Tempered Homogeneous Spaces

Toshiyuki Kobayashi

The Graduate School of Mathematical Sciences
The University of Tokyo
http://www.ms.u-tokyo.ac.jp/˜toshi/

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Tempered Homogeneous Spaces

Part 1. Ideas from Dynamical Approach [I, II]

Part 2. Combinatorial Approach [III]

Part 3. Topological Approach and Quantization [IV]

* Y. Benoist–TK, I (2015), II (2022), III (2021), IV (2023) $+\alpha$.

Part I: Optimal L^q-Exponent $\cdots$ Dynamical Approach

$$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright L^2(X) \quad \text{unitary rep}$$

<u>Basic Problem</u> Find a geometric criterion on (G, X) such that $L^2(X)$ is a <u>tempered representation</u>.

Change of approach beyond "spherical cases".

PDE $\rightsquigarrow$ Dynamical approach

Plan (Part 1)

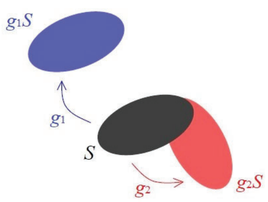
- Methods:
  - Volume estimate; $\mathrm{vol}(gS \cap S)$.
- Concept:
  - Optimal L^q-exponent $q(G; X)$,
    <u>Tempered homogeneous spaces</u>.

Learn from Topology

G : locally compact group

X : locally compact space

<u>Definition</u> A continuous action $G \curvearrowright X$ is called <u>proper</u> if the subset
$$G_S := \{g \in G : S \cap gS \neq \emptyset\}$$
is compact for any compact subset $S \subset X$.



$\Longrightarrow$ $g_1 \notin G_S$, $g_2 \in G_S$

Idea: Quantify proper actions

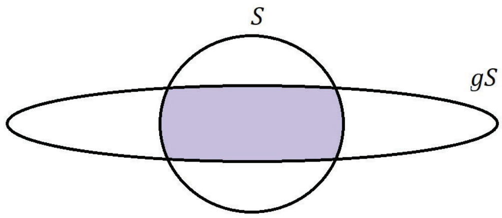
Locally compact group $G \curvearrowright X$ locally compact space

$$G \curvearrowright X \text{ proper } \overset{\text{def}}{\Leftrightarrow} \{g \in G : S \cap gS \neq \emptyset\} \text{ is compact } {}^\forall S \subset X \text{ compact},$$
$$\Leftrightarrow \mathrm{vol}(S \cap gS) \in C_c(G) \qquad {}^\forall S \subset X \text{ compact},$$

where we fix an appropriate Radon measure on X.

Idea: Quantitative estimate for non-proper actions.

Look at asymptotic behavior of $\mathrm{vol}(S \cap gS)$ as g goes to infinity.



Volume estimate $\mathrm{vol}(t \cdot S \cap S)$: example $\quad \mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

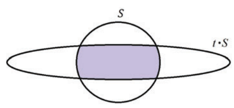
Example Let $\mathbb{R} \ni t$ act on $X = \mathbb{R}^2 \setminus \{(0,0)\}$ by

$$(x, y) \mapsto (e^t x, e^{-t} y)$$

- This action is free, but is not proper.
- Asymptotic behavior of $\mathrm{vol}(S \cap t \cdot S)$.

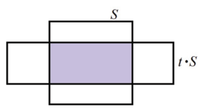
For any compact neighbourhood S of the origin in $\mathbb{R}^2$, one has

$$C_1 e^{-|t|} \leq \mathrm{vol}(t \cdot S \cap S) \leq C_2 e^{-|t|}.$$



For instance, if $S = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$,

$$\mathrm{vol}(t \cdot S \cap S) = 4e^{-|t|}.$$



Almost L^q function

Z: locally compact space equipped with a Radon measure.

Definition A measurable function f on Z is almost L^q if
$$f \in \bigcap_{\varepsilon>0} L^{q+\varepsilon}(Z).$$

Remark For $q \leq q'$, one has
f is almost $L^q \Longrightarrow f$ is almost $L^{q'}$.

We are interested in the best possible q for which f is almost L^q.

Optimal exponent $q(G;X)$ of volume estimate

loc. cpt. gp $G \curvearrowright X$ loc. cpt. space

Equip G with a Haar measure.
Suppose X admits a G-invariant Radon measure.

Definition Let $q(G;X)$ denote the optimal exponent $q > 0$ such that
$$\mathrm{vol}(S \cap gS) \text{ is an almost } L^q\text{-function on } G$$
for every compact subset $S \subset X$.

General Problem Find an explicit formula of $q(G;X)$.

Optimal exponent $q(G;X)$ of volume estimate

loc. cpt. gp $\quad G \curvearrowright X \quad$ loc. cpt. space

$\boxed{\text{General Problem} \quad \text{Find an explicit formula of } q(G;X).}$

$\boxed{\text{Example} \quad q(G;X) = 2 \text{ if } (G,X) = (SL(2,\mathbb{R}),\mathbb{R}^2).}$

Example. L^q-estimate of $\mathrm{vol}(gS \cap S)$ for $SL(2,\mathbb{R}) \curvearrowright \mathbb{R}^2$

$$G = SL(2,\mathbb{R}) \quad \curvearrowright \quad X = \mathbb{R}^2$$

$\boxed{\mathrm{vol}(gS \cap S) \text{ is almost } L^2(G) \text{ for any compact subset } S \subset \mathbb{R}^2.}$

- Cartan decomposition: $G = SL(2,\mathbb{R}) = KAK \ni g = k_1 a_t k_2$ where

$$k_1, k_2 \in K = SO(2),\ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in A.$$

- Volume estimate and Haar measure on $G = KAK$:

$$\mathrm{vol}(gS \cap S) \sim e^{-|t|} \quad \text{(previous example)}.$$

$$dg = \sinh(2t) dk_1 dt dk_2 \sim e^{2|t|} dk_1 dt dk_2.$$

Hence

$$\mathrm{vol}(gS \cap S) \in L^{q+\varepsilon}(G) \iff 2 - q - \varepsilon < 0.$$

Optimal exponent $q(G; X)$ of volume estimate

semismple Lie group $G \curvearrowright X$ manifold

$\boxed{\text{General Problem} \quad \text{Find an explicit formula of } q(G;X).}$

Example: $(G, X) = (SL(2,\mathbb{R}), \mathbb{R}^2)$

Plan:

- Case 1 $\quad X$ is a vector space
- Case 2 $\quad X = G/H \quad H \subset G$ reductive
- Case 3 $\quad X = G/H \quad H$ is any connected subgroup

L^p-estimate of $\mathrm{vol}(gS \cap S) \cdots$ Case 1. $H \curvearrowright V$ linear

Changing notation : $G \curvearrowright X \rightsquigarrow H \curvearrowright V$ (linear)

Let H be a semisimple Lie group, and let $\tau\colon H \to SL_{\mathbb{R}}(V)$ be a homomorphism with a compact kernel.

The optimal exponent $q(H; V)$ for $\mathrm{vol}(gS \cap S)$ to be almost L^q is given as follows.

Theorem A For a linear action $H \curvearrowright V$, one has

$$\underbrace{q(H;V)}_{\text{analysis}} = \underbrace{\beta_V}_{\text{combinatorics}}.$$

β_V : numerical invariant of an H-module $V \cdots$ next page.

Numerical Invariant β_V of Representations V

$\mathfrak{h}$: Lie algebra $/\mathbb{R}$

$\mathrm{Rep}(\mathfrak{h}) :=$ {finite-dimensional representations of $\mathfrak{h}$}

We introduce a numerical invariant β_V of $\mathfrak{h}$-modules V by:

Definition (Numerical Invariant β_V, Benoist–K-, 2015)*

$$\beta : \mathrm{Rep}(\mathfrak{h}) \to \mathbb{R}_{\geq 0}$$
$$V \mapsto \beta_V := \max_{Y \in \mathfrak{h}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$$

- ρ_V, $\rho_{\mathfrak{h}}$ are functions on $\mathfrak{h}$ (to be defined in next slide).
- β_V is a non-negative number.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

Piecewise linear function ρ_V associated to $\tau\colon \mathfrak{h} \to \mathrm{End}(V)$

For a finite-dimensional rep $\tau\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$, we introduce:

Definition (non-negative function ρ_V on the Lie algebra $\mathfrak{h}$)

$$\rho_V : \mathfrak{h} \to \mathbb{R}_{\geq 0}, \quad Y \mapsto \frac{1}{2} \sum |\mathrm{Re}\, \lambda(Y)|.$$

gen. eigenvalues of $\tau(Y) \in \mathrm{End}(V_{\mathbb{C}})$

Let $\mathfrak{a}$ be a maximal split abelian subspace of the Lie algebra $\mathfrak{h}$.

$\rightsquigarrow$
- ρ_V is determined by its restriction to $\mathfrak{a}$,
- The restriction $\rho_V|_{\mathfrak{a}}$ is piecewise linear.

Remark For a reductive $\mathfrak{h}$ and for $(\tau, V) = (\mathrm{ad}, \mathfrak{h})$,
$\rho_{\mathfrak{h}}|_{\mathfrak{a}}$ = twice the usual ρ on the dominant Weyl chamber,
however, our $\rho_{\mathfrak{h}}|_{\mathfrak{a}}$ is not linear whereas the usual ρ is linear.

Numerical Invariant β_V associated to $\tau\colon \mathfrak{h} \to \mathrm{End}(V)$

Let $\mathfrak{h}$ be a semisimple Lie algebra.
Let $\mathfrak{a}$ be a maximally split abelian subspace of $\mathfrak{h}$. Then,

$$\beta_V = \max_{Y \in \mathfrak{a}\setminus\{0\}} \frac{\sum |\text{eigenvalue of } \mathrm{ad}(Y) \in \mathrm{End}(\mathfrak{h})|}{\sum |\text{eigenvalue of } \tau(Y) \in \mathrm{End}(V)|}.$$

<u>Example</u> $H_0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $\mathfrak{a} = \mathbb{R}H_0 \subset \mathfrak{h} = \mathfrak{sl}(2,\mathbb{R}) \curvearrowright V = \mathbb{R}^2$

$$\rho_{\mathfrak{h}}(tH_0) = \tfrac{1}{2}(|2t| + 0 + |-2t|) = 2|t|.$$
$$\rho_V(tH_0) = \tfrac{1}{2}(|t| + |-t|) = |t|.$$
$$\beta_V = \max_{t\neq 0} \frac{2|t|}{|t|} = 2$$

<u>Example</u> $q(G;X) = 2$ if $(G,X) = (SL(2,\mathbb{R}), \mathbb{R}^2)$.

Sketch of Proof for Theorem A: $H \curvearrowright V$ (linear)

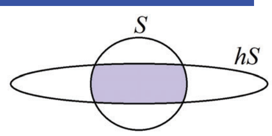
Let H be a semisimple Lie group. Suppose $\tau\colon H \to GL_{\mathbb{R}}(V)$ has a compact kernel. As in the case $(H,V) = (SL(2,\mathbb{R}), \mathbb{R}^2)$, one has

<u>Theorem A</u> For a linear action $H \curvearrowright V$, one has

$$\underset{\text{analysis}}{q(H;V)} = \underset{\text{combinatorics}}{\beta_V}.$$

Proof. • For $H \ni h = k_1 e^Y k_2$, one has

$$\mathrm{vol}(hS \cap S) \sim e^{-\rho_V(Y)}.$$



• For the Haar measure dh on H, one has

$$dh \sim e^{\rho_{\mathfrak{h}}(Y)} dk_1 dY dk_2 \quad \text{(away from wall).}$$

Therefore the $L^{q+\varepsilon}$-estimate of $\mathrm{vol}(hS \cap S)$ amounts to

$$\mathrm{vol}(hS \cap S)^{q+\varepsilon} dh \sim e^{\rho_{\mathfrak{h}}(Y) - (q+\varepsilon)\rho_V(Y)} dk_1 dY dk_2. \qquad \square$$

Optimal exponent $q(G;X)$ of volume estimate

semismple Lie group $G \curvearrowright X$ manifold

$\boxed{\text{General Problem} \quad \text{Find an explicit formula of } q(G;X).}$

Example: $(G,X) = (SL(2,\mathbb{R}), \mathbb{R}^2)$

Plan:

- Case 1 $\quad X$ is a vector space
- Case 2 $\quad X = G/H \quad H \subset G$ reductive
- Case 3 $\quad X = G/H \quad H$ is any connected subgroup

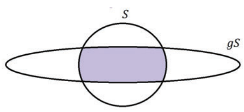
Case 2 $G \curvearrowright X = G/H$ where $\underset{\text{semisimple}}{G} \supset \underset{\text{reductive}}{H}$

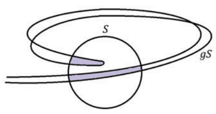
Theorem B* Let G be a semisimple Lie group, H a reductive subgroup, and $X := G/H$. Then one has

$$\underbrace{q(G;X)}_{\text{analysis}} = \underbrace{\beta_{\mathfrak{g}/\mathfrak{h}}}_{\text{combinatorics}} + 1.$$

Local $\rightsquigarrow$ Global

$H \curvearrowright \mathfrak{g}/\mathfrak{h} \qquad G \curvearrowright G/H$



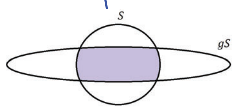


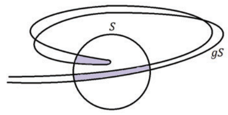
* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

<u>Case 2</u> $G \curvearrowright G/H$ where $\underset{\text{semisimple}}{G} \supset \underset{\text{reductive}}{H}$

> <u>Asymptotic estimate of volume</u>
> For any compact $S \subset G/H$, we want to find $m(g)$ and $M(g)$:
> $$m(g) \le \mathrm{vol}(gS \cap S) \le M(g) \quad \text{for all } g \in G.$$

for $g \in H$





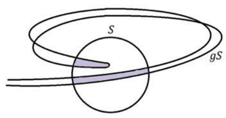
$$H \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g}/\mathfrak{h} \underset{\text{locally}}{\fallingdotseq} G/H.$$

Some difficulties to overcome:

- Need a lower bound $m(g)$ for $g \in G$, not only for $g \in H$.
- An upper bound $M(g)$ is more involved.

<u>Case 2</u> $G \curvearrowright X = G/H$ where $\underset{\text{semisimple}}{G} \supset \underset{\text{reductive}}{H}$

> <u>Theorem B</u>* Let G be a semisimple Lie group, H a reductive subgroup, and $X := G/H$. Then one has
> $$\underset{\text{analysis}}{q(G;X)} = \underset{\text{combinatorics}}{p_{\mathfrak{g}/\mathfrak{h}}} + 1.$$



<u>Key idea</u>: Quantify the proof of the properness criterion** for subgroups L of G acting on G/H.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.
** T. Kobayashi, Proper action on a homogeneous space of reductive type, Math. Ann., **285** (1989), 249–263.

Plan for Part 1

- Methods and elementary examples
  - Optimal constant $q(G;X)$ for L^q-estimate $\mathrm{vol}(gS \cap S)$.
  - Almost L^p-representation, tempered representations.
- Tempered homogeneous spaces.

Almost L^p representations

Almost L^p functions

$\Downarrow$

Almost L^p representations

Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$.

Definition For $p \geq 1$, $(\pi, \mathcal{H})$ is called almost L^p if there is

a dense subspace $D \subset \mathcal{H}$ such that matrix coefficients

for $x, y \in D$ are almost L^p, namely,

$$(\pi(g)x, y)_{\mathcal{H}} \in \bigcap_{\varepsilon>0} L^{p+\varepsilon}(G) \quad {}^\forall x, {}^\forall y \in D$$

Example Suppose that G acts on (X, μ), preserving the measure μ.
Then the unitary rep $G \curvearrowright L^2(X, \mu)$ is almost $L^{q(G;X)}$.

Recall $q(G;X)$ is the optimal constant q for which $\mathrm{vol}(gS \cap S)$ is almost L^q for all compact subset $S \subset X$.

Irreducible tempered reps — semisimple Lie groups

Def A unitary representation π of G is called tempered if $\pi \prec L^2(G)$.

- For a semisimple Lie group G and irreducible $\pi \in \widehat{G}$, tempered representations π have been studied extensively.

Known results on tempered reps and beyond ...

- Many equivalent definitions, *e.g.,* $L^{2+\varepsilon}(G)$,
- Harish-Chandra's theory towards Plancherel formula,
- Knapp–Zuckerman's classification *,
- A cornerstone of Langlands' classification,
- Selberg $\frac{1}{4}$ eigenvalue conjecture (1965-),
- Gan–Gross–Prasad conjecture, $\cdots$

* A. W. Knapp–G. Zuckerman, Classification of irreducible tempered representations of semisimple Lie groups, Ann. Math., (1980), 389–455; 457–501.

Almost L^2 representation *vs* tempered representations

Definition A unitary representation π of G is called tempered if $\pi \prec L^2(G)$.

- For a solvable Lie group G,

all unitary reps π are tempered (Hulanicki, Reiter), but are not always almost L^2.
E.g. the trivial one-dimensional rep is not almost L^p $(1 \le p < \infty)$ if G is non-compact.

- For a semisimple Lie group G, one has

Fact (Cowling–Haagerup–Howe)* One has the equivalence:

π is tempered $\iff$ π is almost L^2.

* M. Cowling–M. Haagerup–R. Howe, Almost L^2 matrix coefficients, J. Reine Angew. Math. **387**, (1988), 97–110.

Towards a temperedness criterion

Problem 2 For which pair $G \supset H$, is the unitary rep of G on $L^2(G/H)$ tempered?

For semisimple Lie groups G, we have already discussed a refinement of Problem 2 as below:

Problem 1 Find the optional constant $q(G;G/H)$ for which $\mathrm{vol}(gS \cap S)$ is almost L^q for all compact subset $S \subset G/H$.

$$q(G;G/H) \le 2 \iff L^2(G/H) \text{ is tempered.}$$

Temperedness criterion in the reductive case

G semisimple Lie group,

H any reductive subgroup.

Since we know from Theorem B that

$$\underbrace{q(G;G/H)}_{\text{analysis}} = \underbrace{\beta_{\mathfrak{g}/\mathfrak{h}}}_{\text{combinatorics}} + 1$$

where $\beta_V = \max_{\mathfrak{h} \ni Y \neq 0} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$ is defined for a linear action $H \curvearrowright V$, one obtains from the volume estimate:

Theorem C* For a pair of real reductive Lie groups, one has $L^2(G/H)$ is G-tempered $\iff \beta_{\mathfrak{g}/\mathfrak{h}} \le 1$.

Remark. $\beta_{\mathfrak{g}/\mathfrak{h}} \le 1 \iff 2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}}$ on $\mathfrak{h}$.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

Easy but Non-Trivial Example

$\beta_V = \max_{\mathfrak{h} \ni Y \neq 0} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$ equals 1 if we take $V = \mathfrak{h}$.

The temperedness criterion

$$L^2(G/H) \text{ is tempered} \iff \beta_{\mathfrak{g}/\mathfrak{h}} \leq 1$$

applied to the very special case where $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{h}$ implies that $L^2(G \times G/\Delta G)$ and $L^2(G_{\mathbb{C}}/G)$ are tempered representations.

<u>Example</u> $L^2(GL(n,\mathbb{C})/GL(n,\mathbb{R}))$,
$L^2(GL(n,\mathbb{C})/U(p,n-p))$
$L^2(GL(2m,\mathbb{C})/GL(m,\mathbb{H}))$
are all tempered.

Optimal exponent $q(G;X)$ of volume estimate

semismple Lie group $G \curvearrowright X$ manifold

<u>General Problem</u> Find an explicit formula of $q(G;X)$.

Plan:

- Case 1 X is a vector space
- Case 2 $X = G/H$ $H \subset G$ reductive
- Case 3 $X = G/H$ <u>H is any connected subgroup</u>

General Case: $\underset{\text{semisimple}}{G} \supset \underset{\text{any}}{H}$

Theorem C can be extended without reductive assumption of H:

Recall $\beta_V = \max_{\mathfrak{h} \ni Y \neq 0} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$ is defined for a linear action $H \curvearrowright V$.

We consider the adjoint action of H on $V := \mathfrak{g}/\mathfrak{h}$.

G semisimple Lie group,
H any connected closed subgroup.

> Theorem D*
> $L^2(G/H)$ is G-tempered $\iff \beta_{\mathfrak{g}/\mathfrak{h}} \leq 1$.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces II, Chicago Univ. Press (2022).

Strategy: Temperedness Criterion

			Method
(Theorem A)	Case 1	$\underset{\text{semisimple}}{G} \curvearrowright \underset{\text{linear}}{V}$	Dynamical approach
(Theorem C)	Case 2	$\underset{\text{semisimple}}{G} \supset \underset{\text{reductive}}{H}$	Local $\rightsquigarrow$ Global
(Theorem D)	Case 3	$\underset{\text{semisimple}}{G} \supset \underset{\text{any}}{H}$	Domination of G-spaces

Key Concept: Domination of G-spaces

$$G \curvearrowright X \rightsquigarrow G \curvearrowright L^2(X),$$
$$G \curvearrowright X_0 \rightsquigarrow G \curvearrowright L^2(X_0).$$

Suppose both X and X_0 admit G-invariant Radon measures.

Definition (2023)* $(X \prec_G X_0)$ We say that X is G-dominated by X_0 if $^{\forall}$ compact $S \subset X$, $^{\exists}$ compact $S_0 \subset X_0$ and $^{\exists}\lambda > 0$ such that

$$\mathrm{vol}(gS \cap S) \leq \lambda\, \mathrm{vol}(gS_0 \cap S_0) \quad {}^{\forall}g \in G.$$

* Y. Benoist–T. Kobayashi, Tempered Homogeneous Spaces IV, J. Inst. Math. Jussieu, **22** (2023), pp. 2879–2906.

Domination of G-spaces and L^q-estimate

We recall $q(G; X)$ is the optimal exponent for which $\mathrm{vol}(gS \cap S)$ is almost L^q for all compact subset $S \subset X$.

Consider two measure-preserving actions $G \curvearrowright X$ and X_0.

Lemma If $X \prec_G X_0$ then $q(G; X) \leq q(G; X_0)$.

One also have

Lemma Let G be a real reductive Lie group, P a minimal parabolic subgroup, and X a G-space.

$$L^2(X) \text{ is } G\text{-tempered} \iff X \prec_G G/P.$$

Inducing a Dominated Action

<u>Question</u> How can we find X_0 such that $X \prec_G X_0$ when we are given a G-space X?

$$G \supset \underset{\text{subgroup}}{F}, \text{ and } F \curvearrowright Z,\ Z_0$$

We set two G-spaces by

$$X := G \times_F Z, \quad X_0 := G \times_F Z_0$$

<u>Proposition</u> $\quad Z \prec_F Z_0 \implies X \prec_G X_0.$

Tempered Homogeneous Spaces

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Part 2. Combinatorial Approach [III]

Part 3. Topological Approach and Quantization [IV]

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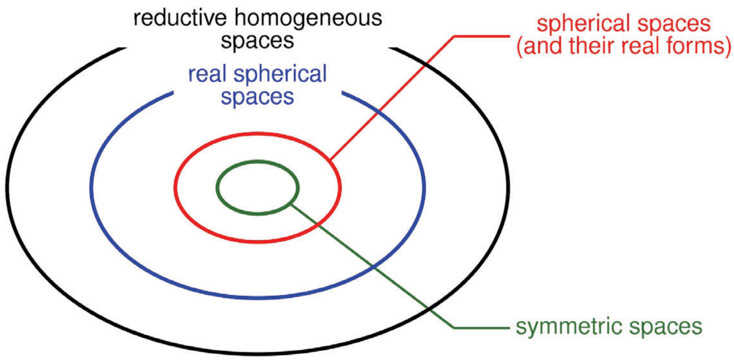
Part II: Classification Theory of Tempered Spaces

Quite surprisingly, it turns out that a complete description of tempered reductive homogeneous spaces G/H is realistic.

Why ?   How ?



* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Reminder: Summary of Part I

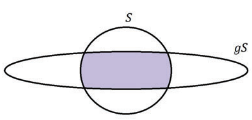
Let H be a semisimple Lie group.
Consider $H \to SL_{\mathbb{R}}(V)$ and $H \subset G$ (reductive).

<u>Theorems A and B</u>* (Part I) ($L^2(G/H)$)

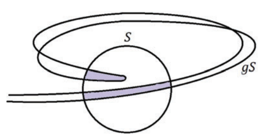
$\beta_V \leq 2 \iff H \curvearrowright L^2(V)$ is tempered.

$\beta_{\mathfrak{g}/\mathfrak{h}} \leq 1 \iff G/H$ is a tempered homogeneous space.

$\Longleftarrow$ easier (local estimate)



$\Longrightarrow$ more difficult (global estimate)



* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

Main theme of Part II

$\boxed{\text{Basic Problem: Classify all non-tempered homogeneous spaces.}}$

$\Updownarrow$

Combinatorial Aspect:
Understand the numerical invariant β_V associated to a finite-dimensional representation $\tau\colon H \to GL_{\mathbb{R}}(V)$.

Classification theory of non-tempered G/H — Strategy

$\boxed{\text{Setting: } G \supset H \text{ both real reductive.}}$

Step 1. Reduction Using the criterion (Theorem B in Part I), we have

1.A. G reductive $\rightsquigarrow$ G simple (perfect)

1.B. G/H non-tempered $\Longrightarrow$ $G_{\mathbb{C}}/H_{\mathbb{C}}$ non-tempered

Step 2. Classify non-tempered $G_{\mathbb{C}}/H_{\mathbb{C}}$ when $G_{\mathbb{C}}$ is complex simple.

2.A. Combinatorics for β_V for simple $H \curvearrowright V$ (irreducible)

2.B. Combinatorics for β_V for reductive $H \curvearrowright V$(reducible)

Step 3. Understand non-tempered $G_{\mathbb{C}}/H_{\mathbb{C}}$ for complex simple $G_{\mathbb{C}}$.

Step 4. Determine which real forms of $G_{\mathbb{C}}/H_{\mathbb{C}}$ are non-tempered.

Classifying non-tempered G/H — Step 1. Reduction

$\boxed{\text{Setting: } G \supset H \text{ both real reductive.}}$

Step 1.A. G reductive $\Rightarrow$ G simple

For $H \subset G = G_1 \times \cdots \times G_n$, we set $H_i := H \cap G_i$.

$$L^2(G/H) \text{ is tempered } \underset{\text{difficult}}{\overset{\text{easy}}{\rightleftarrows}} L^2(G_i/H_i) \text{ is tempered } \quad {}^\forall i.$$

Criterion $\Updownarrow$ (Theorem B) $\qquad\qquad$ $\Updownarrow$ Criterion (Theorem B)

$$2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}} \quad \Longleftrightarrow \quad 2\rho_{\mathfrak{h}_i} \le \rho_{\mathfrak{g}_i} \quad {}^\forall i.$$

$\boxed{\text{Example: If } \mathfrak{h} \cap \mathfrak{g}_i = \{0\}\ {}^\forall i, \text{ then } L^2(G/H) \text{ is tempered.}}$

Classification theory of non-tempered G/H — Strategy

$\boxed{\text{Setting: } G \supset H \text{ both real reductive.}}$

Step 1. Reduction

1.A. G reductive $\rightsquigarrow$ G simple (perfect)

1.B. $G_{\mathbb{C}}/H_{\mathbb{C}}$ tempered $\Longrightarrow$ G/H tempered

Classifying non-tempered G/H — Step 1. Reduction

$\boxed{\text{Setting: } G \supset H \text{ both real reductive.}}$

Step 1.A. G reductive $\Rightarrow$ G simple

Step 1.B. $G_{\mathbb{C}}/H_{\mathbb{C}}$ tempered $\Longrightarrow$ G/H tempered

$$\begin{array}{ccc} L^2(G_{\mathbb{C}}/H_{\mathbb{C}}) \text{ is tempered} & \Longrightarrow & L^2(G/H) \text{ is tempered.} \\ \text{Criterion}\Updownarrow & & \Updownarrow\text{Criterion} \\ 2\rho_{\mathfrak{h}_{\mathbb{C}}} \le \rho_{\mathfrak{g}_{\mathbb{C}}} & \Longrightarrow & 2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}}. \end{array}$$

Classification theory of non-tempered G/H — Strategy

$\boxed{\text{Setting: } G \supset H \text{ both real reductive.}}$

Step 1. Reduction

1.A. G reductive $\Longrightarrow$ G simple (perfect)

1.B. (G, H) real $\Longrightarrow$ $(G_{\mathbb{C}}, H_{\mathbb{C}})$ (useful)

Step 2. Classify non-tempered $G_{\mathbb{C}}/H_{\mathbb{C}}$ when $G_{\mathbb{C}}$ is complex simple.

2.A. Combinatorics for β_V for simple $H \curvearrowright V$ (irreducible)

2.B. Combinatorics for β_V for reductive $H \curvearrowright V$(reducible)

Step 3.

Step 4.

Recall:

Numerical Invariant β_V associated to $\tau \colon \mathfrak{h} \to \mathrm{End}(V)$

Let $\mathfrak{h}$ be a semisimple Lie algebra. Recall that β_V is the optimal constant satisfying

$$\beta_V \rho_V(Y) \geq \rho_{\mathfrak{h}}(Y) \qquad {}^\forall Y \in \mathfrak{h}.$$

Let $\mathfrak{a}$ be a maximally split abelian subspace of $\mathfrak{h}$. Then,

$$\beta_V = \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\sum |\text{eigenvalue of } \mathrm{ad}(Y) \in \mathrm{End}(\mathfrak{h})|}{\sum |\text{eigenvalue of } \tau(Y) \in \mathrm{End}(V)|}.$$

Example: Computation of ρ_V for Standard Rep $V = \mathbb{R}^n$

Short Summary $\tau \colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$

$\rightsquigarrow \rho_V, \rho_{\mathfrak{h}} \cdots$ piecewise linear function on $\mathfrak{a}$

$\beta_V \cdots$ non-negative number

$$\mathfrak{h} := \mathfrak{sl}(n, \mathbb{R}) \supset \mathfrak{a} := \{X = \mathrm{diag}(x_1, \ldots, x_n) : \sum x_i = 0\}.$$

Example 1) $V = \mathbb{R}^n$ (standard representation)

{Eigenvalues of $X \curvearrowright \mathbb{R}^n$} $= \{x_i : 1 \leq i \leq n\}$

$$\rho_V = \frac{1}{2} \sum_{i=1}^{n} |x_i|$$

Example 2) $V = \mathfrak{h}$ (adjoint representation)

{Eigenvalues of $\mathrm{ad}(X)$} $= \{x_i - x_j : 1 \leq i, j \leq n\}$

$$\rho_{\mathfrak{h}} = \sum_{1 \leq i < j \leq n} |x_i - x_j|$$

Example: Computation of β_V for Standard Rep $V = \mathbb{R}^n$

$V = \mathbb{R}^n$: the standard representation of $\mathfrak{h} = \mathfrak{sl}(n, \mathbb{R})$.
Inspecting the roots and weights, we have

$$\beta_V = \max \frac{\rho_{\mathfrak{h}}}{\rho_V} = \max_{x_1+\cdots+x_n=0} \frac{2\sum_{1\le i<j\le n}|x_i-x_j|}{|x_1|+\cdots+|x_n|}.$$

- We claim $\beta_V = 2(n-1)$.
- $\beta_V \ge 2(n-1)$.
  Proof: Take $x = (1, 0, \ldots, 0, -1)$ ("witness vector").
  $$\beta_V \ge \frac{2\sum_{1\le i<j\le n}|x_i-x_j|}{|x_1|+\cdots+|x_n|} = \frac{2\cdot 2(n-1)}{2} = 2(n-1).$$
- $\beta_V \le 2(n-1)$; that is,
  $$\sum_{1\le i<j\le n} |x_i - x_j| \le (n-1)(|x_1| + \cdots + |x_n|).$$

Further Example: $\mathfrak{h} = \mathfrak{gl}(n, \mathbb{R}) \curvearrowright V = \Lambda^3(\mathbb{R}^n)$

Computational Aspects

$$\beta_V = \max_{(x_1,\ldots,x_n)\neq(0,\ldots,0)} \frac{2\sum_{1\le i<j\le n}|x_i - x_j|}{\sum_{1\le i<j<k\le n}|x_i + x_j + x_k|}$$

The explicit value of β_V is given by [BK21]*.

$$\beta_V = \begin{cases} 6, 3, 2, \frac{10}{7}, \frac{10}{9} & \text{if } n = 4, 5, 6, 7, 8, \\ \le 1 & \text{if } n \ge 9. \end{cases}$$

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2021), 833–869.

Observation

As seen in the previous example about the numerical invariant β_V for

$$\mathfrak{gl}(n, \mathbb{R}) \curvearrowright V = \Lambda^3(\mathbb{C}^n),$$

we expect that there are not many reps V such that $\beta_V > 1$.

In other words, we expect that

$$\beta_V \le 1$$

for "most of" $\mathfrak{h}$-modules V; that is,

$$\rho_{\mathfrak{h}}(Y) \le \rho_V(Y) \qquad {}^\forall Y \in \mathfrak{h}.$$

List of V with $\beta_V > 1$: simple Lie algebra $\mathfrak{h}$

Consider the case where

- $\mathfrak{h}$: simple Lie algebra, and
- $\mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$ is irreducible.

Theorem E (BK2021*)
Let $\mathfrak{h} = \mathfrak{sl}(\ell + 1, \mathbb{R})$, and V an irreducible rep $/\mathbb{R}$.
If V or V^* is not in the following list (1)–(4), then $\beta_V \le 1$.

(1) $V = \mathbb{R}^{\ell+1} \quad \Rightarrow \beta_V = 2\ell$.

(2) $V = S^2(\mathbb{R}^{\ell+1}) \quad \Rightarrow \beta_V = \frac{2\ell}{\ell+1}$.

(3) $V = \Lambda^2(\mathbb{R}^{\ell+1}) \quad \Rightarrow \beta_V = \begin{cases} \frac{2(\ell+2)}{\ell} & \text{if } \ell \text{ is even,} \\ \frac{2(\ell+1)}{\ell-1} & \text{if } \ell \text{ is odd.} \end{cases}$

(4) $V = \Lambda^3(\mathbb{R}^{\ell+1}) \quad \Rightarrow \beta_V = 6, 3, 2, \frac{10}{7}, \frac{10}{9}$ if $\ell = 3, 4, 5, 6, 7$.

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2021), 833–869.

Tool

We expect that

$$\boxed{\beta_V} \leq 1$$

for "most of" $\mathfrak{h}$-modules V; that is,

$$\boxed{\rho_{\mathfrak{h}}}(Y) \leq \boxed{\rho_V}(Y) \qquad {}^{\forall}Y \in \mathfrak{h}.$$

- Analyze when $\boxed{\beta_V} > 1$ for a representation (τ, V).

$\cdots$ Finite inequalities on generators of covex polyhedral cones.

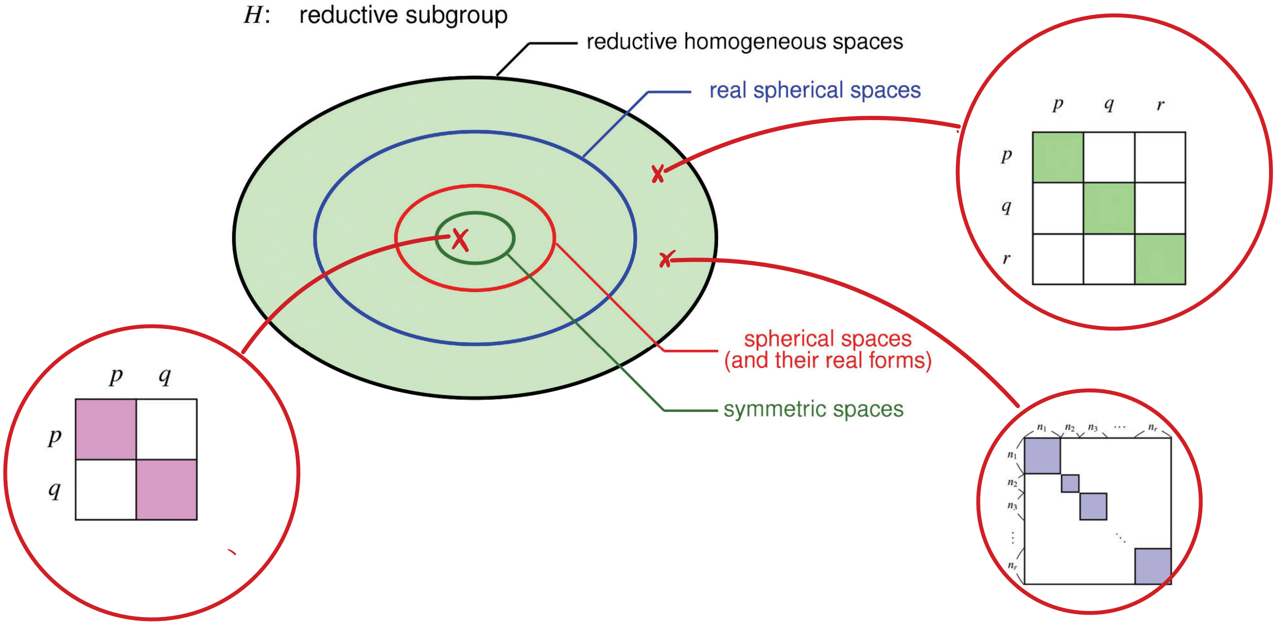
("exponential time" $\Rightarrow$ "polynomial time")

Case 1 $\mathfrak{h}$ simple, (τ, V) irreducible. $\cdots$ Theorem E

$$G/H = GL(n, \mathbb{R})/GL(n_1, \mathbb{R}) \times \cdots \times GL(n_r, \mathbb{R})$$

$$n_1 + n_2 + \cdots + n_r = n$$

G: real reductive groups
H: reductive subgroup



Combinatorics of temperedness criterion — Example

$$G/H = SL(p+q+r,\mathbb{R})/SL(p,\mathbb{R})\times SL(q,\mathbb{R})\times SL(r,\mathbb{R}).$$

Our temperedness criterion $\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}/\mathfrak{h}}$ amounts to the following:

$$\sum_{1\le i<j\le p} |x_i - x_j| + \sum_{1\le i<j\le q} |y_i - y_j| + \sum_{1\le i<j\le r} |z_i - z_j|$$
$$\le \sum_{\substack{1\le i\le p\\ 1\le j\le q}} |x_i - y_j| + \sum_{\substack{1\le j\le q\\ 1\le k\le r}} |y_j - z_k| + \sum_{\substack{1\le k\le r\\ 1\le i\le p}} |z_k - x_i|$$

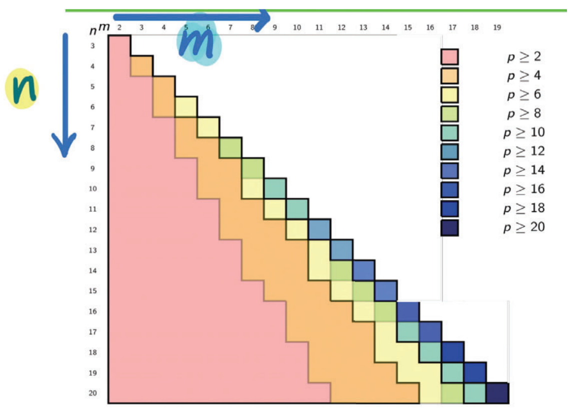
for all $(x_1,\cdots,x_p, y_1,\cdots,y_q, z_1,\cdots,z_r)\in\mathbb{R}^{p+q+r}$ with $\sum x_i = 0$, $\sum y_j = 0$, $\sum z_k = 0$.

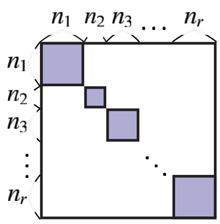
$$\Longleftrightarrow 2\max(p,q,r) \le p+q+r+1.$$

$\cdots$ combinatorics on convex polyhedral cones

Almost L^p representation

Example $G/H = GL(n,\mathbb{R})/GL(n_1,\mathbb{R})\times\cdots\times GL(n_r,\mathbb{R})$
The smallest even integer p for which $L^2(G/H)$ is almost L^p
amounts to $p = 2[\frac{n-1}{2(n-m)}]$ with $m = \max(n_1,\cdots,n_r)$.





* Y. Benoist–Y. Inoue–T. Kobayashi, J. Algebra (2023).

Tool

We expect that

$$\beta_V \leq 1$$

for "most of" $\mathfrak{h}$-modules V; that is,

$$\rho_{\mathfrak{h}}(Y) \leq \rho_V(Y) \qquad {}^\forall Y \in \mathfrak{h}.$$

Classification — feature : " huge factors " in $H_{\mathbb{C}}$

Begin with $G_{\mathbb{C}} \supset H_{\mathbb{C}}$ that are complex reductive.

Theorem F ("huge factor") * Let $G_{\mathbb{C}}$ be a simple Lie group, and $H_{\mathbb{C}}$ reductive subgroup. If $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is non-tempered, then $H_{\mathbb{C}}$ is " huge " in the following sense.

(Type A) If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$, then $\mathfrak{h}_{\mathbb{C}}$ contains

- $\mathfrak{sl}(p, \mathbb{C})$ with $p > \frac{1}{2}(n+1)$ or
- $\mathfrak{sp}(p, \mathbb{C})$ with $2p = n$.

$\cdots$

(Type E_7) If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{e}_7^{\mathbb{C}}$, then $\mathfrak{h}_{\mathbb{C}}$ contains $\mathfrak{d}_6^{\mathbb{C}}$ or $\mathfrak{e}_6^{\mathbb{C}}$.

(Type E_8) If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{e}_8^{\mathbb{C}}$, then $\mathfrak{h}_{\mathbb{C}}$ contains $\mathfrak{e}_7^{\mathbb{C}}$.

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Classification theory — theorem

Quite surprisingly, we can provide a complete description of non-tempered reductive homogeneous spaces G/H.

Theorem G* One can give a complete description of pairs $G \supset H$ of real reductive algebraic groups for which $L^2(G/H)$ is not tempered.

Example For $n_1 + \cdots + n_r \le n$, we consider

$$G/H := GL(n,\mathbb{R})/(GL(n_1,\mathbb{R}) \times \cdots \times GL(n_r,\mathbb{R})).$$

$L^2(G/H)$ is non-tempered $\iff \max_i n_i > \frac{1}{2}(n+1)$.

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Classification theory — theorem

Quite surprisingly, it turns out that a complete description of non-tempered reductive homogeneous spaces G/H is realistic.

Theorem G* One can give a complete description of pairs $G \supset H$ of real reductive algebraic groups for which $L^2(G/H)$ is not tempered.

Example For $p_1 + \cdots + p_r \le p$ and $q_1 + \cdots + q_r \le q$, we consider

$$G/H := SO(p,q)/(SO(p_1,q_1) \times SO(p_2,q_2) \times \cdots \times SO(p_r,q_r)).$$

$L^2(G/H)$ is non-tempered $\iff \max_{p_i q_i \neq 0} (p_i + q_i) > \frac{1}{2}(p+q+2)$.

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Combinatorics for β_V

Very special cases of combinatorics for β_V have already interactions with

- Kazhdan's estimate ($SL(3,\mathbb{R}) \downarrow SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$),
- Tempered subgroup a la Margulis,
- Minimal K-type theory of Vogan,
  (G, H) symmemtric pair, H split
- Plancherel formula for G/H,
  (G, H) semisimple symemtric pair
- Vanishing condition of gen. Borel–Weil–Bott theorem,
  Zuckerman's module $A_{\mathfrak{q}}(\lambda)$ with singular parameter λ,

and more.

Algebraic Aspect of β_V: AGS and AmGS

For a representation $\tau : H \to GL(V)$, we set $(V)_{\mathrm{Ab}} \subset (V)_{\mathrm{Am}}$ by

$$(V)_{\mathrm{Ab}} := \{x \in V : \text{the stabilizer } H_x \text{ is abelian}\},$$
$$(V)_{\mathrm{Am}} := \{x \in V : \text{the stabilizer } H_x \text{ is amenable}\}.$$

Definition (AGS)* V has AGS in $\mathfrak{h}$ if $\{x \in V : H_x \text{ is abelian and reductive}\}$ is dense in V.

Definition (AmGS)* V has AmGS in $\mathfrak{h}$ if $\{x \in V : H_x \text{ is amenable}\}$ is dense in V.

AGS =Abelian Generic Stabilizer,
AmGS =Amenable Generic Stabilizer.

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces III, J. Lie Theory (2021).

Classification theory: generic stabilizers of $H \curvearrowright \mathfrak{g}/\mathfrak{h}$

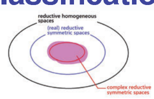
Classification theory includes:

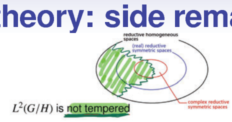
> Theorem H* $G \supset H$: pairs of real reductive algebraic groups.
> Then we have the implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).
> (i) $\mathfrak{g}/\mathfrak{h}$ has (AGS).
> (ii) $L^2(G/H)$ is a tempered unitary representation of G.
> (iii) $\mathfrak{g}/\mathfrak{h}$ has (AmGS).

> Corollary I $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is tempered iff $\mathfrak{g}/\mathfrak{h}$ has (AGS).

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Classification theory: side remarks 1





(a) Let $G_{\mathbb{C}}/H_{\mathbb{C}}$ be a complex reductive symmetric space.
Take a real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$ such that $G_{\mathbb{R}} \cap H_{\mathbb{C}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$. Corollary I in this special case implies that

> $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is $G_{\mathbb{C}}$-tempered $\iff$ $G_{\mathbb{R}}$ is quasi-split.

Vogan's theory on minimal K-types gives an alternative proof:

$2\rho_{\mathfrak{k}_{\mathbb{C}}} \leq \rho_{\mathfrak{g}_{\mathbb{C}}}$ $\underset{\text{Vogan}}{\iff}$ $G_{\mathbb{R}}$ is quasi-split

Lecture 1 $\Updownarrow$ Dynamics combinatorics $\Leftrightarrow$ Corollary I $\Updownarrow$ definition

$L^2(G_{\mathbb{C}}/K_{\mathbb{C}})$ is tempered $(\mathfrak{p}_{\mathbb{C}})_{\mathrm{Ab}}$ is dense in $\mathfrak{p}_{\mathbb{C}}$.

Classification theory: side remarks 2

- Special cases are already non-trivial.

(b) Let G/H be a reductive symmetric space.

The Plancherel theorem* for G/H:

$$L^2(G/H) \simeq \bigoplus_{j=1}^{N} \int_{\nu}^{\oplus} \sum_{\lambda}^{\oplus} \operatorname{Ind}_{L_jN_j}^{G}(\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)})d\nu.$$

$\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)} \cdots$ relative discrete series for $L_j/(L_j \cap H)$.

- Delicate issues arise from $\tau_\lambda^{(j)}$ with "singular" λ.

$L^2(G/H)$ is tempered $\overset{\text{Plancherel}}{\Longleftrightarrow}$ $\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)}$ is a tempered rep of L_j for all λ, a.e. ν

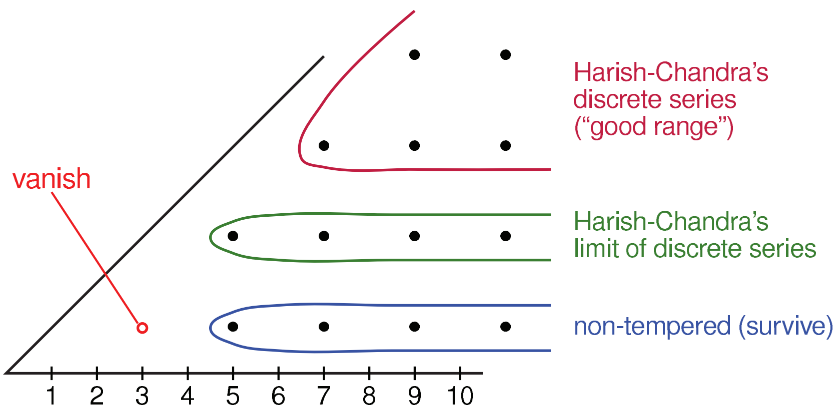
$\Downarrow$ Theorem I $\qquad\qquad$ $\Downarrow$ obvious

$(\mathfrak{g}/\mathfrak{h})_{\mathrm{Am}}$ is dense in $\mathfrak{g}/\mathfrak{h}$ $\overset{\text{Quantization}}{\Longleftrightarrow}$ $\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)}$ is a tempered rep of L_j, ${}^\forall\lambda \gg 0$, a.e. ν

* T. Oshima (1980s); Delorme, Ann. Math. 1998; van den Ban–Schlichtkrull, Invent. Math. 2005.

** Y. Benoist–T. Kobayashi, Tempered homogeneous spaces III, J. Lie Theory (2021).

Discrete series for $Sp(4,1)/Sp(1) \times Sp(3,1)$



Tempered Homogeneous Spaces

Part 1. Ideas from Dynamical Approach [I, II]

Part 2. Combinatorial Approach [III]

Part 3. Topological Approach and Quantization [IV]

* Y. Benoist–TK, I (2015), II (2022), III (2021), IV (2023) $+\alpha$.

Varion Aspects of β_V

– – – Algebraic Aspects : (AGS) and (AmGS)

– – – Topological Aspects : Limit Algebras

– – – Gometric Aspects : Quantization of Coadjoint Orbits

Limit algebras — Formulation

By forgetting the Lie algebra structure of $\mathfrak{g}$, one considers

$$G \overset{\mathrm{Ad}}{\curvearrowright} \mathrm{Gr}(\mathfrak{g}) := \coprod_{m=0}^{\dim \mathfrak{g}} \mathrm{Gr}_m(\mathfrak{g}), \quad \text{(Grassmann variety)}.$$

$\mathfrak{h}$: a subalgebra of $\mathfrak{g}$, with dimension m.

$\rightsquigarrow$ $\mathfrak{h}$ may be regarded as a point of $\mathrm{Gr}_m(\mathfrak{g})$.

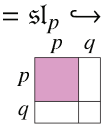
$\mathrm{Gr}(\mathfrak{g}) \underset{\text{submanifold}}{\supset} \mathrm{Ad}(G)\mathfrak{h}$, which may or may not be closed.

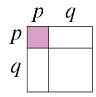
$$\mathrm{Gr}(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty \quad \text{(limit algebra)}$$

<u>Definition</u> (limit algebra) $\mathfrak{h}_\infty$ $(\subset \mathfrak{g})$ is a limit algebra of $\mathfrak{h}$ in $\mathfrak{g}$ if $^\exists$ sequence $g_j \in G$ such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h} = \mathfrak{h}_\infty$ in $\mathrm{Gr}(\mathfrak{g})$.

Deforming Lie algebras to solvable ones

<u>Example</u> $\mathfrak{h} = \mathfrak{sl}_p \hookrightarrow \mathfrak{g} = \mathfrak{sl}_{p+q}$

$q \le p$  does not have a solvable limit.

$q \ge p+1$  has a solvable limit.

<u>Definition</u> (solvable limit algebra) $\mathfrak{h} \subset \mathfrak{g}$ Lie algebras
We say $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$ if
$^\exists g_j \in G$ such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h}$ is a solvable Lie algebra.

Topological Aspect of β_V

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra.

Definition (Limit Algebra) $\mathfrak{h}_\infty$ is a limit algebra if there exists a sequence $g_j \in \mathrm{Int}(\mathfrak{g})$ such that

$$\lim_{j \to \infty} g_j \cdot \mathfrak{h} = \mathfrak{h}_\infty \quad \text{in } \mathfrak{g}.$$

Let $\mathfrak{g}$ be a complex reductive Lie algebra.
Consider the adjoint action of $\mathfrak{h}$ on $V := \mathfrak{g}/\mathfrak{h}$.

Theorem J ([BK23]*)

$$\beta_{\mathfrak{g}/\mathfrak{h}} \leq 1 \iff \mathfrak{h} \text{ has a solvable limit } \mathfrak{h}_\infty \subset \mathfrak{g}.$$

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, (2023).

Variety of all Lie algebras $\mathcal{L}$ and its subset $\mathcal{S}$

Formulation: Consider the variety of all subalgebras in $\mathfrak{g}$.

$$Gr(\mathfrak{g}) \equiv \coprod_{N=0}^{\dim \mathfrak{g}} \mathrm{Gr}_N(\mathfrak{g}) \qquad \cdots \text{ algebraic variety}$$

$\cup$

$\mathcal{L}$:= {subalgebras of $\mathfrak{g}$} $\cdots$ algebraic variety

$\cup$

$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^\exists\mathfrak{h}_\infty$ solvable $\}$

$\cup$

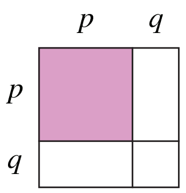
{solvable subalgs} $\cdots$ algebraic variety

Variety of all Lie algebras $\mathcal{L}$ and its subset $\mathcal{S}$

$\mathfrak{g}$: Lie algebra.

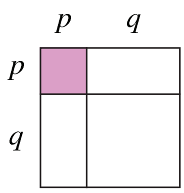
$$\mathcal{L} := \{\text{subalgebras of } \mathfrak{g}\}$$
$$\cup$$
$$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_{\infty} \text{ solvable}\}$$

<u>Question</u> What does $\mathcal{S}$ look like in $\mathcal{L}$?



$p \geq q$

$\notin \mathcal{S} \ni$



$p \leq q-1$

Topology of $\mathcal{S} = \{\mathfrak{h} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_{\infty}$ solvable$\}$

Suppose $\mathfrak{g}$ is an algebraic Lie algebra $/\mathbb{C}$.

<u>Open Problem L</u> Is $\mathcal{S}$ open in $\mathcal{L}$?

<u>Theorem M</u>*

(1) $\mathcal{S}$ is closed in $\mathcal{L}$.

(2) $\mathcal{S}$ is open and closed in $\mathcal{L}$ if $\mathfrak{g}$ is semisimple.

Recall

$$\mathcal{L} := \{\text{subalgebras of } \mathfrak{g}\}$$
$$\cup$$
$$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_{\infty} \text{ solvable}\}$$

Our proof for Theorem M uses unitary representation theory.

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

Geometric quantization and temperedness

$\mathrm{Ad}\colon G \to GL_{\mathbb{R}}(\mathfrak{g})$ adjoint representation.

$\mathrm{Ad}^*\colon G \to GL_{\mathbb{R}}(\mathfrak{g}^*)$ coadjoint representation.

Coadjoint orbit $O_\lambda := \mathrm{Ad}^*(G)\lambda$ for $\lambda \in \mathfrak{g}^*$.

<u>Lemma</u> (Kostant–Kirillov–Souriau)
Every coadjoint orbit O_λ carries a natural symplectic structure.

"Geometric quantization":

$$\mathfrak{g}^* \supset \underbrace{O_\lambda = \mathrm{Ad}^*(G)\lambda}_{\text{symplectic mfd}} \overset{?}{\rightsquigarrow} \underbrace{\pi_\lambda \in \widehat{G}}_{\text{unitary rep}}$$

Expect

$$\mathfrak{g}^*/\mathrm{Ad}^*(G) \fallingdotseq \widehat{G}$$

Geometric quantization and temperedness

"Geometric quantization": $\mathfrak{g}^* \supset \underbrace{O_\lambda = \mathrm{Ad}^*(G)\lambda}_{\text{symplectic mfd}} \overset{?}{\rightsquigarrow} \underbrace{\pi_\lambda \in \widehat{G}}_{\text{unitary rep}}$

$\mathfrak{h}^\perp := \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{h}} \equiv 0\}$

$\mathfrak{g}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{g}^* : \mathrm{Ad}^*(G)\cdot\lambda$ is of maximal dimension$\}$

$$\begin{array}{ccc}
\mathrm{Ad}^*(G)\mathfrak{h}^\perp/\mathrm{Ad}^*(G) & \fallingdotseq & \mathrm{Supp}(L^2(G/H)) \\
\cap & & \cap \\
\mathfrak{g}^*/\mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G} \\
\cup & & \cup \\
\mathfrak{g}^*_{\mathrm{reg}}/\mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G}_{\mathrm{temp}}
\end{array}$$

Geometric quantization and temperedness

"Geometric quantization": $\mathfrak{g}^* \supset \mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda \overset{?}{\rightsquigarrow} \pi_\lambda \in \widehat{G}$

symplectic mfd — unitary rep

$$\begin{array}{ccc} \mathrm{Ad}^*(G)\mathfrak{h}^\perp/\mathrm{Ad}^*(G) & \fallingdotseq & \mathrm{Supp}(L^2(G/H)) \\ \cap & & \cap \\ \mathfrak{g}^*/\mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G} \\ \cup & & \cup \\ \mathfrak{g}^*_{\mathrm{reg}}/\mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G}_{\mathrm{temp}} \end{array}$$

$\mathfrak{h}^\perp := \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{h}} \equiv 0\}$

$\mathfrak{g}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{g}^* : \mathrm{Ad}^*(G)\cdot\lambda$ is of maximal dimension$\}$

We may ask:

Question*
Suppose G is a real reductive Lie group,
and H a connected closed subgroup. Is (i) $\Leftrightarrow$ (ii)?
(i) $G \curvearrowright L^2(G/H)$ is tempered.
(ii) $\mathfrak{g}^*_{\mathrm{reg}} \cap \mathfrak{h}^\perp$ is dense in $\mathfrak{h}^\perp$.

Further interactions for "tempered spaces"

Theorem P Let $\mathfrak{g}$ be a complex reductive Lie algebra.
The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

(i)	(Analysis)	$L^2(G/H)$ is tempered.
(ii)	(combinatorics)	$2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.
(iii)	(Geometric quantization)	$\mathfrak{h}^\perp \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.
(iv)	(Topology)	$\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$.

Application Representation theory $\Longrightarrow$ Topology

Corollary Q (Topology) The property "having solvable limit" is an open and closed condition for subalgebras in a complex reductive Lie algebra $\mathfrak{g}$, namely, $\mathcal{S}$ is open and closed in $\mathcal{L}$.

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

Sketch of Proof for Theorem P: Tempered homogeneous spaces

Thm P Let $\mathfrak{g}$ be a complex reductive Lie algebra.

The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

(i) (unitary rep) $L^2(G/H)$ is tempered.

(ii) (combinatorics) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.

(iii) (orbit method) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.

(iv) (limit algebra) $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$.

$L^2(G/H)$

Analysis (i)

Part 1

dynamical system $\nearrow\!\!\swarrow$ $\nwarrow\!\!\searrow$ geom quantization

Classification $\Longleftarrow$ Algebra (ii) $\Longleftrightarrow$ Geometry (iii)

Part 2 $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ $\qquad$ $O_\lambda = \mathrm{Ad}^*(G)\lambda$ symplectic mfd

Topology (iv)

$\lim_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h}$

References

The theme of the mini-course is joint with Yves Benoist.

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Method (Dynamical System)

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Representation Theory

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Limit algebra, geometric quantization

Tensor product of GL_n (J. Algebra, 2023) Benoist-Inoue-TK