University of Tokyo
Seminary Notes 45

Quantum Symmetry, Differential Geometry of Finite Graphs and Classification of Subfactors

Lectures given at the University of Tokyo by

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Yasuyuki Kawahigashi

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I. Bimodules and the Galois functor

Let (X, μ) be a measure space. Passing from the commutative algebra $L^\infty(X, \mu)$ to an algebra of bounded linear operators on a Hilbert space, we pass from a classical space to a quantum space. Groups are symmetries of classical spaces, then what are the symmetries of quantum spaces ? We will use an analogue of the Galois theory.

Let G be a finite group, α an action of G on a factor N. We assume that α is outer. In operator algebras, a basic construction consists in building a larger algebra out of an algebra N and some operators on N. Here we construct the crossed product algebra $M = N \rtimes_\alpha G = \langle N, (\lambda_g)_{g \in G} \rangle$. (This is similar to a semidirect product of a group.) Then the action α is outer if and only if $N' \cap M = \mathbf{C}$. (This can be taken as a definition of outerness.) We now consider conjugacy for $N \subset M$. This is an analogue of knot isotopy. We say $N \subset M$ is conjugate to $N_1 \subset M_1$ if there is an isomorphism $\theta : M \to M_1$ with $\theta(N) = N_1$. That is, a position of N in M is a conjugacy class. Now we would like to find G out of the position of $N \subset M = N \rtimes G$.

I.1. Bimodules and basis

A *bimodule* is a Hilbert space ${}_A H_B$ with normal $*$-representations of A and B^{op}, the opposite algebra of B, which commute with each other. Let N have a finite trace tr. By abuse of notation, we write ${}_N M_M$ for ${}_N L^2(M, \mathrm{tr})_M$. We say $T : {}_A X_B \to {}_A Y_B$ is an intertwiner if T is in $\mathrm{Hom}({}_A X_B, {}_A Y_B)$, that is, T is linear and $T(axb) = aT(x)B$, $a \in A, x \in X, b \in B$. A bimodule ${}_A X_B$ is called irreducible if the only self-intertwiners are the scalars. Here a self-intertwiner means an element in $\mathrm{Hom}({}_AX_B, {}_AX_B) = \mathrm{End}({}_AX_B) = B(X) \cap A' \cap (B^{\mathrm{op}})'$. If this von Neumann algebra is finite dimensional, we can get an irreducible bimodule by cutting X by a minimal projection in this.

Example. The bimodule ${}_NN_N$ is the standard form of N. Then $\mathrm{End}({}_NN_N) = N' \cap (N^{\mathrm{op}})' \cap B(L^2(N)) = \mathscr{Z}(N)$, the center of N. If N is a factor, then ${}_NN_N$ is irreducible. Now let $M = \langle N, \lambda_g \rangle$ be the crossed product. An element $x \in M$ is written in the form $x = \sum_{g \in G} x_g \lambda_g$, $x_g \in N$. This implies that the irreducible decomposition of ${}_NM_N$ is $\bigoplus_g {}_N(N\lambda_g)_N$. Here each component is irreducible and not mutually equivalent because of outerness of the action. Thus each element $g \in G$ is in one-to-one correspondence to an irreducible bimodule ${}_N(N\lambda_g)_N$. What is the composition law of the group in terms of these bimodules ? The answer is given by ${}_N(N\lambda_g)_N \otimes_N {}_N(N\lambda_h)_N \cong {}_N(N\lambda_{gh})_N$ By an analogue of the Galois theory, we get a correspondence between the position and the dual of the finite group. We obtain natural symmetries of a quantum space.

Now suppose $N \subset M$ and $N' \cap M = \mathbf{C}$, the outerness condition. We try to find a group-like object acting on M so that M is a "crossed product". We assume no algebraic conditions on $N \subset M$. That is, the data are $N \subset M$ and a finite trace $\mathrm{tr} : M \to \mathbf{C}$. If ${}_NX$ is a left N-module, then X is known to be projective, by von Neumann, that is, ${}_NX \cong {}_N(L^2(N)^{\oplus k}p)$, where p is a projection in $\mathrm{Mat}_k(N)$, $k \leq \infty$. Then we write $(\mathrm{Tr}_k \otimes \mathrm{tr})(p) = \dim_N X$, which is invariant though p is not unique. (This number is called a coupling constant.) For $(N \subset M, \mathrm{tr})$, we set $[M : N] = \dim_N M$, which is the Jones index [J].

We also see the modules are not only projective but also "almost" free. For any ${}_NX$, one can find a basis $(\lambda_i)_{i \in I} \subset X$ in the following sense. Denote the

Hilbert space inner product on X by $\langle\ ,\ \rangle$. We have an N-valued inner product $\langle\ ,\ \rangle_N$ defined on a dense subset X_0 of X with $\mathrm{tr}_N(a\langle x_1, x_2\rangle_N) = \langle ax_1, x_2\rangle$, for $x_1, x_2 \in X, a \in N$. (This is a Radon-Nikodym derivative with respect to the trace. Also note that an algebra-valued inner product is necessary for KK-theory.) The system $(\lambda_i)_{i\in I}$ is called a basis for ${}_NX$ if for all $x \in X$ we have $x = \sum_{i\in I}\langle x, \lambda_i\rangle_N\lambda_i$. (See Herman-Ocneanu.) To find a basis, decompose ${}_NX$ into cyclic pieces ${}_N(Nv_i)$ and set

$$\lambda_i = \langle v_i, v_i\rangle_N^{-1/2} v_i = \lim_{\varepsilon\searrow 0}(\langle v_i, v_i\rangle_N + \varepsilon)^{-1/2} v_i,$$

if v_i is bounded in ${}_NX$. Then we get for $av_i \in {}_N(Nv_i)$

$$\begin{aligned}\langle av_i, \lambda_i\rangle_N\lambda_i &= a\langle v_i, \langle v_i, v_i\rangle_N^{-1/2} v_i\rangle_N \langle v_i, v_i\rangle_N^{-1/2} v_i \\ &= a\langle v_i, v_i\rangle_N \langle v_i, v_i\rangle_N^{-1} v_i \\ &= av_i.\end{aligned}$$

Hence for all $x \in X$, we get $\sum_i\langle x, \lambda_i\rangle_N\lambda_i = x$. We define an orthogonal projection $q \in \mathrm{Mat}_{I\times I}(N)$ by $q_{ij} = \langle\lambda_i, \lambda_j\rangle_N$. Then

$$\dim_N X = (\mathrm{Tr}_I \otimes \mathrm{tr})(q) = \sum_i \mathrm{tr}(\langle\lambda_i, \lambda_i\rangle) = \sum_i\langle\lambda_i, \lambda_i\rangle.$$

If $\tilde\lambda_j$ is another basis of ${}_NX$, then $\tilde\lambda_j = \sum\langle\tilde\lambda_j, \lambda_i\rangle_N\lambda_i = \sum_i v_{ji}\lambda_i$, where a partial isometry $v = (v_{ji}) \in \mathrm{Mat}_{J\times I}(N)$ is defined by $v_{ji} = \langle\tilde\lambda_j, \lambda_i\rangle_N$.

For the case of $N \subset M$, a basis for ${}_NM$ was introduced by Pimsner-Popa by different means [PP]. (This argument is also O.K. for type III cases with a little bit of modular automorphism groups.)

For $N \subset M$, construct the standard bimodule ${}_N H_M \equiv {}_N M_M (= {}_N(L^2(M))_M.)$ A bimodule is a "quantum automorphism" in the following sense. For a bimodule ${}_A X_B$, choose a basis (λ_i) for ${}_A X$. Then $\lambda_i b = \sum_j \langle \lambda_i b, \lambda_j \rangle_A \lambda_j$ for $b \in B$. Define a map $\theta = (\theta_{ij}) : B \to \mathrm{Mat}_{I \times I}(A)$ by $\theta_{ij}(b) = \langle \lambda_i b, \lambda_j \rangle_A$. This is a $*$-homomorphism with the properties $\theta(1) = q$, $\theta(B) \subset q(\mathrm{Mat} A)q$.

Example. In the group case $M = N \rtimes G$, set ${}_N X_N = {}_N(N\lambda_g)_N$. Then λ_g is a basis for ${}_N(N\lambda_g)$. Because $\lambda_g x = \alpha_g(x)\lambda_g$, $x \in N$, we get $\langle \lambda_g x, \lambda_g \rangle_N = \alpha_g(x)$ and $\theta = \alpha_g \in \mathrm{Aut}(N)$. This θ is defined uniquely up to inner perturbation. For the bimodule ${}_N M_N$, the homomorphism θ is given by $\bigoplus_g \alpha_g : N \to L^\infty(G) \otimes N \subset \mathrm{Mat}_{G \times G}(N)$. For the bimodule ${}_N M_M$, the homomorphism $\theta : M \to \mathrm{Mat}_{G\times G}(N) = B(L^2(G)) \otimes N$ is given by $\theta(x) = (\alpha_g(x))_g \in L^\infty(G) \otimes N$, $x \in N$, and $\theta(\lambda_g) = \lambda_g \otimes 1 \in B(L^2(G)) \otimes N$. Determining θ for the bimodule ${}_M M_N$ is left as an exercise.

I.2. Group-like structure — cells and connections

In the above sense, bimodules are "quantum homomorphisms". Then what is the group-like global structure ? Recall Weyl's method of making tensor powers of the standard one and decomposing them into pieces. We take tensor powers of $H = {}_N M_M$ and $\bar{H} = {}_M M_N^* = {}_M M_N$. Take a finite power $\cdots H \otimes \bar{H} \otimes H \otimes \bar{H} \cdots$ and decompose it into irreducible components. We have four kinds of bimodules; N-N, N-M, M-N, M-M bimodules. We choose a representative in each equivalence class.

For an irreducible bimodule ${}_M X_M$, take $H \otimes X$, which means ${}_N H_M \otimes_M {}_M X_M$, and decompose it into irreducible N-M sub-bimodules: $H \otimes X \cong \bigoplus_i Y_i$. We write the following graph for this.

$$\begin{tikzcd} X \arrow[d, "H\otimes\cdot"] \\ Y_i \end{tikzcd}$$

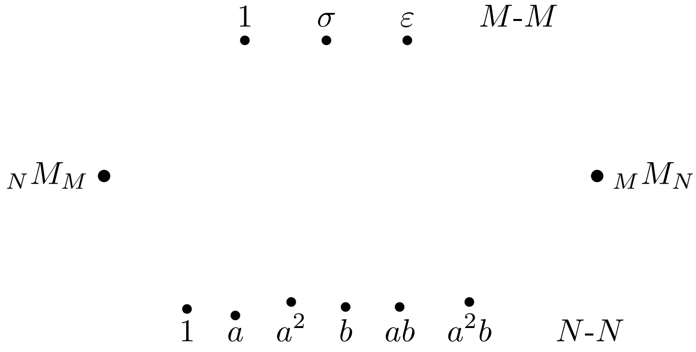
For each edge v_i, choose an N-M intertwiner $T(v_i) : H \otimes X \to Y_i$. (That is, we have partial isometries with mutually orthogonal support.) We draw a following graph.

$$\begin{array}{ccc} & M\text{-}M \text{ bimodules} & \\ & \cdots & \\ H^{\#} \otimes \cdot & & \cdot \otimes H^{\#} \\ N\text{-}M \text{ bimodules} \cdots & & \cdots M\text{-}N \text{ bimodules} \\ \cdot \otimes H^{\#} & & H^{\#} \otimes \cdot \\ & \cdots & \\ & N\text{-}N \text{ bimodules} & \end{array}$$

where $H^{\#}$ stands for H or $\bar{H}$.

The terminology "finite depth" means that only finitely many bimodules appear in repeating this procedure. We also call *depth* the longest distance from $*$ to a vertex on the graph.

Example(Group case). Let $M = N \rtimes G$, and assume $G = S_3$ for simplicity. By Frobenius duality, we get $\mathrm{Hom}(H \otimes X, Y) \cong \mathrm{Hom}(X, \bar{H} \otimes Y)$. Write $S_3 = \{1, a, a^2, b, ab, a^2b\}$, with $a^3 = b^2 = (ab)^2 = 1$, and $\hat{S}_3 = \{1, \varepsilon, \sigma\}$, where ε denotes the sign representation and σ denotes the 2-dimensional representation. Then we get the following graph.



Note that ${}_M M_M$ is obtained as $\bar{H} \otimes H = {}_M(M \otimes_N M)_M$. For $\rho \in \mathrm{Rep}(G)$, set $\xi^\rho_{ij} = \sum_g \rho_{ij}(g^{-1})\lambda_{g^{-1}} \otimes_N \lambda_g \in \bar{H} \otimes H$. If $x \in N$, then $\xi^\rho_{ij} x = x \xi^\rho_{ij}$. We also have

$$\begin{aligned}\xi^\rho_{ij}\lambda_h &= \sum_g \rho_{ij}(g^{-1})\lambda_{g^{-1}} \otimes \lambda_{gh} \\ &= \sum_{k,s} \rho_{is}(h)\rho_{sj}(k^{-1})\lambda_{hk^{-1}} \otimes \lambda_k \\ &= \lambda_h \sum_s \rho_{is}\xi^\rho_{sj}.\end{aligned}$$

Thus, $X^\rho_j = \bigoplus_s M\xi^\rho_{sj}$ is an M-M bimodule. One can show that $\bar{H} \otimes H = \bigoplus_{\rho,j} X^\rho_j$ and $X^\rho_i \cong X^\rho_j$, where the intertwiner $T^\rho_{ij} : X^\rho_i \to X^\rho_j$ is given by $T^\rho_{ij}(\xi^\rho_{si}) = \xi^\rho_{sj}$. Each ${}_M(X^\rho_i)_M$ is irreducible as seen below. Notice that $N' \cap X^\rho_i \equiv \{x \in X^\rho_i \mid ax = xa \text{ for all } a \in N\}$ is a linear span of $\{\xi^\rho_{si}\}_s$. Any intertwiner $T : X^\rho_i \to X^\rho_i$ must map $N' \cap X^\rho_i$ to $N' \cap X^\rho_i$. Now use $T(\xi\lambda_g) = T(\xi)\lambda_g$ and that ρ is irreducible to show that $T \in \mathbf{C}$. Now what is the group structure ? We have the following diagram of N-N bimodules.

$$\begin{tikzcd}
{}_M X_M \arrow[r, "\cdot\otimes\bar{H}", "T_2"'] \arrow[d, "H\otimes\cdot"', "T_1"] & {}_M Y_{2N} \arrow[d, "S_2"', "H\otimes\cdot"] \\
{}_N Y_{1M} \arrow[r, "S_1", "\cdot\otimes\bar{H}"'] & {}_N Z_N,
\end{tikzcd}$$

that is,

$$\begin{tikzcd}
H\otimes X\otimes\bar{H} \arrow[r, "1_H\otimes T_2"] \arrow[d, "T_1\otimes 1_{\bar{H}}"'] & H\otimes Y_2 \arrow[d, "S_2"] \\
Y_1\otimes\bar{H} \arrow[r, "S_1"] & Z.
\end{tikzcd}$$

Because $S_2(1\otimes T_2)(T_1\otimes 1)^*S_1^*$ is a N-N bimodule map from Z to Z, it is a scalar by the irreducibility. We write this

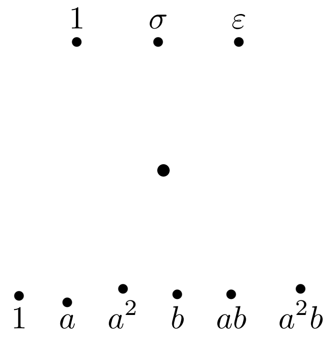
$$W\left(\begin{tikzcd}
& {} \arrow[dl, "T_1"'] \arrow[dr, "T_2"] & \\
{} \arrow[dr, "S_1"'] & & {} \arrow[dl, "S_2"] \\
& {} &
\end{tikzcd}\right)\in\mathbf{C},$$

which depends on choices of the bimodules and the intertwiners. We call four edges, one from each graph, with common vertices, a cell. In the above example, we have 36 cells. A map W from the set of cells to $\mathbf{C}$ is called a connection (or a Boltzmann weight). This is unique up to a gauge coming from the choice of vertices and edges of the graphs just like a cohomological invariant. In the above case,

$$W\left(\begin{tikzcd}
& \sigma \arrow[dl, "i"'] \arrow[dr, "j"] & \\
{} \arrow[dr] & & {} \arrow[dl] \\
& g &
\end{tikzcd}\right)=\sigma_{ij}(g)\in\mathbf{C},$$

as seen below. For a subfactor $R\rtimes H\subset R\rtimes G$, we get the Mackey machine in this way.

In the following graph,



we get the following connection. For the cell

$$\begin{tikzcd}
{}_M(X_j^\rho)_M \arrow[d] & H \otimes {}_N M_M \arrow[l, "T"'] \\
{}_M M_N \arrow[r] & {}_N(N\lambda_g)_N, \arrow[u]
\end{tikzcd}$$

the connection is given by

$$\begin{tikzcd}
\lambda_g \sum_s \sigma_{is}(g)\xi^\sigma_{sj} = \xi^\sigma_{ij}\lambda_g \arrow[d, "k"'] & \lambda_g \arrow[l] \\
\sigma_{ik}(g)\lambda_g \arrow[r] & \sigma_{ik}(g)\lambda_g, \quad \lambda_g. \arrow[u]
\end{tikzcd}$$

Thus $W = \sigma_{ik}(g)$.

I.3. Biunitarity

In the following diagram, let X, Z be fixed and Y vary.

$$\begin{tikzcd}
X \arrow[d, "T_1"'] & \\
Y \arrow[r, "S_1"] & Z.
\end{tikzcd}$$

Then $S_1(T_1 \otimes 1)$'s make a basis for $\mathrm{Hom}(H \otimes X \otimes \bar{H}, Z)$. Similarly, in

$$\begin{tikzcd}
X \arrow[r, "S_2"] & Y_2 \arrow[d, "S_2"'] \\
& Z,
\end{tikzcd}$$

we get a basis for $\mathrm{Hom}(H \otimes X \otimes \bar{H}, Z)$. These mean that $W\left(\begin{tikzcd}[row sep=small, column sep=small] & {} \arrow[dl] \arrow[dr] & \\ i \arrow[dr] & & j \arrow[dl] \\ & {} & \end{tikzcd}\right)$ is unitary when X, Z are fixed. For example, for fixed σ, g in the group case, $(\sigma_{ij}(g))_{ij}$ is unitary.

Similarly, if Y_1, Y_2 are fixed in

$$\begin{tikzcd}
X \arrow[r] & Y_2 \\
Y_1 \arrow[u] \arrow[r] & Z, \arrow[u]
\end{tikzcd}$$

then $W\left(\begin{tikzcd}[row sep=small, column sep=small] & i \arrow[dr] & \\ {} \arrow[ur] \arrow[dr] & & {} \\ & j \arrow[ur] & \end{tikzcd}\right)$ is unitary, where we need a renormalization

$$W\left(\begin{tikzcd}[row sep=small, column sep=small] & i \arrow[dr] & \\ k \arrow[ur] \arrow[dr] & & l \\ & j \arrow[ur] & \end{tikzcd}\right) = \sqrt{\frac{\mu(i)\mu(j)}{\mu(k)\mu(l)}}\,\overline{W\left(\begin{tikzcd}[row sep=small, column sep=small] & i \arrow[dl] \arrow[dr] & \\ k \arrow[dr] & & l \arrow[dl] \\ & j & \end{tikzcd}\right)}.$$

In the group case, $(\frac{|\sigma|^{1/2}}{|G|^{1/2}}\sigma_{ij}(g))_{(\sigma,i,j),g}$ is a unitary, which is $|G| \times |G|$-matrix. That is,

$$\begin{pmatrix} 1_{11}(1) & 1_{11}(a) & 1_{11}(a^2) & \cdots \\ \sigma_{11}(1) & \sigma_{11}(a) & \sigma_{11}(a^2) & \cdots \\ \sigma_{12}(1) & \sigma_{12}(a) & \sigma_{12}(a^2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is unitary (up to renormalization). (cf. Peter-Weyl Theorem.)

The biunitarity axioms states that the both matrices

$$\left(W\left(\begin{tikzcd} & i \arrow[dl] \arrow[dr] & \\ {} \arrow[dr] & & {} \arrow[dl] \\ & j & \end{tikzcd}\right)\right), \left(W\left(\begin{tikzcd} & {} \arrow[dr] & \\ k \arrow[ur] \arrow[dr] & & l \\ & {} \arrow[ur] & \end{tikzcd}\right)\right),$$

are unitary for each fixed (i, j) and (k, l). (Note that this renormalization convention is slightly different from that in [O1].)

In the following we write

$$W\left(\begin{tikzcd} a \arrow[r] \arrow[d] & c \arrow[d] \\ b \arrow[r] & d \end{tikzcd}\right)$$

for one of

$$W\left(\left(\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b \arrow[dr] & & c \arrow[dl] \\ & d & \end{tikzcd}\right)\right), W\left(\left(\begin{tikzcd} & b \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & d \\ & c \arrow[ur] & \end{tikzcd}\right)\right),$$

depending the position of the vertex a. In this notation, the renormalization rule and biunitarity are expressed as follows.

$$W\left(\begin{tikzcd} a \arrow[r] \arrow[d] & c \arrow[d] \\ b \arrow[r] & d \end{tikzcd}\right) = \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}} \cdot \overline{W\left(\begin{tikzcd} b \arrow[r] \arrow[d] & d \arrow[d] \\ a \arrow[r] & c \end{tikzcd}\right)},$$

$$\sum_{\xi,\eta} W\left(\begin{tikzcd} a \arrow[r, "\xi"] \arrow[d, "\zeta"'] & c \arrow[d, "\eta"] \\ b \arrow[r, "\rho"] & d \end{tikzcd}\right) \cdot W\left(\begin{tikzcd} c \arrow[d, "\eta"'] & a \arrow[l, "\xi"'] \arrow[d, "\zeta"] \\ d & b \arrow[l, "\rho'"'] \end{tikzcd}\right) = \delta_{\zeta\cdot\rho,\zeta'\cdot\rho'},$$

where we used the notation

$$W\left(\begin{tikzcd} a \arrow[r] \arrow[d] & c \arrow[d] \\ b \arrow[r] & d \end{tikzcd}\right) = \overline{W\left(\begin{tikzcd} c \arrow[d] & a \arrow[l] \arrow[d] \\ d & b \arrow[l] \end{tikzcd}\right)}.$$

I.4. Indices and Perron-Frobenius eigenvectors

We have

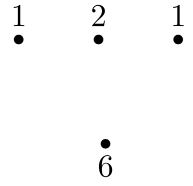
$$\dim({}_N M \otimes_N X) = [M:N] \dim_N X,$$

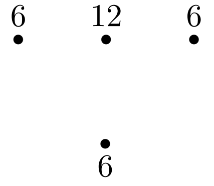
$$\dim({}_N M \otimes_M X) = \dim_N X,$$

where we mean by "dim" the dimension as N-module. By these, we get a relation between the index and an eigenvalue. For the vector μ_0, which assigns the dimension to each bimodule, the adjacency matrix Δ of the graph satisfies

$$\Delta\mu_0(x) = \sum_{\substack{v\in\text{edges} \\ s(v)=x \\ r(v)=y}} \mu_0(y),$$

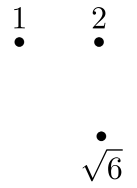
where $s(v)$ and $r(v)$ denote the source and the range of an edge v.

$\mu_0 =$ 

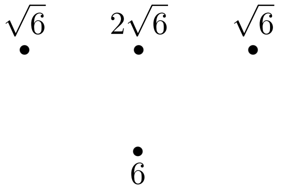
, $\Delta\mu_0 =$ 

Renormalizing μ_0 by multiplying the dim of ${}_M X$ by $[M : N]^{-1/2}$, we get an eigenvector μ for Δ.

$\mu =$



, $\Delta\mu = \sqrt{6}\mu =$

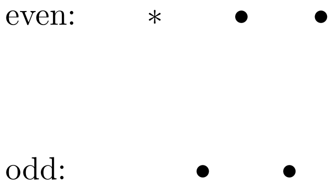


That is, the index is equal to the square of the eigenvalue.

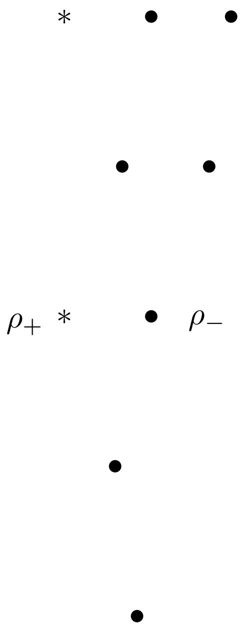
II. String algebra and graph geometry

II.1. Strings and operations

Let $\mathcal{G}$ be an unoriented bipartite finite graph with a distinguished vertex.



We will see how a graph is viewed as a manifold, which is called a *granifold*. The adjacency matrix Δ plays a role of the Laplacian, and the vector μ the harmonic non-negative function. An (oriented) path on $\mathcal{G}$ is a succession of edges: $\xi = (v_1, v_2, \ldots, v_n)$, where $r(v_i) = s(v_{i+1})$. We write $\tilde{v}$ for the edge v with the reversed orientation. A string on $\mathcal{G}$ is a pair of paths, $\rho = (\rho_+, \rho_-)$, with $s(\rho_+) = s(\rho_-)$, $r(\rho_+) = r(\rho_-)$, and $|\rho_+| = |\rho_-|$, where $|\cdot|$ denote the length of a path. We draw a picture as follows to denote a string.



Define an algebra $\mathrm{String}^{(n)}\mathcal{G}$ with the linear basis of the n-strings, that is, the strings with length n. We define a multiplication and a $*$-operation by

$$(\rho_+, \rho_-) \cdot (\eta_+, \eta_-) = \delta_{\rho_-, \eta_+}(\rho_+, \eta_-),$$

$$(\rho_+, \rho_-)^* = (\rho_-, \rho_+),$$

which makes $\mathrm{String}^{(n)}\mathcal{G}$ into a finite dimensional C^*-algebra.

For the moment, we work with strings ρ having $s(\rho) = *$, the given distinguished point on the graph $\mathcal{G}$. Define $\mathrm{tr} : \mathrm{String}^{(n)} \to \mathbf{C}$ by $\mathrm{tr}(\rho) = \delta_{\rho_+, \rho_-} \beta^{-|\rho|} \mu(r(\rho))$, where β is the eigenvalue for μ, $\Delta(\mu) = \beta\mu$. Define an embedding $i_n^{n+k} : \mathrm{String}^{(n)}\mathcal{G} \to \mathrm{String}^{(n+k)}\mathcal{G}$ by $i_n^{n+k}(\rho_+, \rho_-) = \sum_{|\xi|=k}(\rho_+ \cdot \xi, \rho_- \cdot \xi)$, where $\rho_\pm \cdot \xi$ denotes the concatenation of the paths. (Remark that the identity in matrix algebra is written as $\sum e_{ii}$.) This i_n^{n+k} is compatible with the trace tr defined above: $\mathrm{tr}(i_n^{n+k}(\rho)) = \mathrm{tr}(\rho)$.

The above tr is the only trace with this property. Let $A_\infty = (\bigcup_n \mathrm{String}^{(n)}, \mathrm{tr})$ and we complete this as follows. (The GNS construction with respect to tr.) Define an inner product on A_∞ by $\langle x, y\rangle = \mathrm{tr}(xy^*)$ to make a pre-Hilbert space. By completion, we get a Hilbert space $L^2(A_\infty, \mathrm{tr})$. The algebra A_∞ acts by multiplication on the left. Taking a weak closure, we get a von Neumann algebra, which is the (Murray-von Neumann) hyperfinite II_1 factor R.

II.2. Connections and embedding

Now start with four graphs $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ with common vertices.

$$\begin{tikzcd}
\cdots \arrow[r, "\mathcal{G}_2"] \arrow[d, "\mathcal{G}_1"'] & \cdots \arrow[d, "\mathcal{G}_3"] \\
\cdots \arrow[r, "\mathcal{G}_4"'] & \cdots
\end{tikzcd}$$

Assume that we have a vector μ : vertices $\to \mathbf{C}$, which is harmonic for each $\mathcal{G}_i$, such that the eigenvalues $\beta_{\mathcal{G}_i}$ and $\beta_{\mathcal{G}_{i+2}}$ are equal. (The index i is in mod 4.) Further assume we have a biunitary connection W. We then construct string algebras.

$$\begin{tikzcd}
* \arrow[r, "\mathcal{G}_2"] \arrow[d, "\mathcal{G}_1"'] & \bullet \arrow[r, "\tilde{\mathcal{G}}_2"] \arrow[d, "\mathcal{G}_3"] & \bullet \cdots \arrow[d, "\mathcal{G}_1"'] \\
\bullet \arrow[r, "\mathcal{G}_4"'] \arrow[d, "\tilde{\mathcal{G}}_1"'] & \bullet \arrow[r, "\tilde{\mathcal{G}}_4"'] \arrow[d, "\tilde{\mathcal{G}}_3"] & \bullet \cdots \arrow[d, "\tilde{\mathcal{G}}_1"'] \\
\bullet \arrow[r, "\mathcal{G}_2"] & \bullet \arrow[r, "\tilde{\mathcal{G}}_2"] & \bullet \cdots \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

Take Hilbert spaces H_1, H_2 with a orthonormal basis:

$$\begin{tikzcd}
\bullet \arrow[d] & & & \bullet \arrow[r] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet & , & & \bullet
\end{tikzcd}$$

We can define a unitary matrix: $H_1 \to H_2$ by W, which determines the matrix entries. We write the following picture for this unitary.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \arrow[dl, phantom, "\swarrow W"] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

We can use these unitaries for several times for identifying Hilbert spaces. The result does not depend on the order in which we use W.

$$\begin{tikzcd}
* \arrow[r] & \bullet \arrow[d] & & * \arrow[d] & & & * \arrow[d] & & & * \arrow[d] & \\
& \bullet \arrow[d] & & \bullet \arrow[r] & \bullet \arrow[d] & & \bullet \arrow[d] & & & \bullet \arrow[d] & \\
& \bullet \arrow[d] & & & \bullet \arrow[d] & & \bullet \arrow[r] & \bullet \arrow[d] & & \bullet \arrow[d] & \\
& \vdots \arrow[d] & \longrightarrow & & \vdots \arrow[d] & \longrightarrow & & \vdots \arrow[d] & \longrightarrow\cdots & \vdots \arrow[d] & \\
& \bullet & & & \bullet & & & \bullet & & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Thus all the Hilbert spaces of paths of the following form are unitarily equivalent using W.

$$\begin{tikzcd}
\overset{(0,0)}{*} \arrow[d] & & \\
\bullet \arrow[d] & & \\
\bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[d] \\
& & \vdots \arrow[d] \\
& & \underset{(k,n)}{\bullet}
\end{tikzcd}$$

The corresponding string algebras are isomorphic to each other, using $\mathrm{Ad}(W)$. For any $k, n \geq 0$, we have an algebra $A_{k,n}$ given by strings from $* = (0, 0)$ to (k, n). The trace is compatible with this identification.

Note that $A_{k,n}$ is embedded into $A_{k,n+1}$ just like before. Similarly, $A_{k,n} \subset A_{k+1,n}$. We have a double complex of algebras.

$$\begin{array}{cccccc}
 & \vdots & & \vdots & & \\
\cdots & A_{k,n} & \subset & A_{k,n+1} & \cdots & A_{k,\infty} \\
 & \cap & & \cap & & \\
\cdots & A_{k+1,n} & \subset & A_{k+1,n+1} & \cdots & A_{k+1,\infty} \\
 & \vdots & & \vdots & & \\
\cdots & A_{\infty,n} & \subset & A_{\infty,n+1} & \cdots &
\end{array}$$

The inclusion $A_{0,\infty} \subset A_{1,\infty}$ is our string model for $N \subset M$. Note that by definition of identification using W, the coefficient of the embedding $\rho = (\rho_+, \rho_-) \in A_{0,n} \mapsto \sum c_{\rho,\eta}\eta$, where $c_{\rho,\eta} \in \mathbf{C}$, and a string $\eta = (\eta_+, \eta_-) \in A_{k,n}$ is given in the form of vertical k-paths composed with horizontal n-paths, is given by

$$c_{\rho,\eta} = \sum_{\text{configurations}} \prod_{\text{cells}} W(\text{cell}).$$

Here a "configuration" means a choice of cells which fill the following $k \times 2n$-diagram.

$$\begin{tikzcd}
* \arrow[r] \arrow[d, "\eta_+"'] & \bullet \arrow[r, dash, dotted, "\rho_+"] & \bullet \arrow[r] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[r, dash, dotted, "\rho_-"] & \bullet & * \arrow[l] \arrow[d, "\eta_-"] \\
\vdots \arrow[d, "\eta_+"'] & & & \vdots \arrow[d] & & & \vdots \arrow[d, "\eta_-"] \\
\bullet \arrow[r] & \bullet \arrow[r, dash, dotted, "\eta_+"'] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r, dash, dotted, "\eta_-"'] & \bullet & \bullet \arrow[l]
\end{tikzcd}$$

This is just by the definition of the embeddings and matrix multiplication rule. We also denote this coefficient $c_{\rho,\eta}$ by

$$W\left(\begin{tikzcd}
* \arrow[r] \arrow[d] & \cdots & * \arrow[l] \arrow[d] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
\bullet \arrow[r] & \cdots & \bullet \arrow[l]
\end{tikzcd}\right).$$

This is a trace-preserving homomorphism. Most of known subfactors are from this construction. Note that the string algebra construction works for any four graphs with biunitary connection *without* assuming flatness which will be explained in II.5. (This construction is equivalent to that based on commuting squares. The commuting square condition corresponds to biunitarity.) But there is a following example, which does not come from the string algebra.

Example. Choose a non-amenable finitely generated group G and consider the non-commutative Bernoulli shift α of G on the hyperfinite II_1 factor R. This actions is ergodic on the central sequence algebra by Jones [J2]. Consider a subfactor

$$N = \left\{ \begin{pmatrix} x & & & \\ & \alpha_{g_1(x)} & & \\ & & \ddots & \\ & & & \alpha_{g_n}(x) \end{pmatrix} \middle| \; x \in R \right\} \subset M = R \otimes \mathrm{Mat}_{n+1},$$

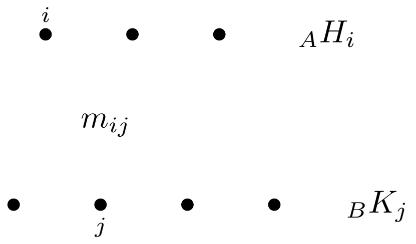
where $g_1, \ldots g_n$ are generators of G. By ergodicity on the central sequence algebra, we get $N^\omega \cap M' = \mathbf{C}$. But for string algebra construction, we always get the sequence of Jones projections $(e_n)_n$ (on the horizontal string algebras) in $N^\omega \cap M'$. (See II.3 below.) Thus this subfactor does not arise from the string algebra construction.

The main theorem states that the Galois functor and the string algebra construction are inverses to each other under good conditions.

Also see Roche [R] for a string algebra construction.

II.3. Bratteli diagrams and Jones projections

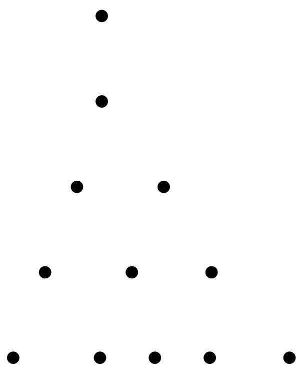
Now we move to local analysis of finite dimensional algebras. Let A be a finite dimensional C^*-algebra. Then A is isomorphic to $\bigoplus_n B(H_n) \cong \bigoplus_n H_n^* \otimes H_n$, where each H_n is a finite dimensional Hilbert space. The irreducible A-modules are, up to isomorphism, ${}_AH_n$'s. If there is a homomorphism $\varphi : A \to B$ with $\varphi(1) = 1$, ${}_AH_i$ is an A-module, and ${}_BK_j$ is a B-module, then K_j becomes an A-module using φ. Then ${}_AK_j \cong \bigoplus_i m_{ij\,A}H_i$, for some integers m_{ij}. We draw the following diagram for this.



For each edge v, choose an intertwiner (partial isometries with mutually orthogonal ranges) $T(v) : {}_AH_i \to {}_AK_j$, that is, with relations

$$T(v)^*T(v) = 1_{H_i}, \qquad \sum_{v, r(v)=j} T(v)T(v)^* = 1_{K_j}.$$

Now suppose one has an inductive system $A_0 = \mathbf{C} \subset A_1 \subset A_2 \subset \cdots$ of finite dimensional algebras. Make a graph $\mathcal{G}$ with vertices on level n corresponding to irreducible A_n-modules and edges corresponding to intertwiners as above.



If an oriented graph $\mathcal{G}$ with a distinguished point is given, we let A_n be the n-string algebra with source $*$. We assume that $\mathcal{G}$ has no dead ends (i.e., every vertex is a source) and every vertex is accessible (i.e., the range of a path starting at $*$). The two functors (correspondences between the system of algebras and the graph) are inverse to each to other.

Start with (A_n), $A_0 = \mathbf{C}$, build its Bratteli diagram $\mathcal{G}$ as above, and denote for each vertex x of $\mathcal{G}$, $T(x)$ the corresponding module and for each edge v, $T(v)$ intertwiner. Now build the string algebra $(\mathrm{String}^{(n)}\mathcal{G})_n$ on $\mathcal{G}$. For every string $\rho = (v_1 \cdot v_2 \cdots v_n, w_1 \cdot w_2 \cdots w_n)$, the map

$$T(\rho) = T(v_n)T(v_{n-1}) \cdots T(v_1)T(w_1)^* \cdots T(w_n)^*$$

is from $T(x)$ to $T(x)$, hence $T(\rho) \in B(T(x)) \subset A_n = \bigoplus_x B(T(x))$.

For the other way, start with a graph which is a Bratteli diagram; it is $\mathbf{N}$-graded, on the level 0 it has one vertex $*$, and every edge v has $\mathrm{level}(r(v)) = \mathrm{level}(s(v)) + 1$, each vertex is accessible, and there are no dead ends.

Build the string algebras $(\mathrm{String}^{(n)}\mathcal{G})$. We want to show that the invariant of this inductive system is again $\mathcal{G}$.

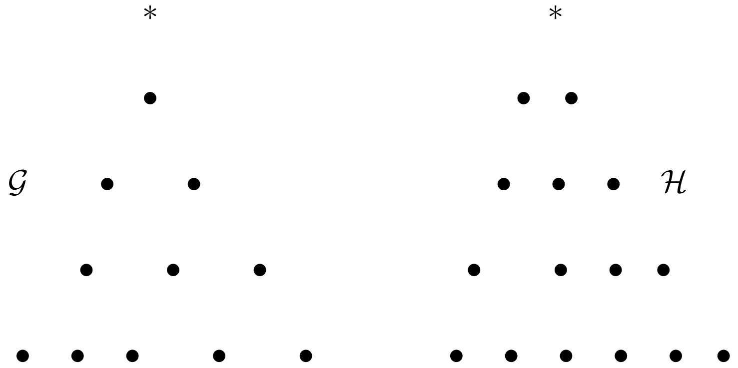
For each vertex x of $\mathcal{G}$, let $\mathrm{Path}^{(n)}_{*,x}\mathcal{G}$ be the Hilbert space with orthonormal basis consisting of the paths on $\mathcal{G}$ from $*$ to x, $n =$ level of x.

Now $\mathrm{Path}^{(n)}_{*,x}\mathcal{G}$ is a left module for $\mathrm{String}^{(n)}\mathcal{G}$ by $(\rho_+, \rho_-) \cdot \xi = \delta_{\rho_-,\xi}\rho_+$. This module is irreducible and

$$\mathrm{String}^{(n)}\mathcal{G} = \bigoplus_{\mathrm{level}(x)=n} (\mathrm{Path}^{(n)}\mathcal{G})^* \otimes \mathrm{Path}^{(n)}\mathcal{G}$$

by the correspondence $(\rho_+, \rho_-) \leftrightarrow \rho_+^* \otimes \rho_-$. The intertwiner $T(v)$ corresponding to an edge $x \overset{v}{\to} y$ maps $\xi \in \mathrm{Path}_x^{(n)}\mathcal{G}$ into $\xi \cdot v \in \mathrm{Path}_y^{(n+1)}\mathcal{G}$, that is, "adding an edge".

The systems $(A_n), (B_n)$ are inductive systems of algebras and $\varphi : \bigcup A_n \to \bigcup B_n$ is a homomorphism with $\varphi(A_n) \subset B_n$. We obtain, besides the graphs, ($\mathcal{G}$ for (A_n) and $\mathcal{H}$ for (B_n)), a graph $\mathcal{I}$ with edges having source in $\mathcal{G}$ and range in $\mathcal{H}$.



For every cell

$$\begin{tikzcd}
x \arrow[r, "\mathcal{I}", "c"'] \arrow[d, "\mathcal{G}"', "a"] & \bullet \arrow[d, "b"', "\mathcal{H}"] \\
\bullet \arrow[r, "d", "\mathcal{I}"'] & y,
\end{tikzcd}$$

we have an intertwiner

$$T(c)^* T(b)^* T(d) T(a) : T(x) \to T(x).$$

Since $T(x)$ is irreducible, this is a scalar, and we write $W\left(\begin{matrix} \swarrow & \searrow \\ \searrow & \swarrow \end{matrix}\right) \in \mathbf{C}$.

If we keep x, y fixed, then $\{T(b)T(c)\}_{b,c}$ and $\{T(d)T(a)\}_{d,a}$ are both basis for $\mathrm{Hom}_{A_n}(T(x), T(y))$, so $W\left(\begin{tikzcd}[sep=small] & x \arrow[dr] & \\ {} \arrow[ur] \arrow[dr] & & {} \\ & y \arrow[ur] & \end{tikzcd}\right)$ gives a unitary matrix. Now $\varphi(\rho_+, \rho_-) = \sum c_{\rho,\eta}(\eta_+, \eta_-)$, where $(\rho_+, \rho_-) \in \mathrm{String}^{(n)}\mathcal{G}$, $(\eta_+, \eta_-) \in \mathrm{String}^{(n)}\mathcal{H}$, and

$$c_{\rho,\eta} = \sum_{\text{configuration}} \prod_{\text{cells}} W(\text{cell}).$$

A configuration is for the following diagram.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\rho_+"'] & \cdots \arrow[r] & \bullet \arrow[d, "\eta_+"] \\
\vdots & & \vdots \\
\bullet \arrow[r] \arrow[u, "\rho_-"] & \cdots \arrow[r] & \bullet \arrow[u, "\eta_-"']
\end{tikzcd}$$

Here we have a similar convention for the cells in the lower half of the diagram to that in I.3. That is,

$$W\left(\begin{tikzcd} b \arrow[r] & d \\ a \arrow[u] \arrow[r] & c \arrow[u] \end{tikzcd}\right) = \overline{W\left(\begin{tikzcd} a \arrow[r] \arrow[d] & c \arrow[d] \\ b \arrow[r] & d \end{tikzcd}\right)}$$

(Compare this to partition functions in statistical mechanics.) Let $\mathcal{G}$ be a connected finite unoriented graph with a distinguished vertex x and consider the Laplacian $\Delta \equiv \Delta_{\mathcal{G}}$, and let μ be its unique harmonic measure $\Delta\mu = \beta\mu$ on vertices of $\mathcal{G}$ with $\mu(*) = 1$. We construct *Jones projections* $e_n \in \mathrm{String}_*^{(n+1)}$ by

$$e_n = \sum_{\substack{|\alpha|=n-1 \\ |v|=|w|=1}} \frac{\mu(r(v))^{1/2}\mu(r(w))^{1/2}}{\mu(r(\alpha))}(\alpha \cdot v \cdot \tilde{v}, \alpha \cdot w \cdot \tilde{w}).$$

Then we get, by simple computations,

$$\text{(I)}\qquad \begin{gathered} e_i e_j = e_j e_i, \qquad |i-j| \neq 1, \\ e_i = e_i^* = e_i^2, \\ e_i e_{i\pm 1} e_i = \beta^{-2} e_i. \end{gathered}$$

Recall that

$$\mathrm{tr}(\rho) = \delta_{\rho_+,\rho_-} \beta^{-|\rho|} \mu(r(\rho))$$

is a trace on String$(\mathcal{G})$. This give a conditional expectation

$$\begin{aligned} &E_{n-1} : \mathrm{String}^{(n)}\mathcal{G} \to \mathrm{String}^{(n-1)}\mathcal{G}, \\ &\mathrm{tr}(ab) = \mathrm{tr}(aE_{n-1}(b)), \qquad a \in \mathrm{String}^{(n-1)}\mathcal{G}, \;\; b \in \mathrm{String}^{(n)}\mathcal{G}. \end{aligned}$$

Expressing $\rho = (\rho_+, \rho_-)$ as $\rho_+ = \eta_+ \cdot v_+$, $\rho_- = \eta_- \cdot v_-$, $|v_+| = |v_-| = 1$, we get

$$E_{n-1}(\rho) = \beta^{-1} \delta_{v_+,v_-} \mu(r(v_+)) \mu(s(v_+))^{-1} \cdot (\eta_+, \eta_-).$$

Now we also have

$$\text{(II)}\qquad \begin{gathered} e_n \in (\mathrm{String}_*^{(n-1)}\mathcal{G})' \cap \mathrm{String}_*^{(n+1)}\mathcal{G}, \\ e_n \rho e_n = E_{n-1}(\rho) e_n, \qquad \text{for all } \rho \in \mathrm{String}^{(n)}\mathcal{G}. \end{gathered}$$

Theorem. *If the inductive system* (A_n) *of finite dimensional algebras is endowed with a trace tr, and there are projections* $e_n \in A'_{n-1} \cap A_{n+1}$ *satisfying (I), (II), then* A_n *is isomorphic to the string algebras on an unoriented graph* $\mathcal{G}$.

II.4. Infinite graphs

Let $\mathcal{G}, \mu, \beta$ as above. Even if $\mathcal{G}$ is not finite but just locally finite, there exits an eigenvector μ for $\Delta_{\mathcal{G}}$ as follows. Let $\mathcal{G}_n$ be a system of increasing graphs, containing $*$, whose union is $\mathcal{G}$. Then there exists μ_n for each n such that

$$\Delta_{\mathcal{G}_n} \mu_n = \beta_n \mu_n,$$

$$\mu_n(*) = 1.$$

Because $\{\mu_n(x)\}_n$ is bounded at each x, there exists a weak limit μ with $\Delta_{\mathcal{G}} \mu = (\lim \beta_n) \mu = \beta \mu$, where $\beta = \|\Delta_{\mathcal{G}}\|$. But unlike finite graph cases, this μ is not unique as the following example for the graph A_∞ shows.

$$* = 1 \text{———} 2 \text{———} 3 \text{———} 4 \text{———} \cdots, \qquad \Delta \mu = 2\mu,$$

$$* = 1 \text{———} 3 \text{———} 8 \text{———} 21 \text{———} \cdots, \qquad \Delta \mu = 3\mu,$$

For any unoriented graph $\mathcal{G}$, setting

$$m = \sup_{x \in \mathrm{vertex}(\mathcal{G})} \#\{e \in \mathrm{edge}(\mathcal{G}) \mid s(e) = x\},$$

we get $m^{1/2} \leq \|\mathcal{G}\| \leq m$. In general, if $\Delta \mu = \beta \mu$, we get $\beta \geq \|\Delta_{\mathcal{G}}\|$.

II.5. Transport and flatness

We consider an auxiliary graph $\mathcal{H}$ in addition to $\mathcal{G}$, and now our four graphs are $\mathcal{G}_1 = \mathcal{G}, \mathcal{G}_2 = \mathcal{H}, \mathcal{G}_3 = \mathcal{G}$, and $\mathcal{G}_4 = \mathcal{H}$. Let W be a biunitary connection on these. Our basic observation is that strings are similar to tensors. For example, a string $\rho = (v_1 \cdot v_2 \cdot v_3, w_1 \cdot w_2 \cdot w_3)$ corresponds to a tensor $dv_1 \otimes dv_2 \otimes dv_3 \otimes \frac{d}{dw_3} \otimes \frac{d}{dw_2} \otimes \frac{d}{dw_1}$, where the first three components are covariant and the other three are contravariant. The string product corresponds to the tensor product as associative algebras.

Transport along (ξ_+, ξ_-) is defined the formula

$$\mathrm{Transp}_{(x,y)}^{(\xi_+,\xi_-)}(\rho) = \sum c_{\rho,\eta}^{(\xi_+,\xi_-)} \eta,$$

where $x = s(\rho) = s(\xi_+) = s(\xi_-)$, $y = r(\xi_+) = r(\xi_-) = s(\eta)$, and the coefficient $c_{\rho,\eta}^{(\xi_+,\xi_-)} \in \mathbf{C}$ is determined by conjugation using W as before. This is *not* a homomorphism for general ξ_+, ξ_-. If we make ξ_+, ξ_- vary and consider the map

$$\rho \mapsto \sum_{\xi_+,\eta_+,\xi_-,\eta_-} (\text{coefficient})(\xi_+ \cdot \eta_+, \xi_- \cdot \eta_-),$$

then it is a homomorphism, but not for a fixed (ξ_+, ξ_-). The map $\mathrm{Transp}_{x,y}^{(\xi_+,\xi_-)}$ is just completely positive in general.

A *field of strings* on $\mathcal{G}$ is a map $x \in \mathrm{vertex}(\mathcal{G}) \mapsto f_x \in \mathrm{String}_x \mathcal{G}$. They form an algebra naturally. A field f is said to be *flat* if for all vertices x, y of $\mathcal{G}$ which can be joined by $\mathcal{H}$ and any paths ξ_+, ξ_- of the same length from x to y on $\mathcal{H}$, we have

$$\mathrm{Transp}_{x,y}^{(\xi_+,\xi_-)}(f_x) = \delta_{\xi_+,\xi_-} f_y.$$

Flat fields form a $*$-subalgebra of all the fields. Note that the restriction of flat fields $(f_x)_x \mapsto f_* \in \mathrm{String}_*\mathcal{G}$ is an injective homomorphism by flatness. If this restriction for flat fields on $\mathcal{G}_1^{(\mathrm{even})}$ with respect to the horizontal graph $\mathcal{G}_2 \cdot \tilde{\mathcal{G}}_2$ is surjective in the string algebra construction, we say the connection on the four graphs is flat. Note that such a flat field of k-strings is characterized by the property that it is embedded into a vertical k-string algebra $A'_{0,2n} \cap A_{k,2n}$ with the same form of strings for all $2n \geq$ depth. (Also see the proof of the theorem in II.6.) By this, flatness of a connection means that $A_{k,0} \subset A'_{0,n} \cap A_{k,n}$ for all k, n. This condition is equivalent to saying that $A_{0,\infty}$ and $A_{\infty,0}$ commute. (In this way, it is easy to see a connection arising via the Galois functor satisfies flatness.) Note that $A'_{0,2n} \cap A_{k,2n}$ is spanned by elements of the form $\sum_\zeta (\zeta \cdot \eta_+, \zeta \cdot \eta_-)$, where ζ is a horizontal $2n$-path and η is a vertical k-string. Thus, seeing a coefficient of a vertical k-string (ξ_+, ξ_-) embedded in $A_{k,2n}$, flatness can be stated in the form

$$(*) \quad W \left(\begin{tikzcd} * \arrow[r] \arrow[d, "\zeta_+"'] & \overset{\xi_+}{\cdots} \arrow[r] & \bullet \arrow[d] & \overset{\xi_-}{\cdots} \arrow[l] & * \arrow[l] \arrow[d, "\zeta_-"] \\ \vdots \arrow[d, "\zeta_+"'] & & \vdots \arrow[d] & & \vdots \arrow[d, "\zeta_-"] \\ x \arrow[r] & \underset{\eta_+}{\cdots} \arrow[r] & \bullet & \underset{\eta_-}{\cdots} \arrow[l] & y \arrow[l] \end{tikzcd} \right) = \delta_{\zeta_+,\zeta_-} C_{\xi,\eta},$$

where $C_{\xi,\eta} \in \mathbf{C}$ depends only on $\xi = (\xi_+, \xi_-), \eta = (\eta_+, \eta_-)$. This form is mentioned in Remarks on page 153 of [O1].

Example. The field $id^{(n)}$ is given by $\sum_{s(\xi)=x, |\xi|=n} (\xi, \xi)$ at every point x. Using the biunitarity $\sum W(i,j)\overline{W(k,j)} = \delta_{i,j}$, we get

$$\mathrm{Transp}_{x,y}^{(\xi_+,\xi_-)} id_x^{(n)} = \delta_{\xi_+,\xi_-} id_y^{(n)},$$

which shows flatness, because we have

$$\sum_{\zeta} W\left(\begin{tikzcd} x \arrow[r, "\xi_+"] \arrow[d, "\zeta"'] & y \arrow[d, "\eta_+"] \\ \bullet & \bullet \\ x \arrow[r, "\xi_-"] \arrow[u, "\zeta"] & y \arrow[u, "\eta_-"'] \end{tikzcd}\right) = \delta_{\eta_+,\eta_-}\delta_{\xi_+,\xi_-}.$$

(The reason is as follows. Around the left middle •, we get, by biunitarity,

$$\sum_{v,w} W\left(\begin{tikzcd} \cdot \arrow[r] \arrow[d, "v"'] & \cdot \arrow[d] \\ \bullet \arrow[r, "w"] & \cdot \\ \cdot \arrow[r] \arrow[u, "v"] & \cdot \arrow[u] \end{tikzcd}\right) = \delta_{\substack{\rightarrow\\ \ \downarrow}, \substack{\ \uparrow\\ \rightarrow}},$$

where v, w are edges. Repeating this procedure, we get the above formula.)

Example. Let $e_{n,x}$ be the n-th Jones projection at x and e_n the corresponding field. This field is flat. By the same kind of computation as above, we get flatness of e_n. Note that the coefficients of the Jones projections come from the renormalization rule:

$$W\left(\begin{tikzcd} a \arrow[r, "\xi_1"] & b \\ c \arrow[u, "\tilde{\eta}_1"] \arrow[r, "\eta_2"'] & d \arrow[u, "\tilde{\xi}_2"'] \end{tikzcd}\right) = \sqrt{\frac{\mu(a)\mu(d)}{\mu(b)\mu(c)}}\, W\left(\begin{tikzcd} a \arrow[r, "\xi_1"] \arrow[d, "\eta_1"'] & b \arrow[d, "\xi_2"] \\ c \arrow[r, "\eta_2"'] & d \end{tikzcd}\right),$$

for four edges $\xi_1, \xi_2, \eta_1, \eta_2$.

Remark. Because for $k \geq$ depth, $A_{k+1,0}$ is generated by $A_{k,0}$ and the Jones the projection, which is flat, we need to check the formula $(*)$ for flatness only for $2n \times 2k$-diagrams with $k, n \leq$ depth. Thus flatness for a given connection on finite graphs can be checked by finite times of computations. This corresponds to a remark on page 154 of [O1].

Starting from $\mathcal{G}, \mathcal{H}, W$, we get a subfactor $N = A_{0,\infty} \subset M = A_{1,\infty}$. From this subfactor, we get a principal graph $\mathcal{K}$ via the Galois functor. We will later show that this $\mathcal{K}$ is given in terms of flatness. If the algebra of flat fields are spanned by e_n, then the graph $\mathcal{K}$ is A_∞. This situation often happens in generic cases and means that the subfactor is non-amenable in the sense that spanning condition does not hold. (See [O1].)

The smallest known index value above 4 for an irreducible subfactor is $4.026\cdots$, and this is obtained for $\mathcal{G} = E_{10}$, $\mathcal{H} = P(E_{10})$, where P is a certain polynomial of degree 10. (See [HSO]).

Theorem. *This subfactor with index*$= 4.026\cdots$ *has* A_∞ *as a principal graph, In other words, the higher relative commutants consist of only Jones projections. Hence this subfactor is non-amenable.*

The proof will be given in IV.4.

Example. Consider the following subfactor.

$$N = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & \alpha(x) & 0 \\ 0 & 0 & \beta(x) \end{pmatrix} \middle| \, x \in R \right\} \subset M = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \middle| \, a, b, c, d, e, f, g, h, i \in R \right\}. \blacksquare$$

Here α, β gives an outer action of F_2, the free group of two generators. Then $N' \cap M = \mathbf{C}^3$, and the higher relative commutants are given by the Cayley graph.

The norm of this graph is $1+\sqrt{2}$, and the eigenvalue is $3 > 1+\sqrt{2}$.

Example. Consider the following subfactor.

$$N = \left\{ \begin{pmatrix} x & 0 \\ 0 & \alpha(x) \end{pmatrix} \,\middle|\, x \in R \right\} \subset M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in R \right\}.$$

We have index $= 4$. If $\alpha^n = \mathrm{Ad}(u) \in \mathrm{Int}(R)$ for some $n > 0$, then the graph obtained by the Galois functor is four times $\tilde{A}_{2n-1}$. (See IV.1 for this notation.) We have a one-parameter family of biunitarity connection on this graph parametrized by λ. Flatness forces $\lambda^{2n} = 1$. If $\alpha(u) = \gamma u$ in the above, then $\gamma^n = 1$. (Connes' obstruction.) Here we have $\gamma = \lambda^2$. If we use a λ without $\lambda^{2n} = 1$ for construction, then we get a free α and the graph A_∞ as an invariant.

In the basic construction,

$$M_0 \subset M_1 \subset \mathrm{End}({}_{M_0}M_1) \cong M_1 \otimes_{M_0} M_1,$$

we get $\mathrm{End}(M_1) \cong B(L^2(M_1))$ and $\mathrm{End}({}_{M_0}M_1) \cong M_0' \cap B(L^2(M_1))$. (cf. Sauvageot.) In the above isomorphism, we get $e_0 \leftrightarrow 1 \otimes_{m_0} 1$ and $1 \leftrightarrow \sum \lambda_i^* \otimes_{M_0} \lambda_i$, where λ_i's make a basis for ${}_{M_0}M_1$. We then have

$$M_0' \cap M_2 = M_0' \cap \mathrm{End}({}_{M_0}M_1) = \mathrm{End}({}_{M_0}M_{1M_0}).$$

In this way, we get a correspondence between the towers of relative commutants and endomorphisms of bimodules.

We have chosen a basis in $\mathrm{End}({}_{M_0}M_{kM_0})$. This basis is $\mathrm{String}_*^{(2k)}$:

$$\begin{aligned} M_0' \cap M_n &\cong \mathrm{String}_*^{(n)} (\text{the principal graph}), \\ M_1' \cap M_{n+1} &\cong \mathrm{String}_*^{(n)} (\text{the dual graph}). \end{aligned}$$

II.6. Computation of towers of relative commutants and flatness

Consider a subfactor $A_{0,\infty} = M_0 \subset A_{1,\infty} = M_1$ constructed as the string algebra on four graphs. Using the Jones projection on the vertical algebra, we can easily show that the inclusion

$$A_{0,\infty} = M_0 \subset A_{1,\infty} = M_1 \subset A_{2,\infty} = M_2$$

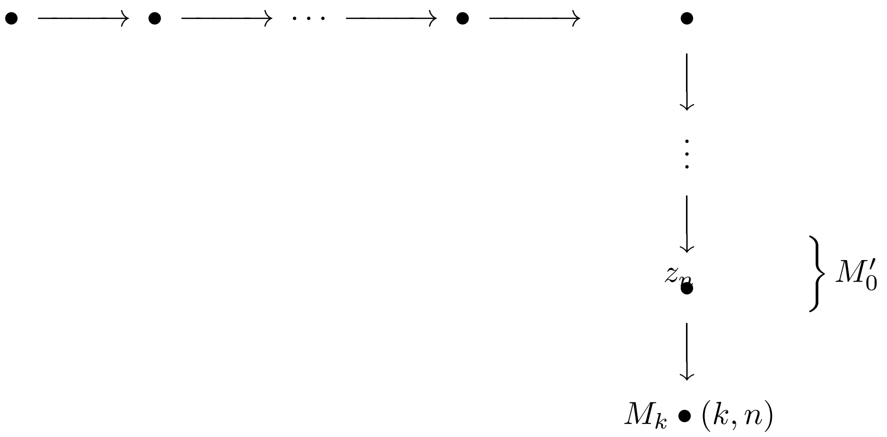
is standard, that is, the third algebra is obtained as the basic construction from the first two. Similarly,

$$A_{0,\infty} = M_0 \subset A_{1,\infty} = M_1 \subset A_{2,\infty} = M_2 \subset A_{3,\infty} = M_3 \cdots$$

is the Jones tower. The problem is what $M_0' \cap M_k = A_{0,\infty}' \cap A_{k,\infty}$ is. We show the following theorem.

Theorem. *In the string algebra construction from finite graphs $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$, the relative commutant $A'_{0,\infty} \cap A_{k,\infty}$ is given by fields of flat k-strings on $\mathcal{G}_1^{(\mathrm{even})}$ [resp. $\mathcal{G}_3^{(\mathrm{odd})}$] with respect to horizontal graph $\mathcal{G}_2 \cdot \tilde{\mathcal{G}}_2$ [resp. $\tilde{\mathcal{G}}_2 \cdot \mathcal{G}_2$]. In other words, $A'_{0,\infty} \cap A_{k,\infty} = (A'_{0,n} \cap A_{k,n}) \cap (A'_{0,n+2} \cap A_{k,n+2})$ for all $n \geq$ depth.*

Proof. Let $z^0 \in A'_{0,\infty} \cap A_{k,\infty}$ and set $z_n = E_{A_{k,n}}(z^0)$. Then $z_n \in A'_{0,n} \cap A_{k,n}$.



Note that for $2n \geq$ depth each z_{2n} is in the copy of k-string field algebra A on $\mathcal{G}_1^{(\mathrm{even})}$ at the $2n$-th place. Define a map φ_{2n} to be the copying of strings from $A'_{0,2n} \cap A_{k,2n}$ to this finite dimensional algebra A. We apply compactness argument to this finite dimensional algebra A as follows. By compactness, we may assume that there is a sequence $\{n_j\}$ such that $\varphi_{2n_j}(z_{2n_j}) \to z$, $\varphi_{2n_j+2}(z_{2n_j+2}) \to z'$ for some $z, z' \in A$ as $j \to \infty$. (Note that $\|z_n\|$ is bounded.) Consider the following diagram.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[rr, phantom, bend left, "\text{length } 2"] & \bullet \arrow[r] & \bullet \arrow[d] \\
A \vdots \arrow[d] & & \vdots A \arrow[d] \\
\bullet \arrow[r] \arrow[rr, phantom, bend right, "\text{length } 2"'] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

That is, we have two A's vertically, and two 2-string algebras horizontally. Because $\|z_{2n} - z_{2n+2}\|_2 \to 0$ as $n \to \infty$, we know that $z \cdot id^{(2)} =$
$$\begin{tikzcd}
\bullet \arrow[d, "z"'] & \\
\bullet \arrow[r, "id^{(2)}"'] & \bullet
\end{tikzcd}$$
is identified to $id^{(2)} \cdot z' =$
$$\begin{tikzcd}
\bullet \arrow[r, "id^{(2)}"] & \bullet \arrow[d, "z'"] \\
& \bullet
\end{tikzcd}$$
via connection W. We write $z \cdot id^{(2)} = id^{(2)} \cdot z'$ for this. We will show that z and z' are the same as strings in A. Let e be the Jones projection in the upper horizontal string algebra in the above diagram. By flatness of the Jones projection, this e has the same form as the original e in the lower horizontal string algebras after identification via W, that is, $e \cdot id_{(k)} = id^{(k)} \cdot e$, where $id^{(k)}$ denotes the identity field of k-strings in A. On the other hand, we can show easily $z' \cdot e = e \cdot z'$ under W by induction on k. (The computation for this is same as that for flatness of the Jones projection.) This implies

$$(z \cdot id^{(2)}) \times (id^{(k)} \cdot e) = (id^{(2)} \cdot z') \times (e \cdot id^{(k)}) = e \cdot z' = z' \cdot e = (z' \cdot id^{(2)}) \times (id^{(k)} \cdot e),$$

here "×" means multiplication in the string algebra and "·" means concatenation of strings. Taking a conditional expectation to the left vertical algebra A in the above diagram, we get $z = z'$ in A. This means that for any j we get

$$\begin{tikzcd}
\bullet \arrow[d, "z"'] & & & \bullet \arrow[r, "id^{(2j)}"'] & \bullet \arrow[d, "z"] \\
\bullet \arrow[r, "id^{(2j)}"] & \bullet & = & & \bullet
\end{tikzcd}$$

under identification via W, where $id^{(2j)}$ means the identity in the horizontal algebra for $\mathcal{G}_2 \cdot \tilde{\mathcal{G}}_2 \cdots \tilde{\mathcal{G}}_2$. Let (f_x) be a field of strings on $\mathcal{G}_1^{(\text{even})}$ corresponding to z. We will show this field is flat with respect to the horizontal graph $\mathcal{G}_2 \cdot \tilde{\mathcal{G}}_2$. Let x, y be vertices on $\mathcal{G}_1^{(\text{even})}$. We have shown that $z \cdot id^{(2j)} = id^{(2j)} \cdot z$ under W, which implies

$$\sum_{\substack{s(\xi_+)=s(\xi_-)=x \\ r(\xi_+)=r(\xi_-)=y \\ |\xi_+|=|\xi_-|=2j}} (\xi_+, \xi_-) \cdot \mathrm{Transp}_{x,y}^{(\xi_+,\xi_-)}(f_x) = \sum_{\substack{s(\xi)=x, r(\xi)=y \\ |\xi|=2j}} (\xi, \xi) \cdot f_y.$$

Thus for two paths ξ_+, ξ_- on $\mathcal{G}_2$ with even length $2j$ from x to y, we get $\mathrm{Transp}_{x,y}^{(\xi_+,\xi_-)}(f_x) = \delta_{\xi_+,\xi_-} f_y$, which shows flatness of the field (f_x) by definition. We denote by $z \in A'_{0,2n} \cap A_{k,2n}$ the elements denoted by this flat field. It is clear that $z^0 = \lim_j z_{2n_j} = z$.

Conversely, if we have a flat field $z = (f_x)$ of k-strings on $\mathcal{G}_1^{(\text{even})}$ with respect to horizontal graph $\mathcal{G}_2 \cdot \tilde{\mathcal{G}}_2$, this can be embedded in $A'_{0,2n} \cap A_{k,2n}$ for all sufficiently large n with the same form of strings. We denote this by z. It is trivial that $z \in A'_{0,\infty} \cap A_{k,\infty}$.

The statement for $\mathcal{G}_3^{\text{odd}}$ is proved similarly. Q.E.D.

Note that the above theorem contains so-called Wenzl's lemma as a particular case. Indeed, suppose $\mathcal{G}_1^{(\text{even})}$ has a vertex x_0 which has only one edge from x_0.

Because the restriction of the fields:$(f_x)_x \mapsto f_{x_0}$ is an algebra isomorphism by flatness, we get $A'_{0,\infty} \cap A_{1,\infty} \subset \mathrm{String}^{(1)}_{x_0}\mathcal{G}_1 = \mathbf{C}$.

The theorem says that if a connection is flat, we get the same graph back via the Galois functor, which is the Range Theorem in [O1].

Here we show another method of a proof based on amenability of the horizontal graph in the sense that $E_{(id^{(n-1)}\cdot \mathrm{Flat}_m)'}(\tilde{e}_{n-1})$ converges to a scalar $\tilde{\tau}$ in L^2-norm as $m \to \infty$ for all n, where Flat_m denote the field algebra of flat m-stings on the horizontal algebra with respect to vertical transport, $\tilde{e}_n$ is the Jones projection on the horizontal string algebra and $\tilde{\tau} = \beta_{\mathcal{G}_2}^{-2}$.

Suppose $z \in A'_{0,\infty} \cap A_{k,\infty}$ and set $z_n = E_{A_{k,n}}(z)$ again. Now for $n \geq$ depth, $z_n \in A_{k,n}$ can be written as $\sum_{\mathrm{finite}} a_i \tilde{e}_{n-1} b_i$, where $a_i, b_i \in A_{k,n-1}$, $\tilde{e}_{n-1} \in A'_{k,n-2} \cap A_{k,n}$ as well as $\tilde{e}_{n-1} \in A'_{0,n-2} \cap A_{0,n}$ (by flatness of the Jones projection). Note that $z_{n-1} = E_{A_{k,n-1}}(z_n) = \tilde{\tau} \sum a_i b_i$. If $x \in id^{(n-1)} \cdot \mathrm{Flat}_m$, then x can be regarded as an element in $A'_{k,n-1} \cap A_{k,n-1+m}$ by flatness, hence we get $a_i x = x a_i$, that is, $a_i, b_i \in (id^{(n-1)} \cdot \mathrm{Flat}_m)'$. Now we get by amenability that

$$\begin{aligned}
\|z - z_n\|_2 &\geq \|E_{(id^{(n-1)}\cdot \mathrm{Flat}_m)'}(z - z_n)\|_2 \\
&= \|z - E_{(id^{(n-1)}\cdot \mathrm{Flat}_m)'}(z_n)\|_2 \\
&= \|z - \sum a_i E_{(id^{(n-1)}\cdot \mathrm{Flat}_m)'}(e_{n-1}) b_i\|_2 \\
&\to \|z - \tilde{\tau} \sum a_i b_i\|_2, \qquad \text{as } m \to \infty \\
&= \|z - z_{n-1}\|_2.
\end{aligned}$$

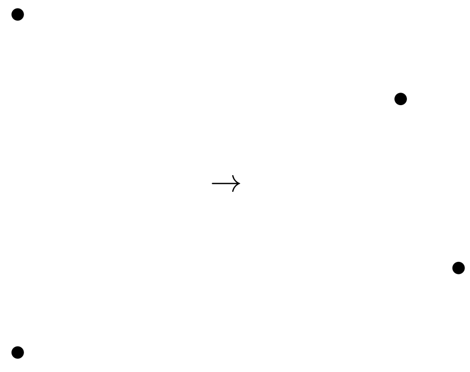
This implies $\|z - z_n\|_2 \geq \|z - z_{n-1}\|_2 \geq 0$. Because the sequence $\{\|z - z_n\|_2\}_n$ is an increasing sequence of positive numbers converging to 0, all the terms must be 0. Then we get the conclusion as in the proof of the theorem above.

We get the amenability for a finite graph is obtained by Perron-Frobenius theory, because we can write down an explicit formula for $E_{(id^{(n-1)}\cdot\mathrm{Flat}_m)'}(\tilde{e}_{n-1})$ as in [O2]. The last part of [J] shows the lack of amenability for A_∞. This amenability is related to random walk and path ergodicity. (See [O1, O2].)

II.7. Fourier transform and convolutions

Flatness is an algebraic property. Flat fields form an algebra; if f, g are flat, then the product fg is also flat. If $f \in \mathrm{Flat}^{(n)}$, then $E_{\mathrm{Flat}^{(n-1)}}(f) \in \mathrm{Flat}^{(n-1)}$. Concatenations of flat fields are also flat; if f, g are flat, then the concatenation $f \cdot g$ is also flat. Here the concatenation is defined by $(\rho_+, \rho_-) \cdot (\eta_+, \eta_-) = (\rho_+ \cdot \eta_+, \rho_- \cdot \eta_-)$ and extended linearly. The embedding of $\mathrm{String}^{(n)}$ into $\mathrm{String}^{(n+k)}$ is a concatenation by $id^{(k)}$. The map $f \mapsto id^{(1)} \cdot f - f \cdot id^{(1)}$ is similar to Lie derivation.

For group case, we get $\mathrm{String}^{(2)}_* = \ell^\infty(G)$. In this case, rotation with 90° on this algebra gives the Fourier transform. We remark that rotation of a flat field is also a flat field, but we need a number $(\mu(s(\rho))/\mu(r(\rho)))^{1/2}$ to get a flat field. This is an analogue of reversing the orientation: $dx_i \to \dfrac{d}{dx_i}$.



The identity $id^{(2)}$ is given by $\sum_{|\xi|=|\eta|=1}(\xi \cdot \eta, \xi \cdot \eta)$, thus the rotation is given by the formula $\sum(\text{coefficient})(\xi \cdot \tilde{\xi}, \tilde{\eta} \cdot \eta)$. This is β^{-1} times the Jones projection.

Thus, the Jones projection is equal to the Fourier transform of $id^{(2)}$ up to a scalar. For the group case, $L^\infty(G)$ is spanned by f_g, a projection corresponding to $g \in G$. In this case, the Jones projection is f_1. Now the left regular representation algebra $L(G) = \mathbf{C}[G]$ is generated by λ_g, a translation corresponding to $g \in G$. Then the Jones projection is $|G|^{-1} \sum_g \lambda_g$.

For the tower $M_0 \subset M_1 \subset M_2 \subset M_3$ and $x \in M_1' \cap M_3$, the Fourier transform $\mathcal{F}(x)$ is given by $\mathcal{F}(x) = E_{M_2}(xe_0e_1) \in M_0' \cap M_2$. Here E_{M_2} is an analogue to the integral, and e_0e_1 is an analogue to the coefficient for the integration.

Example. If $M_1 = M_0 \rtimes G$, the crossed product by an outer action, then $M_0' \cap M_2 = L^\infty(G), M_1' \cap M_3 = L(G), M_0' \cap M_3 = B(L^2(G))$.

Suppose $f = (f_+^1 \cdot f_+^2, f_-^1 \cdot f_-^2), g = (g_+^1 \cdot g_+^2, g_-^1 \cdot g_-^2) \in \mathrm{String}^{(2n)}$ with $|f_+^1| = |f_+^2| = |f_-^1| = |f_-^2| = |g_+^1| = |g_+^2| = |g_-^1| = |g_-^2| = n$. Then the operation $(f,g) \to \delta_{f_+^2, \tilde{g}_+^1} \delta_{f_-^2, \tilde{g}_-^1} (f_+^1 \cdot g_+^2, f_-^1 \cdot g_-^2)$ with a certain coefficient defines the *convolution*. The convolution of flat fields is again flat. The convolution in $L^\infty(G)$ is given by $f_g * f_h = f_{gh}$, that is, the Fourier transform converts the convolution into the multiplication, as expected.

Let $x, y \in M_0' \cap M_2 = \mathrm{End}({}_N M_N) = N' \cap (M \otimes_N M)$. What is $x * y$? Note that the product of $a \otimes_N b$ and $c \otimes_N d$ is given by $aE_N(bc) \otimes_N d$ as compositions of endomorphisms. The convolution is given by $(a \otimes_N b) * (c \otimes_N d) = ca \otimes_N bd$. Though $(a \otimes_N b)^* = b^* \otimes_N a^*$, another involutions is given by $(a \otimes_N b)^\circ = a^* \otimes_N b^*$. In case $M_0' \cap M_3$ is a factor, these give a Kac algebra structure. (See Herman-Ocneanu for discrete Kac algebra and Herman-Nest-Ocneanu for type III cases.) The Haar weight on $M_0' \cap M_2$ is given by restriction of a weight giving a conditional expectation. The algebras $M_0' \cap M_2$ and $M_1' \cap M_3$ are dual to each other.

For a subfactor $M_0 \subset M_1$, there is a problem whether there is any intermediate subfactor P, $M_0 \subset P \subset M_1$. A subfactor is “simple” if there is no intermediate subfactor, as an analogue of a simple group.

Theorem. *There is a one-to-one correspondence between intermediate subfactors* $M_0 \subset P \subset M_1$ *and non-zero projections* $p \in \mathit{Proj}(M_0' \cap M_2)$ *such that* $\mathcal{F}(p)$ *is a scalar multiple of a projection.*

The trivial intermediate subfactors $M_0 \subset M_0 \subset M_1$ and $M_0 \subset M_1 \subset M_1$ correspond to the Jones projection and 1, respectively.

Choosing a Jones projection $e_{-1} \in M_0$ for the downward basic construction, we get $\mathcal{F}(p) = E_{M_0}(pe_{-1}e_0)$. For example, if $p = e_0$, then $E_{M_0}(e_0e_{-1}e_0) = \tau E_{M_0}(e_0) = \tau^2$, and if $p = 1$, then $E_{M_0}(e_{-1}e_0) = \tau e_{-1}$.

Example. Set $M = M_0 \subset M \otimes M_k(\mathbf{C}) = M_1$. Then $M_2 = M \otimes M_k(\mathbf{C}) \otimes M_k(\mathbf{C})$ and $M_0' \cap M_2 = M_k(\mathbf{C}) \otimes M_k(\mathbf{C})$. The explicit formulas for the product and the convolution in this situation are left as an exercise. Writing $H_k = \mathbf{C}^k$, we can write $M_0' \cap M_2 = H_k \otimes \bar{H}_k \otimes H_k \otimes \bar{H}_k$. Then the Fourier transform is a rotation of these four factors. The Fourier transform maps the identity $\sum \xi_i \otimes \bar{\xi}_i \otimes \xi_j \otimes \bar{\xi}_j$ to $\sum \bar{\xi}_i \otimes \xi_j \otimes \bar{\xi}_j \otimes \xi_i = \sum e_{ij} \otimes e_{ji}$, which is the Jones projection up to a scalar. In this case, intermediate subfactors are of the form $M \subset M \otimes \mathrm{Mat}_l \subset M \otimes \mathrm{Mat}_k$, $k = lm$. A projection in $\mathrm{Mat}_k \otimes \mathrm{Mat}_k$ have the property that $\mathcal{F}(p)$ is a projection up to a scalar exactly when $p = P_{\mathrm{Mat}_l}$. In this case, $\mathcal{F}(p) = P_{\mathrm{Mat}_l'} = P_{\mathrm{Mat}_m}$.

II.8. Knot invariant

The braid group B_n has generators $g_1, \ldots, g_{n-1}$ and the relations

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, \\ g_i g_j &= g_j g_i, \qquad |i-j| \geq 2. \end{aligned}$$

The group B_n is a semidirect product of free groups and S_n as follows. Let P_n be the set of the braids which do not permute endpoints. (These are called pure braids.) Then $P_n = F_{n-1} \ltimes F_{n-2} \ltimes \cdots$, and we have an exact sequence

$$0 \to P_n \to B_n \to S_n \to 0.$$

We get a link by taking a closure of a braid. By Alexandar's theorem, this procedure is surjective. Markov's theorem asserts that closures of two braids give the same link if and only if the two braids are equivalent under the equivalence relation generated by the following two Markov moves.

$$\begin{aligned} &\text{I.} \quad v \cdot w \sim w \cdot v, \\ &\text{II.} \quad v \cdot g_n^{\pm 1} \sim v, \qquad v \in B_n. \end{aligned}$$

We choose complex numbers A, B so that the map $\pi : g_i \mapsto Ae_i + B$ gives a homomorphism from $\langle g_i \rangle_i$ to $\langle 1, e_1, e_2, \ldots \rangle$.

Then the trace tr on $\mathrm{String}^{(n)}$, suitably normalized, yields a knot invariant. Markov move I corresponds to the trace property, and Markov move II to conditional expectation property.

III. Central sequences and asymptotic inclusions.

III.1. Asymptotic inclusions

Let

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\infty = \overline{\bigcup M_i}^{w} .$$

We call the subfactor $M_0 \vee (M_0' \cap M_\infty) \subset M_\infty$ *asymptotic inclusion.*

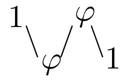
Theorem. *The principal graph for the asymptotic inclusion (for a subfactor with finite depth) is the connected component of* $* = {}_M M_M$ *in the following.*

$$\begin{tikzcd} ({}_M(X_1)_M, {}_M(X_2)_M) \arrow[d] \\ {}_M X_M \end{tikzcd}$$

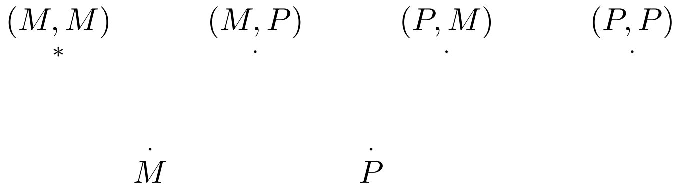
Here X_1, X_2, X *are* M*-*M *bimodules and the number of arrows is a multiplicity of* X *in* $X_1 \otimes_M X_2$*.*

Example. Consider the following graph A_4.

A_4 :



Here $\beta = \varphi = 1.618\cdots$ with $1 + \varphi = \varphi^2$. In this case, we have two kinds of M-M bimodules M and P with the multiplication rules: $M \otimes M = M, M \otimes P \cong P \otimes M \cong P, P \otimes P \cong M \otimes P$. Thus the graph for the asymptotic inclusion is as follows.



This graph is D_6.

Using $e_{-1} \in M'_{-2} \cap M_0$ and $e_{-1} M_0 e_{-1} \cong e_{-1} M_{-2}$, M_{-2} is identified with $id^{(2)} \cdot$Flat. Strings in M_∞ are strings of arbitrary length in both upward and downward direction. The trace tr in M_∞ is given by $\mathrm{tr}(\rho) = \delta_{\rho_+, \rho_-} \beta^{-|\rho|} \mu(r(\rho)) \mu(s(\rho))$. Multiplication in M_∞ is done by transport and the usual multiplication. We get

$$[M_\infty : M_0 \vee (M'_0 \cap M_\infty)] = \sum_{X:\text{irreducible}} (\dim {}_N X_N)^2.$$

Example. In the above example of A_4, we get $[M_\infty, M_0 \vee (M'_0 \cap M_\infty)] = 1 + \varphi^2 = 4\cos^2(\pi/10)$. (Note that the Coxeter number for D_6 is 10.)

Setting $A_{k,l} = M'_k \cap M_l$, we get a commuting square

$$\begin{array}{ccc} A_{-n,0} \vee A_{0,n} & \subset & A_{-n,n} \\ \cap & & \cap \\ A_{-m,0} \vee A_{0,m} & \subset & A_{-m,m}, \end{array}$$

where $0 \leq n \leq m$. This commuting square approximates $M_0 \vee (M'_0 \cap M_\infty) \subset M_\infty$.

III.2. Central sequences

Let $N \subset M$ be a subfactor of the hyperfinite II_1 factor with finite index and finite depth. Let $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$ be a free ultrafilter over $\mathbf{N}$. Set

$$M^\omega = \{(x_n)_n \mid x_n \in M, \sup \|x_n\| < \infty\}/\sim,$$

$$M_\omega = M' \cap M^\omega = \{(x_n)_n \mid [x_n, y] \to 0 \text{ for all } y \in M\}/\sim,$$

where $\sim$ denotes the equivalence relation meaning the termwise difference goes to zero when $n \to \omega$. We get the inclusion $N_\omega \cap M_\omega = N_\omega \cap M' \subset M_\omega$.

Example. Here G denotes a finite group. If $M = N \rtimes G$, then $N_\omega \cap M^\omega = (N_\omega)^G$ and the conditional expectation from M_ω onto $N_\omega \cap M_\omega$ is given by $(x_n)_n \mapsto (E_N(x_n))_n$.

If $N = M^G$ for an outer action of G, then $N_\omega \cap M_\omega = (M_\omega)^G$ and the conditional expectation from M_ω onto $N_\omega \cap M_\omega$ is again given by $(x_n)_n \mapsto (E_N(x_n))_n$, where $E_N(X) = E_{M^G}(x) = |G|^{-1} \sum_g \alpha_g(x)$.

Choose a tunnel:

$$\cdots \subset M_{-n} \subset \cdots \subset M_0 \subset M_1 \subset \cdots M_\infty.$$

The Jones projection e_k satisfies $e_k \in M'_{k-1} \cap M_{k+1}$. Because $(M_0)_\omega \cap (M_{-1})_\omega = M'_0 \cap M^\omega_{-1}$ and $e_{-1} \in M_0$, we get $M'_0 \cap (M_{-1})^\omega \subset M'_0 \cap (M_{-2})^\omega$. Similarly, we get $M'_0 \cap (M_{-1})^\omega \subset M'_0 \cap (M_{-3})^\omega$ and so on. Thus we get $(M_0)_\omega \cap (M_{-1})_\omega = \bigcap_k (M_{-k})^\omega$. We also need the following theorem.

Central Freedom Lemma. *Let M be a type II_1 von Neumann algebra and $P \subset N \subset M$. If P is hyperfinite and $P \cap P' \subset N' \cap N$, then*

$$(P' \cap N^\omega)' \cap M^\omega = P \vee (N' \cap M)^\omega.$$

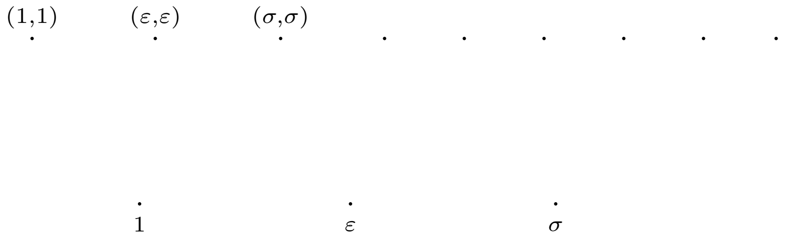
Note that the inclusion "$\supset$" in the above is trivial. By these, we can show the following.

Theorem. *A subfactor $(M_0)_\omega \cap (M_{-1})_\omega \subset (M_0)_\omega$ has the same Galois invariant as the asymptotic inclusion $M_0 \vee (M_0' \cap M_\infty) \subset M_\infty$.*

III.3. Group case

We work on crossed product algebras by finite group actions in detail.

Let $M = N \rtimes G$. For simplicity, assume $G = S_3$. Then the principal graph for the central sequence algebras are obtained by $(\sigma_1, \sigma_2) \in \mathrm{Irr}(G \times G)$ as follows.



For a subfactor $M^G \subset M$, the central sequence algebra has the same invariant as the original subfactor, but not for $N \subset N \rtimes G$. For $N = R \rtimes H \subset R \rtimes G = M$, with H a subgroup of G, we get a principal graph from $\mathrm{Irr}(G)$ and $\mathrm{Irr}(H)$. Thus, if $M = N \rtimes G$, then the subfactor $N_\omega \cap M_\omega \subset M_\omega$ has the same invariant as $P \rtimes G \subset P \rtimes (G \times G)$, where P is some factor and G sits in $G \times G$ diagonally.

Note that we have a tower

$$M = N \rtimes G = \langle N, \lambda_g \rangle \subset M_1 = \langle M, f_g \rangle \subset M_2 = \langle M_1, \rho_g \rangle.$$

Here the action of λ_g, ρ_g is given by $\mathrm{Ad}(\lambda_g)(f_h) = f_{gh}$ and $\mathrm{Ad}(\rho_g)(f_h) = f_{hg^{-1}}$. For the extension

$$N_\omega \cap M_\omega \subset M_\omega \subset \langle M_\omega, f_1 \rangle,$$

we get $\langle M_\omega, f_1 \rangle \subset (N' \cap M_1^\omega)^{\mathrm{Ad}(\lambda_g \cdot \rho_g)}$ because $[M_\omega, \lambda_g] = 0$ ($\lambda_g \in M$), $[M_\omega, \rho_g] = 0$, and $\mathrm{Ad}(\lambda_g \cdot \rho_g)(f_1) = f_{g1g^{-1}} = f_1$. The central freedom lemma implies the action $\mathrm{Ad}(\lambda_g \cdot \rho_g)$ is outer, and we get $\langle M_\omega, f_1 \rangle = (N' \cap M_1^\omega)^{\mathrm{Ad}(\lambda_g \cdot \rho_g)}$ because these two have the same index in $N' \cap M_1^\omega$. Note that $G \times G$ acts by $\mathrm{Ad}(\lambda_g \cdot \rho_h)$. Then

$$(N' \cap M_1^\omega)^{\mathrm{Ad}(\lambda_g \cdot \rho_h)} = (N' \cap M^\omega)^{\mathrm{Ad}(\lambda_g)} = M' \cap M^\omega = M_\omega.$$

Setting $P = N' \cap M_1^\omega$, we get an outer action $\mathrm{Ad}(\lambda_g \cdot \rho_h)$ on P. We have a tower

$$N_\omega \cap M_\omega \subset M_\omega = P^{G \times G} \subset \langle M_\omega, f_1 \rangle = P^G,$$

where G sits in $G \times G$ diagonally.

Furthermore, G acts on N_ω outerly, so there is a partition of unity $r_g \in N_\omega$, $\sum_g r_g = 1$, $\alpha_g(r_h) = r_{gh}$ by non-commutative Rohlin's lemma. We set $p_{g,h} = r_g \cdot f_{gh}$, then this is a partition of unity and, we get

$$\mathrm{Ad}(\lambda_k \cdot \rho_l)(p_{g,h}) = r_{kg} f_{kghl^{-1}} = p_{kg,hl^{-1}}.$$

We can find a family of unitaries $\bar{\lambda}_{g,h} \in (N' \cap M_1^\omega)^{G\times G} = M_\omega$ such that $\mathrm{Ad}(\bar{\lambda}_{g,h})(p_{k,l}) = p_{kg^{-1},hl}$ and $\bar{\lambda}_{g,h}\bar{\lambda}_{k,l} = \bar{\lambda}_{gk,hl}$. Set $R = \{p_{1,1}\}' \cap M_\omega$. We see $R \subset N_\omega$. Indeed, if $x \in M_\omega$ satisfies $[x, r_1 f_1] = 0$, then $[x, f_1] = [x, \sum_g \mathrm{Ad}(\lambda_g \cdot \rho_g)(f_1 r_1)] = 0$ by $\mathrm{Ad}(\lambda_g \cdot \rho_g)(x) = x$, hence $x \in N_\omega \cap M_\omega$. Now M_ω is expressed as $\langle R, \bar{\lambda}_{g,h}\rangle$, the crossed product $R \rtimes (G \times G)$. Because

$$\mathrm{Ad}(\bar{\lambda}_{g,g})(f_1) = \mathrm{Ad}(\bar{\lambda}_{g,g})(\sum_k p_{k,k^{-1}}) = \sum_k p_{kg^{-1},gk^{-1}} = f_1,$$

we get $\bar{\lambda}_{g,g} \in \{f_1\}' \cap M_\omega = N_\omega \cap M_\omega$. Hence we get

$$\langle R, \bar{\lambda}_{g,g}\rangle \subset N_\omega \cap M_\omega \subset \langle R, \bar{\lambda}_{g,h}\rangle = M_\omega.$$

But by the equality

$$[\langle R, \bar{\lambda}_{g,h}\rangle : M_\omega \cap N_\omega] = [\langle R, \bar{\lambda}_{g,h}\rangle : \langle R, \bar{\lambda}_{g,g}\rangle] = |G|,$$

we get

$$\langle R, \bar{\lambda}_{g,g}\rangle = N_\omega \cap M_\omega.$$

Hence the central sequence algebras are of the form $R \rtimes G \subset R \rtimes (G \times G)$, where G sits in $G \times G$ diagonally.

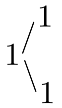
IV. Computation of paragroups of small order

IV.1. Main result for index< 4

By Perron-Frobenius theory, since the matrix Δ has non-negative entries, Δ has a unique non-negative eigenvector. This eigenvalue is equal to the operator norm of Δ and also equal to $[M : N]^{1/2}$, hence we get $[M : N] = \|\Delta\|^2$. If $[M : N] < 4$, then $\|\Delta\| < 2$, and by a result of Kronecker, all the self-adjoint irreducible matrices Δ which have non-negative integer entries and $\|\Delta\| < 2$ are known. (Notice that if $\mathcal{G}$, $\mathcal{H}$ are graphs, $\mathcal{G} \subsetneq \mathcal{H}$, and $\mathcal{G}$ is finite, then $\|\Delta_{\mathcal{G}}\| < \|\Delta_{\mathcal{H}}\|$, since $\|\Delta_{\mathcal{G}}\|$ is an eigenvalue of $\mu_{\mathcal{G}}$.)

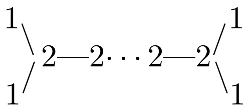
Here we have List 1 of graphs with eigenvectors for the eigenvalue 2.

$\tilde{A}_n$:



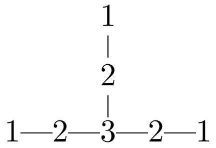
$\Delta\mu = 2\mu,$

$\tilde{D}_n$:



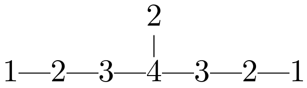
$\Delta\mu = 2\mu,$

$\tilde{E}_6$:



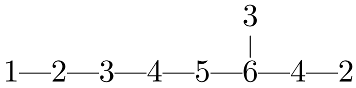
$\Delta\mu = 2\mu,$

$\tilde{E}_7$:



$\Delta\mu = 2\mu,$

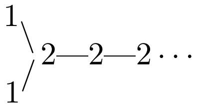
$\tilde{E}_8$:



$\Delta\mu = 2\mu,$

$A_\infty :$ $1—2—3—4—5\cdots$ $\Delta\mu = 2\mu,$

$D_\infty :$



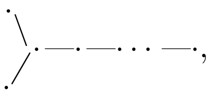
$\Delta\mu = 2\mu,$

$A_{-\infty,\infty} :$ $\cdots 1—1—1—1—1\cdots$ $\Delta\mu = 2\mu,$

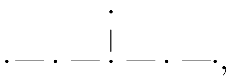
Theorem. *The only graphs which do not contain a graph in List 1 are finite and are in List 0.*

$A_n :$ $\cdot — \cdot — \cdots — \cdot,$

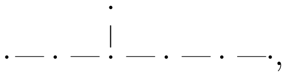
$D_n :$



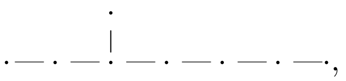
$E_6 :$



$E_7 :$



$E_8 :$



Proof. Let $\mathcal{G}$ be a graph which does not contain any in List 1. Because it does not contain $A_{-\infty,\infty}$, it is finite. Because it does not contain $\tilde{A}_n$, there are no cycles. There are no 4-points and at most one 3-point because it does not contain $\tilde{D}_n$. If there is a 3-point, the three legs from it are not too long because $\mathcal{G}$ does not contain $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. Q.E.D.

Thus if $[M : N] < 4$, then $[M : N]^{1/2}$ is equal to an eigenvalue of one in List 0, which is $2\cos\frac{\pi}{(\text{Coxeter number})}$. Here the Coxeter number is defined to be $n+1$ for A_n, $2n-2$ for D_n, $12, 18, 30$ for E_6, E_7, E_8, respectively. For A_n having vertices $v_1, v_2, \ldots, v_n$, the vector μ defined by $\mu(v_k) = \sin k\frac{\pi}{n+1}$ has an eigenvalue $2\cos\frac{\pi}{n+1}$ because

$$\sin\frac{(k-1)\pi}{n+1} + \sin\frac{(k+1)\pi}{n+1} = 2\cos\frac{\pi}{n+1}\sin\frac{k\pi}{n+1}.$$

(This suggests a relation between a minimal model in conformal field theory (CFT) and a *paragroup*, which is the name for the invariant appearing from the Galois functor.)

The only graphs which do not contain *strictly* a graph in List 1 are those in List 1 and List 0. This means that the other graphs have norm bigger than 2.

We have seen the Galois functor assigning a graph to a subfactor $N \subset M$. If the graph is infinite, then the Perron-Frobenius eigenvector is actually a part of the obtained data, since it is not unique.

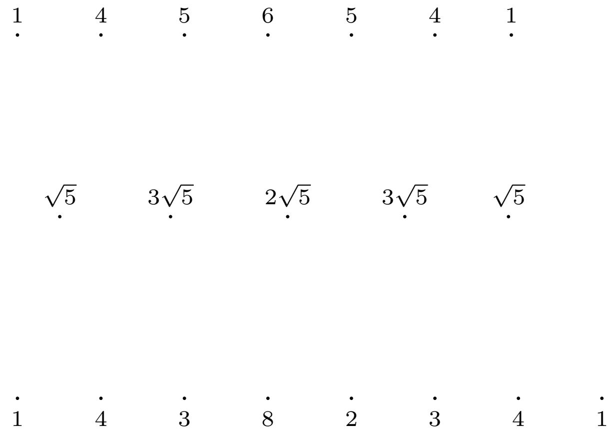
The problem here is finding all the possible invariants for subfactor with index$<$ 4.

Note that the Perron-Frobenius eigenvalue of the adjacency matrix of the graph must be less than 2, hence each graph is one of Dynkin diagrams of type A, D, E. Any paragroup (in duality form) has a distinguished initial point $*$. First we have

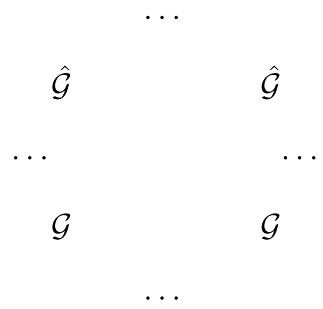
$$\mathrm{End}({}_M M_M) \cong \mathcal{Z}(M) = \mathbf{C},$$

$$\mathrm{End}({}_N M_M) \cong N' \cap \mathrm{End}(M_M) \cong N' \cap M = \mathbf{C}.$$

We have a distinguished homomorphism in $\mathrm{Hom}({}_M M_M, {}_N M \otimes_M M_M)$. This is like an axiom for a unit. For the biunitarity axiom, the number of paths for making rows and columns are equal, and the Perron-Frobenius eigenvectors must match at the common vertices. This implies that the dual graph is equal to the principal graph in each case. (In general cases index$\geq$ 4, this does not hold any more as shown by the following example S_5/S_4.)



The paragroups with $[M : N] < 4$ are of the following form:

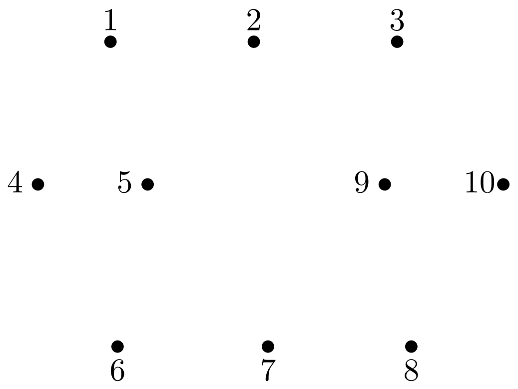


where $\mathcal{G}$ and $\hat{\mathcal{G}}$ are the same as a graph, and one of Dynkin diagrams of type A, D, E. For each A_n, there is only one connection on it. For D_n, there are two connections, but there is only one up to a graph isomorphism (of switching two endpoints next to the triple point). For E_6, E_7, E_8, there are two connection. But (geometrical) flatness of the connection is not satisfied by the connections on D_{odd}, E_7. Then the conclusion is that there is one subfactor for A_n, D_{2n}, and two subfactors for E_6, E_8, which are opposite conjugate but not conjugate to each other. Notice similarity to CFT. Also note that $\text{End}({}_N M_M) \cong N' \cap M$ and $N' \cap M = \mathbf{C}$ is equivalent to that the vertex $*$ has the only one neighbour.

IV.2. Concrete computations of connections

We determine the connection in each case. (cf. IRF model.) Recall that we have a gauge for the connection coming from the choice of edges.

We now work on an example A_5. We label each vertex as follows.



Fixing vertices 1,6, we get the following connection values by the rule of renormalization.

$$W\left(\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ 4 \arrow[dr] & & 9 \arrow[dl] \\ & 6 & \end{tikzcd}\right) = 1, \qquad W\left(\begin{tikzcd} & 1 \arrow[dr] & \\ 4 \arrow[ur] \arrow[dr] & & 9 \\ & 6 \arrow[ur] & \end{tikzcd}\right) = \sqrt{\frac{1}{\sqrt{3}}} \cdot \sqrt{\frac{1}{\sqrt{3}}} \cdot 1 = \frac{1}{\sqrt{3}}.$$

Then fix vertices 1,7 next. By changing the choice of $(4,7)$, we may assume that the connection for the left below is 1.

$$W\left(\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ 4 \arrow[dr] & & 9 \arrow[dl] \\ & 7 & \end{tikzcd}\right) = 1, \qquad W\left(\begin{tikzcd} & 1 \arrow[dr] & \\ 4 \arrow[ur] \arrow[dr] & & 9 \\ & 7 \arrow[ur] & \end{tikzcd}\right) = \frac{\sqrt{2}}{\sqrt{3}}.$$

Fixing vertices 2,6, we get the following similarly.

$$W\left(\begin{tikzcd} & 2 \arrow[dl] \arrow[dr] & \\ 4 \arrow[dr] & & 9 \arrow[dl] \\ & 6 & \end{tikzcd}\right) = 1, \qquad W\left(\begin{tikzcd} & 2 \arrow[dr] & \\ 4 \arrow[ur] \arrow[dr] & & 9 \\ & 6 \arrow[ur] & \end{tikzcd}\right) = \frac{\sqrt{2}}{\sqrt{3}}.$$

Fixing vertices 4,9, we get a 2×2-matrix. By unitarity of $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} & * \end{pmatrix}$, we get

$$W\left(\begin{tikzcd} & 2 \arrow[dr] & \\ 4 \arrow[ur] \arrow[dr] & & 9 \\ & 7 \arrow[ur] & \end{tikzcd}\right) = -\frac{1}{\sqrt{3}} \text{ and hence } W\left(\begin{tikzcd} & 2 \arrow[dl] \arrow[dr] & \\ 4 \arrow[dr] & & 9 \arrow[dl] \\ & 7 & \end{tikzcd}\right) = -\frac{1}{2}.$$

Fixing 5,9, and using a gauge for $(5, 2)$, we get

$$W\left(\begin{tikzcd} & 2 \arrow[dr] & \\ 5 \arrow[ur] \arrow[dr] & & 9 \\ & 7 \arrow[ur] & \end{tikzcd}\right) = 1, \qquad W\left(\begin{tikzcd} & 2 \arrow[dl] \arrow[dr] & \\ 5 \arrow[dr] & & 9 \arrow[dl] \\ & 7 & \end{tikzcd}\right) = \frac{\sqrt{3}}{2}.$$

Fixing 4,10, and using a gauge for $(7, 10)$, we get

$$W\left(\begin{tikzcd} & 2 \arrow[dr] & \\ 4 \arrow[ur] \arrow[dr] & & 10 \\ & 7 \arrow[ur] & \end{tikzcd}\right) = 1, \qquad W\left(\begin{tikzcd} & 2 \arrow[dl] \arrow[dr] & \\ 4 \arrow[dr] & & 10 \arrow[dl] \\ & 7 & \end{tikzcd}\right) = \frac{\sqrt{3}}{2}.$$

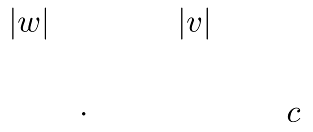
Fixing 7,2, we get a unitary 2×2-matrix and $W\left(\begin{tikzcd} & 2 \arrow[dl] \arrow[dr] & \\ 5 \arrow[dr] & & 10 \arrow[dl] \\ & 7 & \end{tikzcd}\right) = \frac{1}{2}$ similarly.

This computation works for all A_n, and we get a unique connection, up to gauge choice, on each A_n. The Perron-Frobenius eigenvector is given by $k \mapsto s(k)/s(1)$, where a function $s(k)$ is defined by $s(k) = \sin \dfrac{k\pi}{n+1}$.

If one meets a 3-point, as in D_n, E_n, then the previous argument works up to the 3-point. At the 3-point, one has a 3×3-matrix (a_{ij}). The values of $|a_{ij}|$ are completely determined. (See IV.4.) For each row and column, we have a freedom of gauge. We can, for example, make the first row and the first column real.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & x & y \\ a_{31} & z & t \end{pmatrix}, a_{11}, a_{12}, a_{13}, a_{21}, a_{31} \in \mathbf{R}_{+}.$$

We need to choose x, y. (Then z, t are determined by unitarity.) Because $|x|, |y|$ are given, and we have an equation $-xa_{12} - ya_{13} = a_{21}a_{11}$, if we call $v = -a_{12}x \in \mathbf{C}, w = -a_{13}y \in \mathbf{C}, c = a_{21}a_{11} \in \mathbf{R}_{+}$, then our equation is $v + w = c$, with $|v|, |w|$ given.



The above picture shows that this problem has at most two solutions (conjugate to each other). We have three cases; There are no solutions; There is one (self-conjugate) solution; There are two solutions (conjugate to each other). The computation is not difficult for D_n. We shall display a solution for each A_n, D_n, E_n. Note that if we find two conjugate solutions, then they are the only solution because we have at most one 3-point. (This was noted independently by Pasquier.)

Use an identification of the four graphs. (The identification is not canonical, but we choose one. Consider $\mathbf{Z}_3 \cong \hat{\mathbf{Z}}_3$ for example.) For a vertex v, call $\mu(v)$ the Perron-Frobenius eigenvector. Let N be the Coxeter number of the graph and β

the Perron-Frobenius eigenvalue $2\cos\pi/N$. We set $\varepsilon = i\exp(\pi i/2N)$. Our formula for W is

$$W\left(\begin{array}{ccccc} & & i & & \\ k & \swarrow & & \searrow & l \\ & \searrow & & \swarrow & \\ & & j & & \end{array}\right) = \delta_{kl}\varepsilon + \rho\delta_{ij}\bar{\varepsilon}, \qquad \rho = \left(\frac{\mu(k)\mu(l)}{\mu(i)\mu(j)}\right)^{1/2}.$$

Note that this satisfies the renormailization rule. We check unitarity as follows.

$$\sum_l W\left(\begin{array}{ccccc} & & i & & \\ k & \swarrow & & \searrow & l \\ & \searrow & & \swarrow & \\ & & j & & \end{array}\right)\overline{W\left(\begin{array}{ccccc} & & i & & \\ m & \swarrow & & \searrow & l \\ & \searrow & & \swarrow & \\ & & j & & \end{array}\right)} = ?.$$

We consider two cases.

Case 1. $(i \neq j.)$ We get $? = \sum_l \delta_{kl}\varepsilon\delta_{ml}\bar{\varepsilon} = \delta_{km}$.

Case 2. $(i = j.)$ We get

$$\begin{aligned} ? &= \sum_l \delta_{kl}\varepsilon\delta_{ml}\bar{\varepsilon} + \sum_l \delta_{kl}\varepsilon^2\frac{\mu(l)^{1/2}\mu(m)^{1/2}}{\mu(i)} \\ &\quad + \sum_l \frac{\mu(k)^{1/2}\mu(l)^{1/2}}{\mu(i)}\delta_{ml}\bar{\varepsilon}^2 + \sum_l \frac{\mu(k)^{1/2}\mu(m)^{1/2}\mu(l)}{\mu(i)^2} \\ &= \delta_{km} + \frac{\mu(k)^{1/2}\mu(m)^{1/2}}{\mu(i)}(\varepsilon^2 + \bar{\varepsilon}^2 + \frac{\sum_l \mu(l)}{\mu(i)}). \end{aligned}$$

But now $\varepsilon^2+\bar{\varepsilon}^2+\dfrac{\sum_l \mu(l)}{\mu(i)} = \varepsilon^2+\bar{\varepsilon}^2+\beta = 0$ by the definition of ε. The other unitarity can be verified similarly. This computation also works even if $N = \infty$ for $\tilde{A}, \tilde{D}, \tilde{E}$. If we take a complex conjugate on a connection, we get an anti-automorphism. (cf. Yang-Baxter equation.)

The graph D_n with $*$-distinguished point has a $\mathbf{Z}_2$ symmetry. This switches W and $\bar{W}$.

Example. The graph D_4 corresponds to the group $\mathbf{Z}_3$. In this case, a connection W is given by a unitary 3×3-matrix

$$W = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \bar{\alpha} \\ 1 & \bar{\alpha} & \alpha \end{pmatrix},$$

where $\alpha \in \mathbf{C}$ satisfies $\alpha + \alpha^2 + 1 = 0$. The symmetry switches the second and the third columns.

For D_n, there is only one solution, up to graph automorphism. For E_6, E_7, E_8, we have two anti-isomorphic but non-isomorphic solutions.

IV.3. Flatness

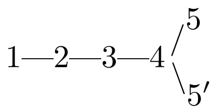
Our result on flatness of the connections given above is as follows.

Theorem. *The connections defined above for A_n are flat. The connections for D_{2n} are flat, and for D_{4k} two connections are conjugate to each other by switching the endpoints, and connections for D_{4k+2} are self-conjugate. Connections for E_6, E_8 are flat. Connections for E_7 and D_{odd} are not flat.*

The result for D_{odd} means that composition table of bimodule multiplications for D_{odd} is impossible.

Consider D_6 for example.

D_6 :



For flatness, we only need to check

$$W\left(\begin{tikzcd} * \arrow[r, "\xi"] \arrow[d, "\eta"'] & 5 \arrow[d] & * \arrow[l, "\xi"'] \arrow[d, "\eta'"] \\ 5 \arrow[r, "\zeta"'] & x & 5' \arrow[l, "\zeta'"] \end{tikzcd}\right) = 0,$$

where $\xi = \eta = (1—2—3—4—5)$, $\eta' = (1—2—3—4—5')$. The reason is as follows. The string algebras $A_{k,0}$, $k \leq 3$, are generated by Jones projections and $A_{k,0}$, $k \geq 5$, are generated by $A_{k-1,0}$ and the Jones projection, thus we may assume $|\xi| = 4$ in the above diagram. Moreover, if $r(\xi) = 1, 2, 3, 4$, then the formula is valid again by flatness of the Jones projections. Hence we may assume that $\xi = \eta = (1—2—3—4—5)$ or $\xi = \eta = (1—2—3—4—5')$, and by

$$(1—2—3—4—5, 1—2—3—4—5) + (1—2—3—4—5', 1—2—3—4—5') \in \langle e_i \rangle,$$

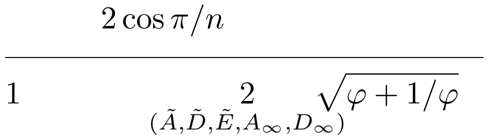
we need consider only the former case. Because the string $(\xi, \xi) \in A_{5,0}$ commutes with horizontal Jones projections again by their flatness, it is enough to assume $\xi = \eta$ and $eta' = (1—2—3—4—5')$. Using that

$$W\left(\begin{tikzcd} * \arrow[r, "\xi"] \arrow[d, "\eta"'] & 5 \arrow[d] & * \arrow[l, "\xi"'] \arrow[d, "\eta'"] \\ 5 \arrow[r, "\zeta"'] & x & 5' \arrow[l, "\zeta'"] \end{tikzcd}\right) + W\left(\begin{tikzcd} * \arrow[r, "\xi"] \arrow[d, "\eta"'] & 5' \arrow[d] & * \arrow[l, "\xi"'] \arrow[d, "\eta'"] \\ 5 \arrow[r, "\zeta"'] & x & 5' \arrow[l, "\zeta'"] \end{tikzcd}\right) = 0,$$

obtained by flatness of the Jones projections, and that $W(2 \times 2\text{-cell})$ can be adjusted to be real by choice of gauges, we can show the desired formula.

IV.4. Non-amenability of the E_{10} subfactor

It is known that the possible values of the Perron-Frobenius eigenvalues for (possibly infinite) graphs are given by the following picture. (See Appendix I of [GHJ].)

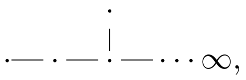


$(1+\varphi=\varphi^2)$

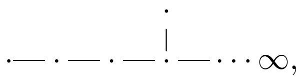
The interval between 2 and $\sqrt{\varphi+1/\varphi}$, where φ is the golden ratio, is given by the following picture.

The first value above 4 is given by E_{10}, and the first accumulation point is given by E_∞:

E_∞ :



The second accumulation point and the value $\sqrt{\varphi+1/\varphi}$ are given by the following, respectively.



Here we give a proof for the theorem on non-amenability of the E_{10}-subfactor in II.5.

Proof. First note that E_{10} is the only finite graph with the given Perron-Frobenius eigenvalue. (See Proposition I.3.4 in Appendix I of [GHJ].) If the graph is infinite, then an eigenvalue is greater than or equal to the norm of the graph, so the possibility of the principal graph now is limited to one of $E_{10}, A_\infty, D_\infty$. Then the lower part of the paragroup (the dual graph) should be the same as the principal graph because the Perron-Frobenius eigenvectors must match. Thus it is enough to eliminate possibilities of E_{10}, D_∞. For this purpose, suppose the principal graph has a triple point a, and the dual graph a corresponding triple point a'. We show that the biunitarity axiom gives a contradiction. Set $\beta = [M : N]^{1/2} > 2$. Let $(c_{ij})_{i,j=1,2,3}$ be the 3×3-unitary matrix given by $W\left(\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b_i \arrow[dr] & & b'_j \arrow[dl] \\ & a' & \end{tikzcd}\right)$. If $i \neq j$, then we can apply the biunitarity axiom to a 1×1-matrix and use the renormalization rule to get $|c_{ij}| = \dfrac{\mu(b_i)^{1/2}\mu(b'_i)^{1/2}}{\mu(a)^{1/2}\mu(a')^{1/2}}$. By

$$|c_{11}|^2 + |c_{12}|^2 + |c_{13}|^2 = 1,$$

$$\mu(b'_2) + \mu(b'_3) = \beta\mu(a) - \mu(b'_1),$$

we get

$$|c_{11}|^2 = \frac{\mu(a)^2 + \mu(b_1)\mu(b'_1) - \beta\mu(a)\mu(b_1)}{\mu(a)^2}.$$

Applying the same formula to c_{11}, c_{21}, c_{31}, we get $\mu(b_1) = \mu(b'_1)$, and hence

$$|c_{11}|^2 = 1 - \frac{\beta\mu(b_1)}{\mu(a)} + \left(\frac{\mu(b_1)}{\mu(a)}\right)^2.$$

Setting $x_i = \mu(b_i)/\mu(a)$, we get

$$(|c_{ij}|) = \begin{pmatrix} \sqrt{1 - \beta x_1 + x_1^2} & \sqrt{x_1 x_2} & \sqrt{x_1 x_3} \\ \sqrt{x_2 x_1} & \sqrt{1 - \beta x_2 + x_2^2} & \sqrt{x_2 x_3} \\ \sqrt{x_3 x_1} & \sqrt{x_3 x_2} & \sqrt{1 - \beta x_3 + x_3^2} \end{pmatrix}.$$

Because $c_{11}\bar{c}_{21} + c_{12}\bar{c}_{22} + c_{13}\bar{c}_{23} = 0$, $x_i > 0$ and $\beta > 2$, we get

$$\begin{aligned} x_3 &= \sqrt{x_1x_3}\sqrt{x_2x_3}/\sqrt{x_1x_2} \\ &\leq \sqrt{1-\beta x_1 + x_1^2} + \sqrt{1-\beta x_2 + x_2^2} \\ &\leq 1 - x_1 + 1 - x_2, \end{aligned}$$

which implies $2 < \beta = x_1 + x_2 + x_3 \leq 2$, a contradiction. Q.E.D.

Note that the above proof eliminates a large class of graphs for small index values> 4. (That is, if the upper graph and the lower graphs are the same, it has no cycle of length 4 and the Perron-Frobenius eigenvalue is bigger than 2, then there is no biunitary connection on this system.)

IV.5. Paragroups for index> 4

Recall Jones' construction of subfactors $\langle e_i \rangle$ in the string algebra of a graph $\mathcal{G}$ at a vertex x. (See [GHJ].) Okamoto [Ok] showed that $[\text{String}(\mathcal{G}) : \langle e_i \rangle] < \infty$ if and only if the graph $\mathcal{G}$ is one of $A, D, E, A_\infty, D_\infty$. He also computed principal graphs for these. The smallest index value arising in this way is $3 + \sqrt{3}$ and this is the smallest known value> 4 for which a flat connection exists, in other words, this is a subfactor with finite depth with the smallest known index value> 4. It is also conjectured that this is actually the smallest value above 4 as an index value of an irreducible subfactor with finite (or amenable) depth.

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