13]
Unique gapped ground state satisfies the split property

(ii) A projective representation can be associated naturally to β-invariant pure split state. ['01 Matsui]

$\nu = 1$ case

The invariant should be $H^2(G, \mathrm{U}(1))$-valued.
It is well known that $H^2(G, \mathrm{U}(1))$ shows up naturally from projective representation

$$u_g u_h = \sigma(g, h) u_{gh}, \quad g, h \in G.$$

A projective representation was already there!

(i) **Thm** [T. Matsui '13]
Unique gapped ground state satisfies the split property

(ii) A projective representation can be associated naturally to β-invariant pure split state. ['01 Matsui]

$\Rightarrow h(\Phi) \in H^2(G, \mathrm{U}(1))$.

Theorem (O '20)

The index $h(\Phi)$ is an invariant of our classification $\sim_\beta$.

How about $\nu = 2$ case??

Any form of split property for unique gapped ground state in 2-dimensional quantum spin systems is not known. Even if it was known, it is apriori not clear how to use it to define an index.

How about $\nu = 2$ case??

Any form of split property for unique gapped ground state in 2-dimensional quantum spin systems is not known. Even if it was known, it is apriori not clear how to use it to define an index.

So I decided to take an advantage of the situation of the problem.

Definition (Symmetry Protected Topological (SPT) phases)

Each equivalence class of $\mathcal{P}^0_{U.G.\beta}$ with respect to $\sim_\beta$ is called Symmetry Protected Topological (SPT) phases.

How about $\nu = 2$ case??

Any form of split property for unique gapped ground state in 2-dimensional quantum spin systems is not known. Even if it was known, it is apriori not clear how to use it to define an index.

So I decided to take an advantage of the situation of the problem.

Definition (Symmetry Protected Topological (SPT) phases)

Each equivalence class of $\mathcal{P}^0_{U.G.\beta}$ with respect to $\sim_\beta$ is called Symmetry Protected Topological (SPT) phases.

Fix any trivial $\Phi_0 \in \mathcal{P}_{U.G.}$. (Its ground state ω_{Φ_0} is of product form.)
Then for any $\Phi \in \mathcal{P}^0_{U.G.\beta}$, we have

$$\Phi \sim \Phi_0.$$

How about $\nu = 2$ case??

Any form of split property for unique gapped ground state in 2-dimensional quantum spin systems is not known. Even if it was known, it is apriori not clear how to use it to define an index.

So I decided to take an advantage of the situation of the problem.

Definition (Symmetry Protected Topological (SPT) phases)

Each equivalence class of $\mathcal{P}^0_{U.G.\beta}$ with respect to $\sim_\beta$ is called Symmetry Protected Topological (SPT) phases.

Fix any trivial $\Phi_0 \in \mathcal{P}_{U.G.}$. (Its ground state ω_{Φ_0} is of product form.)
Then for any $\Phi \in \mathcal{P}^0_{U.G.\beta}$, we have

$$\Phi \sim \Phi_0.$$

$\Rightarrow$ This means that ω_Φ has short range entanglement!

Automorphic Equivalence

Automorphic Equivalence (Hasting's adiabatic Lemma)

Theorem (Hastings-Wen '04, Bachmann et.al. '12 Nachtergaele et.al. '19, Moon-O '20)

For a smooth path of interactions $\Phi(s)$ in $\mathcal{P}_{U.G.}$, there is a smooth path of automorphisms α_s, satisfying a concrete differential equation given by some time-dependent local interactions such that

$$\omega_{\Phi(s)} = \omega_{\Phi(0)} \circ \alpha_s, \quad s \in [0, 1]$$

Because α_s is given by local interactions,

we can cut α_s and factorize it.

The factorization property of α_1 : $\nu = 1$ case

$$\alpha_1 = (\text{inner}) \circ (\alpha_L \otimes \alpha_R) \,.$$

The factorization property of α_1 : $\nu = 2$ case

$$H_L := \left\{(x,y) \in \mathbb{Z}^2 \mid x \le -1\right\}, \quad H_R := \left\{(x,y) \in \mathbb{Z}^2 \mid 0 \le x\right\},$$
$$C_\theta := \left\{(x,y) \in \mathbb{Z}^2 \mid |y| \le \tan\theta \cdot |x|\right\}, \quad 0 < \theta < \frac{\pi}{2}.$$

Proposition

For any $0 < \theta < \frac{\pi}{2}$, there is $\alpha_L \in \mathrm{Aut}\,\mathcal{A}_{H_L}$, $\alpha_R \in \mathrm{Aut}\,\mathcal{A}_{H_R}$, and $\Theta \in \mathrm{Aut}\,\mathcal{A}_{(C_\theta)^c}$ such that

$$\alpha_1 = (\mathrm{inner}) \circ (\alpha_L \otimes \alpha_R) \circ \Theta.$$

For any interaction Φ in SPT phase $\mathcal{P}^{0}_{U.G.\beta}$, there is an automorphism α_1 such that

(i) $\omega_\Phi = \omega_{\Phi_0} \circ \alpha_1$, and

(ii) α_1 factorize nicely.

Inspiration from operator algebra cocycle action [Connes'77] [Jones '80]

Still, how can we derive $H^3(G, \mathrm{U}(1))$?

Let $\{\gamma_g\}_{g\in G}$ be automorphisms on a factor $\mathcal{M}$.
Assume that there are unitaries $\{u(g,h)\}_{g,h\in G} \subset \mathcal{U}(\mathcal{M})$ such that

$$\gamma_g\gamma_h = \mathrm{Ad}\left(u(g,h)\right)\gamma_{gh}.$$

Then by the associativity, we have

$$\gamma_g\gamma_h\gamma_k = \mathrm{Ad}\left(u(g,h)\right)\gamma_{gh}\gamma_k = \mathrm{Ad}\left(u(g,h)u(gh,k)\right)\gamma_{ghk}$$
$$\gamma_g\gamma_h\gamma_k = \gamma_g\,\mathrm{Ad}\left(u(h,k)\right)\gamma_{hk} = \mathrm{Ad}\left(\gamma_g\left(u(h,k)\right)\right)\gamma_g\gamma_{hk}$$
$$= \mathrm{Ad}\left(\gamma_g\left(u(h,k)\right)u(g,hk)\right)\gamma_{ghk}$$

$\Rightarrow \exists c \in C^3(G, \mathrm{U}(1))$ s.t.

$$u(g,h)u(gh,k) = c(g,h,k)\gamma_g\left(u(h,k)\right)u(g,hk).$$

It turns out that

$$c \in Z^3(G, \mathrm{U}(1)) \Rightarrow [c]_{H^3(G,\mathrm{U}(1))} \in H^3(G, \mathrm{U}(1))$$

Remark:

If $\gamma_g = \mathrm{Ad}(u_g)$, (for example if $\mathcal{M}$ is of finite dimension), then we may take $u(g,h)$ in

$$\gamma_g \gamma_h = \mathrm{Ad}\left(u(g,h)\right) \gamma_{gh}.$$

as $u_g u_h u_{gh}^*$. Substituting this to

$$u(g,h)u(gh,k) = c(g,h,k)\gamma_g\left(u(h,k)\right)u(g,hk),$$

we obtain

$$\left(u_g u_h u_{gh}^*\right)\left(u_{gh} u_k u_{ghk}^*\right) = c(g,h,k)\,\mathrm{Ad}\left(u_g\right)\left(u_h u_k u_{hk}^*\right)\cdot u_g u_{hk} u_{ghk}^*,$$

hence $c(g,h,k) = 1$.

Naive consideration

If we have automorphisms $\{\gamma_g\}_{g\in G}$ on $\mathcal{A}$ such that $\gamma_g\gamma_h\gamma_{gh}^{-1}$ is inner, then we get some $[c] \in H^3(G, \mathrm{U}(1))$.

Naive consideration

If we have automorphisms $\{\gamma_g\}_{g\in G}$ on $\mathcal{A}$ such that $\gamma_g\gamma_h\gamma_{gh}^{-1}$ is inner, then we get some $[c] \in H^3(G, \mathrm{U}(1))$.

But how we get such a thing?

Naive consideration

If we have automorphisms $\{\gamma_g\}_{g\in G}$ on $\mathcal{A}$ such that $\gamma_g\gamma_h\gamma_{gh}^{-1}$ is inner, then we get some $[c] \in H^3(G, \mathrm{U}(1))$.

But how we get such a thing?

Hint from former works on index of gapped pahses

Cut the system into two!

Naive consideration

If we have automorphisms $\{\gamma_g\}_{g\in G}$ on $\mathcal{A}$ such that $\gamma_g\gamma_h\gamma_{gh}^{-1}$ is inner, then we get some $[c]\in H^3(G,\mathrm{U}(1))$.

But how we get such a thing?

Hint from former works on index of gapped pahses

Cut the system into two!
Bulk-edge correspondence in free Fermions
F. Pollmann, A. Turner, E. Berg, and M. Oshikawa '10
O' 20,   Bachmann-Bols-DeRoeck-Fraas '20,

Consider the action of G given by

$$\beta_g^U := \mathrm{id}_{\mathcal{A}_{H_D}} \otimes \bigotimes_{x\in H_U} \mathrm{Ad}\left(U(g)\right), \quad g\in G.$$

Good situation

If there are automorphisms $\{\gamma_g\}_{g\in G}$ on $\mathcal{A}$ such that $\gamma_g\gamma_h\gamma_{gh}^{-1}$ is inner, i.e., $\gamma_g\gamma_h\gamma_{gh}^{-1} = \mathrm{Ad}\,(u(g,h))$ then we get some $[c] \in H^3(G, \mathrm{U}(1))$.

For any interaction Φ in SPT phase $\mathcal{P}^0_{U.G.\beta}$, there is an automorphism α_1 such that $\omega_\Phi = \omega_{\Phi_0} \circ \alpha_1$.

Lemma

Suppose that

$$\alpha_1\beta_g^U\alpha_1^{-1}\left(\beta_g^U\right)^{-1} = (\mathrm{inner})\left(\eta_{g,L}\otimes\eta_{g,R}\right)$$

Set $\gamma_g := \eta_{g,R}\beta_g^{U,R}$. *Then we have* $\gamma_g\gamma_h\gamma_{gh}^{-1} = \mathrm{Ad}\,(u(g,h))$, *with some unitary. Hence we obtain some* $[c] \in H^3(G, \mathrm{U}(1))$.

Proof

$$\begin{aligned}\mathrm{id} &= \alpha_1\beta_g^U\alpha_1^{-1}\circ\alpha_1\beta_h^U\alpha_1^{-1}\circ\left(\alpha_1\beta_{gh}^U\alpha_1^{-1}\right)^{-1}\\ &= (\mathrm{inner})\circ\left(\gamma_{g,L}\gamma_{h,L}\gamma_{gh,L}^{-1}\right)\otimes\left(\gamma_{g,R}\gamma_{h,R}\gamma_{gh,R}^{-1}\right)\end{aligned}$$

Good situation : example (Dijkgraaf-Witten model)

Let $\mathbb{C}^d = l^2(G)$, spanned by CONS $\{|g\rangle \mid g \in G\}$, and $U_g |h\rangle = |gh\rangle$.
Fix any $\varphi \in Z^3(G, \mathrm{U}(1))$ and set

$$\nu(g_0, g_1, g_2, g_3) := \varphi(g_0^{-1} g_1, g_1^{-1} g_2, g_2^{-1} g_3).$$

$$\Rightarrow \frac{\nu(g_1, g_2, g_3, g_4)\nu(g_0, g_1, g_3, g_4)\nu(g_0, g_1, g_2, g_3)}{\nu(g_0, g_2, g_3, g_4)\nu(g_0, g_1, g_2, g_4)} = 1.$$

For each $\mathbb{S}_N := \{S_{(x,y)}\}_{(x,y)\in\Lambda_N} : \Lambda_N \to G$, a configuration, set

$$\psi_N(\mathbb{S}_N) := \prod_{(x,y)\in[-N+1,N-1]^2} \frac{\nu\left(e, s(x,y), s(x+1,y), s(x+1,y+1)\right)}{\nu\left(e, s(x,y), s(x,y+1), s(x+1,y+1)\right)}, \text{ and}$$

$$V_N := \sum_{\mathbb{S}_N} \psi_N(\mathbb{S}_N) \left|\mathbb{S}_N\right\rangle \left\langle \mathbb{S}_N\right| .$$

$\exists \alpha_1$ such that

$$\alpha_1(A) = \lim_N \mathrm{Ad}(V_N)(A), \quad A \in \mathcal{A}.$$

$$\alpha_1 \beta_g^U \alpha_1^{-1} \left(\beta_g^U\right)^{-1}(A) = \lim_{N\to\infty} \mathrm{Ad}(V_N \beta_g^U(V_N^*))(A) \Rightarrow V_N \beta_g^U\left(V_N^*\right)?$$

Good situation : example

$$\frac{\nu(e, s(x,y), s(x+1,y), s(x+1,y+1))}{\nu(g, s(x,y), s(x+1,y), s(x+1,y+1))} = \frac{\nu(e, g, s(x+1,y), s(x+1,y+1))\nu(e, g, s(x,y), s(x+1,y))}{\nu(e, g, s(x,y), s(x+1,y+1))}$$

$$\frac{\nu(e, s(x,y), s(x,y+1), s(x+1,y+1))}{\nu(g, s(x,y), s(x,y+1), s(x+1,y+1))} = \frac{\nu(e, g, s(x,y+1), s(x+1,y+1))\nu(e, g, s(x,y), s(x,y+1))}{\nu(e, g, s(x,y), s(x+1,y+1))}.$$

Good situation : example

As a result, in this model, $\alpha_1 \beta_g^U \alpha_1^{-1} \left(\beta_g^U\right)^{-1}$ is localized around $y = 0$, and can be cut into left and right modulo (inner).
$\Rightarrow$ We may apply

Lemma

Suppose that

$$\alpha_1 \beta_g^U \alpha_1^{-1} \left(\beta_g^U\right)^{-1} = (\mathrm{inner}) \left(\eta_{g,L} \otimes \eta_{g,R}\right)$$

Set $\gamma_g := \eta_{g,R} \beta_g^{U,R}$*. Then we have* $\gamma_g \gamma_h \gamma_{gh}^{-1} = \mathrm{Ad}\left(u(g,h)\right)$*, with some unitary. Hence we obtain some* $[c] \in H^3(G, \mathrm{U}(1))$.

and obtain

$$[\varphi] \in H^3(G, \mathrm{U}(1)).$$

$H^3(G, \mathrm{U}(1))$-valued index

Lemma

Suppose that $\alpha_1 \beta_g^U \alpha_1^{-1} \left(\beta_g^U\right)^{-1} = (\mathrm{inner}) \left(\eta_{g,L} \otimes \eta_{g,R}\right)$. *Set* $\gamma_g := \eta_{g,R} \beta_g^{U,R}$. *Then we have* $\gamma_g \gamma_h \gamma_{gh}^{-1} = \mathrm{Ad}\left(u(g,h)\right)$, *with some unitary. Hence we obtain some* $[c] \in H^3(G, \mathrm{U}(1))$.

Unfortunately, it is not clear if α_1 satisfies such a property, in general.

$H^3(G, \mathrm{U}(1))$-valued index

Lemma

Suppose that $\alpha_1 \beta_g^U \alpha_1^{-1} \left(\beta_g^U\right)^{-1} = (\mathrm{inner}) \left(\eta_{g,L} \otimes \eta_{g,R}\right)$. *Set* $\gamma_g := \eta_{g,R} \beta_g^{U,R}$. *Then we have* $\gamma_g \gamma_h \gamma_{gh}^{-1} = \mathrm{Ad}\,(u(g,h))$, *with some unitary. Hence we obtain some* $[c] \in H^3(G, \mathrm{U}(1))$.

Unfortunately, it is not clear if α_1 satisfies such a property, in general.
However, "some weaker version" is true, and from this, we can derive an element $h(\Phi) \in H^3(G, \mathrm{U}(1))$.

Theorem (O' 20+)

$h(\Phi)$ *is independent of choice of* α_1.
$(\omega_\Phi = \omega_{\Phi_0} \circ \alpha_1.)$

Furthermore,

Theorem (O' 20+)

$h(\Phi)$ *is an invariant of the SPT phases.*

Thank you