HARISH-CHANDRA'S ADMISSIBILITY THEOREM AND BEYOND

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Abstract. This article is a record of the lecture at the centennial conference for Harish-Chandra. The admissibility theorem of Harish-Chandra concerns the restrictions of irreducible representations to maximal compact subgroups. In this article, we begin with a brief explanation of two directions for generalizing his pioneering work to *non-compact* reductive subgroups: one emphasizes discrete decomposability with the finite multiplicity property, while the other focuses on finite/uniformly bounded multiplicity properties. We discuss how the recent representation-theoretic developments in these directions collectively offer a powerful method for the new spectral analysis of standard locally symmetric spaces, extending beyond the classical Riemannian setting.

Keywords and phrases: Harish-Chandra, reductive group, branching problem, locally symmetric space, multiplicity, unitary representation.

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Contents

1 Admissible Restriction à la Harish-Chandra

Harish-Chandra's celebrated *admissibility theorem* of real reductive Lie groups G (Theorem 1.1 below) concerns the restriction of representations of G to maximal compact subgroups K. This theorem has become foundational for an algebraic theory of infinite-dimensional representations of G using the notion of *Harish-Chandra modules*. It also plays a crucial role in proving that reductive groups G are of type I in the Murray–von Neumann sense (see [D77]), which implies that irreducible decompositions of any unitary representations of G are essentially unique.

An analogous statement to Harish-Chandra's admissibility does not generally hold for the restriction to reductive symmetric pairs such as $(GL(n,\mathbb{R}), O(p,n-p))$.

In this exposition, we investigate two avenues of research inspired by Harish-Chandra's admissibility theorem for the last three decades. We do so in a more general setting, focusing on the restriction to *non-compact* subgroups. In Section 5, we demonstrate that these generalizations provide a new tool to explore the spectral theory of standard *pseudo-Riemannian* locally symmetric spaces.

1.1. Reminder: Harish-Chandra's admissibility theorem.

Let G be a linear real reductive Lie group, and let K be a maximal compact subgroup. We denote by $\mathrm{Irr}(K)$ the set of equivalence classes of irreducible (finite-dimensional) representations of the compact group K.

Harish-Chandra proved the following fundamental result [HC53, HC54b], combining with Segal's theorem [Se52].

Theorem 1.1 (Harish-Chandra's admissibility theorem)**.** *For any irreducible unitary representation* Π *of* G*, one has the following finite-multiplicity property:*

$$[\Pi|_K : \pi] < \infty \qquad \text{for any } \pi \in \mathrm{Irr}(K). \tag{1.1}$$

Here $[\Pi|_K : \pi]$ *denotes the multiplicity of the representation* π *occurring in the restriction of* Π *to* K*.*

1.2. From Riemannian to reductive symmetric pairs.

One may think of Harish-Chandra's admissibility as a theorem about the restriction of representations of G to K, symbolically written as $G \downarrow K$. The pair (G, K) is referred to as a *Riemannian symmetric pair* from a differential-geometric perspective, as the homogeneous space G/K carries a G-invariant Riemannian metric such that the geodesic symmetry at every point defines a global isometry. A typical example is $(G, K) = (GL(n, \mathbb{R}), O(n))$. More generally, pairs such as $(GL(n, \mathbb{R}), O(p, q))$ where $p + q = n$ have a similar geometric property, and are referred to as *reductive symmetric pairs*.

Definition 1.2. Let G be a real reductive Lie group, σ an involutive automorphism of G, and G' an open subgroup of the fixed point group G^σ. The pair (G, G') is called a *reductive symmetric pair.*

Here are typical examples of reductive symmetric pairs.

Example 1.3. (1) The Riemannian symmetric pair (G, K) corresponding to the Cartan involution θ.
(2) The pair $({}^{\backprime}G \times {}^{\backprime}G, \operatorname{diag} {}^{\backprime}G)$. Let σ be the involutive automorphism of the direct product group $G = {}^{\backprime}G \times {}^{\backprime}G$ defined by $\sigma(x, y) = (y, x)$. Then the fixed-point subgroup $G^\sigma = \operatorname{diag} {}^{\backprime}G$.
(3) The pair $(GL(n, \mathbb{R}), O(p, q))$ with $p + q = n$.
(4) The pair $(O(p, q), O(p_1, q_1) \times O(p_2, q_2))$ with $p_1 + p_2 = p$, $q_1 + q_2 = q$.

In Example 1.3 (1), the decomposition of a representation Π of the group G with respect to the Riemannian symmetric pair (G, K) is referred to as the *K-type formula.*

In Example 1.3 (2), the tensor product representation $\pi_1 \otimes \pi_2$ of two representations π_1 and π_2 of the group ${}^{\backprime}G$ can be viewed as the restriction of the outer tensor product $\pi_1 \boxtimes \pi_2$ of the group $G = {}^{\backprime}G \times {}^{\backprime}G$ to the diagonal subgroup $\operatorname{diag} {}^{\backprime}G$.

Thus, the restriction of a representation of the group G with respect to a symmetric pair (G, G') can be considered as a generalization of these two examples.

We consider two avenues for generalizing Harish-Chandra's admissibility theorem for pairs $G \supset G'$ of reductive Lie groups, particularly for reductive symmetric pairs:

- G'-admissible restriction (Section 2),
- Finite multiplicity property (Section 3).

The former theme focuses on the property of the irreducible decomposition of the restriction $\Pi|_{G'}$ not containing continuous spectrum when the representation Π of G is unitary. In contrast, the latter allows Π to be non-unitary and focuses only on the finite-multiplicity property.

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2 Admissible Restriction to Non-Compact Subgroups

Let $G \supset G'$ be a pair of real reductive Lie groups, particularly when G' is a *non-compact subgroup*, and Π an irreducible unitary representation of G. In what follows, we shall use the upper case letter Π for representations of a group G, and the lower case letter π for those of the subgroup G'. Our main concern in this section is how and when the restriction $\Pi|_{G'}$ behaves like Harish-Chandra's admissibility theorem (Theorem 1.1) for $\Pi|_K$.

2.1. Admissible restriction $G \downarrow G'$ for a non-compact subgroup.

By a theorem of Mautner [Mt50], the restriction $\Pi|_{G'}$ of any unitary representation Π decomposes into a direct integral of irreducible unitary representations of G'. The irreducible decomposition (*the branching law*) usually contains continuous spectrum when G' is non-compact.

The following notion of *G'-admissible restriction* was introduced in [Ko94] for a *non-compact* subgroup G', partly inspired by Harish-Chandra's admissibility theorem:

Definition 2.1. The restriction $\Pi|_{G'}$ is said to be *G'-admissible* if it can be decomposed discretely into a direct sum of irreducible unitary representations π of G':

$$\Pi|_{G'} \simeq \sum_{\pi \in \widehat{G'}}^{\oplus} m_\pi \pi \qquad \text{(discrete sum)}$$

with the multiplicity $m_\pi := [\Pi|_{G'} : \pi]$ is finite for every $\pi \in \widehat{G'}$.

Here $\widehat{G'}$ denotes the unitary dual of the group G', that is, the set of equivalence classes of irreducible unitary representations of G'.

The key aspect of Definition 2.1 is that we require not only the absence of continuous spectrum but also the finiteness of each multiplicity m_π.

When the subgroup G' is compact, the discrete decomposability of the restriction $\Pi|_{G'}$ is automatically guaranteed, thus the main concern is the finiteness of each multiplicity m_π. Harish-Chandra's admissibility (Theorem 1.1) corresponds to the case $G' = K$, stating that any $\Pi \in \widehat{G}$ is K-admissible, in our terminology.

Remark 2.2. The absence of continuous spectrum is also formalized algebraically in the category of $(\mathfrak{g}, K)$-modules without requiring the unitarizability in [Ko98b]. See also Corollary 2.4.

2.2. Restriction to compact subgroups K' $(\subset K)$.

Our interest is in analyzing the restriction $\Pi|_{G'}$ of $\Pi \in \widehat{G}$ for a pair of real reductive Lie groups $G \supset G'$.

The key idea in [Ko94] is first to focus on the K'-structure of the G-module Π, where K and K' are maximal compact subgroups of G and G', respectively, see below.

$$\begin{array}{ccccl} G & \supset & G' & & \text{real reductive groups} \\ \cup & & \cup & & \\ K & \supset & K' & & \text{maximal compact subgroups} \end{array}$$

Theorem 2.3 (Criterion for K'-admissibility)**.** *Suppose* $\Pi \in \widehat{G}$. *Then the following two conditions (i) and (ii) on the triple* (Π, G, K') *are equivalent:*

(i) (multiplicity) $[\Pi|_{K'} : \pi] < \infty$ *for any* $\pi \in \mathrm{Irr}(K')$.

(ii) (geometry) $\mathrm{AS}_K(\Pi) \cap C_K(K') = \{0\}$.

The condition (ii) in Theorem 2.3 uses two closed cones $\mathrm{AS}_K(\Pi)$ and $C_K(K')$ in the dual space of a Cartan subalgebra of $\mathfrak{k}$. $\mathrm{AS}_K(\Pi)$ is the asymptotic K-support of the representation Π. There are only finitely many possibilities for $\mathrm{AS}_K(\Pi)$ among $\Pi \in \mathrm{Irr}(G)$. The closed cone $C_K(K')$ is the momentum set associated with the Hamiltonian action on the cotangent bundle $T^*(K/K')$; see [Ko05b, Chap. 6] for a detailed exposition.

Theorem 2.3 holds without assuming that the representation Π is unitary. The implication (i) $\Rightarrow$ (ii) was proved in full generality in [Ko98a] based on an estimate of the singularity spectrum of hyperfunction characters (or the wavefront set of distribution characters). An alternative and algebraic proof was given in [Ko21b]. The converse implication (ii) $\Rightarrow$ (i) was originally given in [Ko02, Ko05b] with a sketch of the proof, while the full proof is provided in [Ko21b].

2.3. Admissible restriction $G \downarrow G'$.

By Theorem 2.3, we obtain the discrete decomposability of the restriction of $\Pi \in \widehat{G}$ with respect to $G \downarrow G'$.

Corollary 2.4 (Criterion for admissible restriction)**.** *Suppose that a triple* (Π, G, G') *such that* $\Pi \in \widehat{G}$ *and* $G \supset G'$ *satisfies one of (therefore, any of) the equivalent conditions in Theorem 2.3.*

(1) ([Ko94]) *The restriction* $\Pi|_{G'}$ *is* G'*-admissible (Definition 2.1).*

(2) ([Ko98b]) *The underlying* $(\mathfrak{g}, K)$*-module* Π_K *is discretely decomposable as a* $(\mathfrak{g}', K')$*-module (Remark 2.2).*

Remark 2.5. In the case $K' = K$, one has clearly $C_K(K') = \{0\}$. Thus Corollary 2.4 (1) can be viewed as a generalization of Harish-Chandra's admissibility (Theorem 1.1).

2.4. Examples of admissible restrictions $\Pi|_{G'}$.

For $\Pi \in \widehat{G}$ and $G \supset G'$, we collect typical examples of the triples (Π, G, G') such that the restriction $\Pi|_{G'}$ is G'-admissible, namely, it decomposes discretely with finite multiplicity, as if it were Harish-Chandra's admissibility for the restriction to a maximal compact subgroup K.

Example 2.6 (Theta correspondence)**.** Let G be the metaplectic group $Mp(n, \mathbb{R})$, a double covering group of the symplectic group $Sp(n, \mathbb{R})$ of rank n, and $G' = G'_1 \cdot G'_2$ be a compact dual pair. Then the restriction $\Pi|_{G'}$ of the oscillator representation, also referred to as the Segal–Shale–Weil representation, is G'-admissible. Moreover, the branching law is multiplicity-free [H79].

Example 2.7. The tensor product of any two holomorphic discrete series representations decomposes discretely, and each multiplicity is finite [R79]. Moreover, the multiplicities are uniformly bounded [Ko07]. Its generalization to reductive symmetric pairs along with some explicit branching laws can be found in [Ko07].

2.5. Classification theory: Admissible restriction $\Pi|_{G'}$.

Corollary 2.4 together with a necessary condition for the algebraic discrete decomposability (Remark 2.2) given in [Ko98b] provides a family of triples (Π, G, G') where $\Pi \in \widehat{G}$ and $G \supset G'$ such that the restriction $\Pi|_{G'}$ is G'-admissible, namely, it decomposes discretely decomposable with finite multiplicity.

Some classification results of such triples (Π, G, G') highlighted the following cases:

- ([KOy15]) The tensor product $\Pi_1 \otimes \Pi_2$ for $\Pi_1, \Pi_2 \in \widehat{G}$.
- ([KOy12]) (G, G') is a reductive symmetric pair, Π is a discrete series representation of G, or more generally, when the underlying $(\mathfrak{g}, K)$-module Π_K is Zuckerman's derived functor module $A_{\mathfrak{q}}(\lambda)$.
- ([KOy15]) (G, G') is a reductive symmetric pair, and Π is the minimal representation of G.
- ([DGV17]) (G, G') is a non-symmetric pair where $G' = SL(2, \mathbb{R})$, Π is a discrete series representation of G.

Example 2.8. In Section 5, we give yet another geometric setting such that the restriction $\Pi|_{G'}$ is G'-admissible if

- X is a G-space on which G' acts properly and spherically,
- $\Pi \in \widehat{G}$ occurs in the space $\mathcal{D}'(X)$ of distributions.

This geometric setting is motivated by new spectral theory on pseudo-Riemannian locally symmetric spaces and the G'-admissibility is proved without relying on Theorem 2.3, see Theorem 5.13 below.

For some explicit discrete branching laws, see [Ko94, GW00, Ko07, O24]; for further topics on discretely decomposable restrictions, see [Ki24, Ko24].

3 Bounded/Finite Multiplicity Pairs for Restriction

Let $G \supset G'$ be a pair of real reductive Lie groups. In the previous section, we focused on G'-admissible restrictions of irreducible unitary representations Π, as a way to generalize Harish-Chandra's admissibility to non-compact subgroups by requiring discrete decomposability and finiteness of multiplicities. The absence of continuous spectrum facilitates an algebraic approach to the restriction $\Pi|_{G'}$ of the unitary representation Π, even when G' is non-compact.

In this section, we focus solely on the multiplicity by dropping the requirement of discrete decomposability.

We will see that the finite-multiplicity property is not obvious even for reductive symmetric pairs, such as

$$(GL(n, \mathbb{R}), O(p, q)) \text{ or } (GL(n, \mathbb{R}), GL(p, \mathbb{R}) \times GL(q, \mathbb{R})) \text{ when } p + q = n.$$

We also explore a stronger condition referred to as *bounded multiplicity property*, which does not generally hold even for restrictions related to Riemannian symmetric pairs (G, K) but it still holds for some reductive symmetric pairs (Theorem 3.12). These perspectives lead us to yet another avenue of the restriction problem.

3.1. Reminder: smooth representations of moderate growth.

It is observed that irreducible continuous representations Π of real reductive groups can exhibit wild behavior if they are not unitary. Even when $G = \mathbb{R}$, there exists an infinite-dimensional irreducible representation of the abelian group G on a Banach space, as a consequence of a counterexample to the invariant subspace problem by Enflo, *cf.* [E87]. Harish-Chandra's admissibility provides a guiding principle for identifying "reasonable" classes of continuous representations of reductive Lie groups avoiding such counterexamples, defined as below.

Definition 3.1 (admissible representation)**.** A continuous representation Π of G is said to be *admissible* (K*-admissible* for later terminology) if

$$[\Pi|_K : \pi] < \infty \qquad \text{for every } \pi \in \mathrm{Irr}(K).$$

To be rigorous about 'multiplicities' for infinite-dimensional representations, we need to specify the topology of the representation spaces. For this purpose, let G be a real reductive Lie group, and $\mathcal{M}(G)$ denote the category of smooth admissible representations of finite length with moderate growth, which are defined on Fréchet topological vector spaces [Wa88, Chap. 11]. Casselman–Wallach's theory shows that there is a natural category equivalence between $\mathcal{M}(G)$ and the category of $(\mathfrak{g}, K)$-modules of finite length.

Let $\mathrm{Irr}(G)$ denote the set of irreducible objects in $\mathcal{M}(G)$.

If Π is an admissible continuous representation of finite length on a Banach space, then the representation Π^∞ acting on the Fréchet space of C^∞-vectors belongs to $\mathcal{M}(G)$. In particular, this yields a natural injection:

$$\underset{\text{unitary dual}}{\widehat{G}} \hookrightarrow \mathrm{Irr}(G), \quad \Pi \mapsto \Pi^\infty.$$

3.2. Multiplicity of the restriction $\Pi|_{G'}$.

We use the category $\mathcal{M}(G)$ to define the "multiplicity" in the restriction $\Pi|_{G'}$.

Definition 3.2 (Symmetry Breaking Operator)**.** Let $\Pi \in \mathcal{M}(G)$ and $\pi \in \mathcal{M}(G')$. A continuous G'-homomorphism from $\Pi|_{G'}$ to π is referred to as a *symmetry breaking operator*. Let $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$ denote the vector space of symmetry breaking operators. The *multiplicity* of π in the restriction $\Pi|_{G'}$ is defined by its dimension, that is,

$$[\Pi|_{G'} : \pi] := \dim \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\}. \tag{3.1}$$

The definition of the multiplicity (3.1) in the category $\mathcal{M}(G)$ coincides with the multiplicity in the category of unitary representations if the restriction is G'-admissible (Definition 2.1).

3.3. Comparison: $GL(n,\mathbb{R}) \downarrow O(n)$ *vs* $GL(n,\mathbb{R}) \downarrow O(p,n-p)$**.**
Harish-Chandra's admissibility theorem (Theorem 1.1) concerns the restriction with respect to a Riemannian symmetric pair

$$G \supset K, \qquad \text{e.g., } GL(n,\mathbb{R}) \supset O(n)$$

and asserts that

$$[\Pi|_K : \pi] < \infty \quad \text{for any } \Pi \in \mathrm{Irr}(G) \text{ and for any } \pi \in \mathrm{Irr}(K).$$

In contrast, for a reductive symmetric pair

$$G \supset G', \ e.g., \ GL(n,\mathbb{R}) \supset O(p,n-p)$$

it may occur that

$$[\Pi|_{G'} : \pi] = \infty \quad \text{for some } \Pi \in \mathrm{Irr}(G) \text{ and } \pi \in \mathrm{Irr}(G').$$

The classification of reductive symmetric pairs (G,G') having the finite multiplicity property

$$[\Pi|_K : \pi] < \infty \quad \text{for any } \Pi \in \mathrm{Irr}(G) \text{ and for any } \pi \in \mathrm{Irr}(G').$$

has been established in [KM14] using the criterion in Theorem 3.10 below. We will explain the background of the theory.

3.4. Spherical vs real spherical.
Let $G_{\mathbb{C}}$ be a complex reductive Lie group, and $X_{\mathbb{C}}$ a connected complex manifold on which $G_{\mathbb{C}}$ acts holomorphically.

Definition 3.3. The $G_{\mathbb{C}}$-space $X_{\mathbb{C}}$ is *spherical* if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$.

Example 3.4 ([Wo69])**.** Complex reductive symmetric spaces are spherical.

In search of a broader framework for global analysis on homogeneous spaces than those known in the late 1980s (e.g. group manifolds, semisimple symmetric spaces), the author advocated introducing the following concept from the viewpoint of the finite multiplicity property.

Definition 3.5 ([Ko95])**.** Let X be a connected C^∞ manifold on which a real reductive Lie group G acts as diffeomorphisms. We say X is *real spherical* if a minimal parabolic P of G has an open orbit in X.

In what follows, we assume that G is a real reductive group with complexification $G_{\mathbb{C}}$, and that H is an algebraic reductive subgroup of G with complexification $H_{\mathbb{C}}$. We write $X = G/H$ and $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$.

Remark 3.6. It is important to emphasize that the following two notions differ:

- real forms of spherical spaces,
- real spherical spaces.

The former is stronger than the latter; namely, if $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical then its real form G/H is always real spherical ([KOt13, Lem. 3.5]). The converse is not necessarily true. For instance, any homogeneous G/H is real spherical if G is compact, but its complexification $G_{\mathbb{C}}/H_{\mathbb{C}}$ is not necessarily spherical.

Remark 3.7. (1) The definition of real sphericity of $X = G/H$ in Definition 3.5 is equivalent to the condition that $\#(P\backslash G/H) < \infty$ (Kimelfeld, Matsuki, Bien). This extends a theorem by Brion and Vinberg, which asserts that a $G_{\mathbb{C}}$-space $X_{\mathbb{C}}$ is spherical if and only if $\#(B\backslash X_{\mathbb{C}}) < \infty$.
(2) For compact H, Akhiezer–Vinberg [AV99] proved that G/H is a weakly symmetric space in the sense of Selberg if and only if $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical.
(3) For compact G, Tanaka [Tn22] proved that $X_{\mathbb{C}}$ is G-strongly visible in the sense of [Ko05b] if and only if $X_{\mathbb{C}}$ is $G_{\mathbb{C}}$-spherical.

3.5. Analytic view of real spherical spaces.

A fundamental requirement in non-commutative harmonic analysis for a G-space X is that the space of functions on X should be well-controlled by the group G. To be rigorous, we formalize the degree of control of the group G in terms of the multiplicity $\dim \mathrm{Hom}_G(\Pi, C^\infty(X))$, that is, the number of times each irreducible representation $\Pi \in \mathrm{Irr}(G)$ occurs in $C^\infty(X)$.

The following theorem provides a geometric criterion for the finiteness of the multiplicity:

Theorem 3.8 (Finite Multiplicity Space)**.** *Let G be a reductive Lie group, H a reductive algebraic subgroup of G, and $X = G/H$. Then the following two conditions on the pair (G, H) are equivalent.*

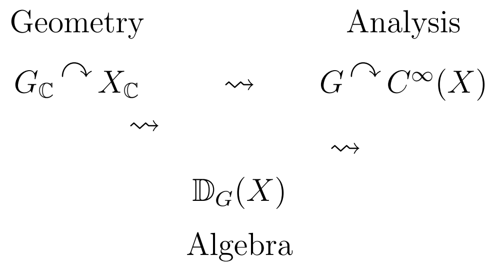
(i) (Global analysis) $\dim \mathrm{Hom}_G(\Pi, C^\infty(X)) < \infty$ *for every* $\Pi \in \mathrm{Irr}(G)$.
(ii) (Geometry) *X is G-real spherical.*

In the original proof ([KOt13, Thm. A]), we consider a more general setting for the space of distribution sections of equivariant vector bundles over $X = G/H$, where H is not necessarily reductive, and not only gives a qualitative result (the equivalence in Theorem 3.8) but also provides quantitative results. Specifically, we give an upper estimate of the multiplicity by using hyperfunction boundary value maps for partial differential equations. For a lower estimate, we generalize an idea of the classical Poisson transform, see [Ko14, Sec. 6.1] for more details. These estimates from the above and below establish a necessary and sufficient condition for the uniform boundedness of the multiplicity, which gives a stronger degree of control of the group G over the function space of X, as discussed in the next section.

3.6. Analytic view of spherical spaces.

A discovery in [Ko05a, KOt13] reveals the fact that the uniform boundedness property of the multiplicity in $C^\infty(X)$ is determined solely by the complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$. This is in sharp contrast to the finiteness criterion established in Theorem 3.8.

The results are summarized in the following theorem, which shows a harmony of analysis, geometry, and algebra:



Theorem 3.9 (Criterion for Uniformly Bounded Multiplicity)**.** *Let $G \supset H$ be a pair of real reductive Lie groups, and $X = G/H$. Then the following conditions (i), (ii), (iii), and (iii)′ on the pair (G, H) are equivalent:*

(i) (Global analysis) *There exists a constant* $C > 0$ *such that*

$$\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \leq C \quad \textit{for all } \pi \in \mathrm{Irr}(G). \tag{3.2}$$

(ii) (Complex geometry) $X_{\mathbb{C}}$ *is* $G_{\mathbb{C}}$*-spherical.*
(iii) (Ring structure) *The algebra* $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ *is a commutative ring.*
(iii) $'$(Ring structure) *The algebra* $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ *is a polynomial ring.*

The proof of the equivalence (iii) $\Leftrightarrow$ (iii)$'$ is given in Knop [Kn94], and the equivalence (ii) $\Leftrightarrow$ (iii) was established earlier, see Vinberg [Vi01] and references therein.

In contrast to conditions (ii), (iii), and (iii)$'$, depending solely on the complexifications $(G_{\mathbb{C}}, H_{\mathbb{C}})$, the objects in (i) such as $\mathrm{Irr}(G)$ and $C^\infty(G)$ are strongly dependent on the choice of the real forms G and X of $G_{\mathbb{C}}$ and $X_{\mathbb{C}}$. The equivalence (i) $\Leftrightarrow$ (ii) was established by the author in collaboration with T. Oshima [KOt13].

3.7. Restriction $G \downarrow G'$ with finite multiplicity property.

We apply Theorem 3.8 to the homogeneous space $(G \times G')/\operatorname{diag} G' \simeq G$ to study the restriction of representations with respect to $G \downarrow G'$.

Theorem 3.10 (Finite Multiplicity Pairs for Restriction, [Ko14])**.** *The following two conditions (i) and (ii) for a pair of real reductive groups* $G \supset G'$ *are equivalent:*
(i) (Representation theory) $[\Pi|_{G'} : \pi] < \infty$ *for every* $\Pi \in \mathrm{Irr}(G)$*, and for every* $\pi \in \mathrm{Irr}(G')$*.*
(ii) (Geometry) $(G \times G')/\operatorname{diag} G'$ *is real spherical.*

Example 3.11 (Harish-Chandra's admissibility)**.** The geometric condition (ii) in Theorem 3.10 holds when $G' = K$ by the Gauss–Iwasawa decomposition $G = KAN$. The representation-theoretic condition (i) corresponds to Harish-Chandra's admissibility when $G' = K$.

A complete classification of reductive symmetric pairs (G, G'), where G' is *non-compact*, satisfying the geometric condition (ii) was established by the author in collaboration with Matsuki [KM14].

3.8. Restriction $G \downarrow G'$ with uniformly bounded multiplicity property.

As in Theorem 3.10, applying Theorem 3.9 to the homogeneous space $(G \times G')/\operatorname{diag} G'$ yields a necessary and sufficient condition for the uniform boundedness property of the restriction $G \downarrow G'$:

Theorem 3.12 (Uniformly Bounded Multiplicity Pairs for Restriction)**.** *Let $G \supset G'$ be a pair of real reductive groups. Then the following four conditions (i), (ii), (iii), and (iii)′ are equivalent:*

(i) (Representation Theory) $\sup_{\Pi \in \operatorname{Irr}(G)} \sup_{\pi \in \operatorname{Irr}(G')} [\Pi|_{G'} : \pi] < \infty$.

(ii) (Complex Geometry) $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag} G'_{\mathbb{C}}$ *is spherical.*

(iii) (Ring) *The algebra $U(\mathfrak{g}_{\mathbb{C}})^{G'}$ is a commutative ring.*

(iii)′ (Ring) *The algebra $U(\mathfrak{g}_{\mathbb{C}})^{G'}$ is a polynomial ring.*

See [Ko14] for some other equivalent conditions, as well as for the proof of the equivalence (i) ⇔ (ii). See also [Ko95, KOt13]. There are a few pairs (G, G') satisfying (i)–(iii)′ but in which the supremum in (i) is greater than one. However, for most of the important cases, a sharper estimate for (ii) ⇒ (i) holds, that is, the supremum in (i) equals one (Sun–Zhu [SZ12]).

In the trivial case where $G = G'$, the finiteness condition (i) is evident. The sphericity condition (ii) is guaranteed by the Bruhat decomposition, while the ring structure (iii)′ is established via the Harish-Chandra isomorphism [HC58].

The classification of such complex pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ was provided by Krämer [Kr76] and Kostant in 1970s if $\mathfrak{g}_{\mathbb{C}}$ is simple, specifically $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ being $(\mathfrak{sl}(n,\mathbb{C}), \mathfrak{gl}(n-1,\mathbb{C}))$, $(\mathfrak{so}(n,\mathbb{C}), \mathfrak{so}(n-1,\mathbb{C}))$, up to abelian factors and possibly considering outer automorphisms.

3.9. Example: $O(p,q) \downarrow O(p_1,q_1) \times O(p_2,q_2)$.

We examine the finiteness criterion (Theorem 3.10) and the uniform boundedness criterion (Theorem 3.12) in the context of the symmetric pair $(G, G') = (O(p,q), O(p_1,q_1) \times O(p_2,q_2))$ where $p_1 + p_2 = p$ and $q_1 + q_2 = q$, and $p + q \geq 5$.

Our criteria tell us the following equivalences:

- $\sup_{\Pi \in \operatorname{Irr}(G)} \sup_{\pi \in \operatorname{Irr}(G')} [\Pi|_{G'} : \pi] < \infty \iff p_1 + q_1 = 1$ or $p_2 + q_2 = 1$;
- $[\Pi|_{G'} : \pi] < \infty$ for every $\Pi \in \operatorname{Irr}(G)$ and for every $\pi \in \operatorname{Irr}(G')$
  $\iff p_1 + q_1 = 1,\ p_2 + q_2 = 1,\ p = 1,\ q = 1$, or G' compact.

The proof of Theorem 3.10 shows that for general values of p_1, p_2, q_1, q_2, it can occur that the multiplicity

$$[\Pi|_{G'} : \pi] = \infty$$

for some principal series representations $\Pi \in \mathrm{Irr}(G)$ and $\pi \in \mathrm{Irr}(G')$, which contrasts with the case where G' is compact (Harish-Chandra's admissibility).

3.10. Question: Bounded multiplicity $\Pi|_{G'}$ for "small" Π.
By inspecting the above examples, one sees that refining the question should broaden the concept of "good classes" for branching problems. Thus, we consider a triple (Π, G, G'), where $\Pi \in \mathrm{Irr}(G)$ and $G \supset G'$, instead of just a pair (G, G'), as in the formulation of the admissible restriction (Corollary 2.4). We now pose the following question.

Question 3.13. Given a reductive symmetric pair $G \supset G'$, does there exist at least one infinite-dimensional $\Pi \in \mathrm{Irr}(G)$ with the following bounded multiplicity property?

$$\sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty.$$

3.11. An affirmative answer to Question 3.13.
In [Ko07], we provided an affirmative answer to Question 3.13 in the context of the reductive symmetric pair (G, G') given by Hermitian Lie groups such as $(U(p,q), U(p_1,q_1) \times U(p_2,q_2))$ where $p_1 + p_2 = p$ and $q_1 + q_2 = q$ using the theory of *visible actions* on complex manifolds. For the general case, we have proved the following result.

Theorem 3.14 ([Ko22a]). *Let G be a simply connected, non-compact, real semisimple Lie group. There exist a constant $C \equiv C(G) > 0$ and an infinite-dimensional irreducible representation Π of G such that*

$$\sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] \leq C$$

for all symmetric pairs (G, G').

Example 3.15 (tensor product). There exist a constant $C > 0$ and infinite-dimensional irreducible representations Π_1, Π_2 of G such that

$$[\Pi_1 \otimes \Pi_2 : \Pi] \leq C \quad \text{for every } \Pi \in \mathrm{Irr}(G).$$

It is important to note that Theorem 3.14 establishes the uniformly bounded multiplicity property. Thus, even in the special case of $G' = K$ in Theorem 3.14, the result does not follow from Harish-Chandra's admissibility theorem, which guarantees only individual finiteness but not the uniformly bounded multiplicity property.

Another example of $\Pi \in \widehat{G}$ that gives an affirmative answer to Question 3.13 is the minimal representation:

Theorem 3.16 ([Ko22b])**.** *Let G be a real reductive Lie group. If the associated variety of $\Pi \in \mathrm{Irr}(G)$ is the minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}^*$, then there exists a constant $C \equiv C(\Pi) > 0$ such that*

$$\sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] \leq C$$

for all reductive symmetric pairs (G, G').

3.12. Methods of proof.

The original approach in [KOt13] to prove the finiteness/uniform boundedness for multiplicities under certain geometric conditions (*e.g.*, (ii) $\Rightarrow$ (i) in Theorem 3.8, (ii) $\Rightarrow$ (i) in Theorem 3.9, and (ii) $\Rightarrow$ (i) in Theorem 3.10, etc.) was to use partial differential equations with regular singularities at the boundary of a specific compactification. Further approaches used by Tauchi, Kitagawa, Kobayashi, Aisenbud–Gourevich, Tanaka and other researchers include holonomic $\mathcal{D}$-modules [Tu22], visible actions on complex manifolds [Ko05a, Tn24], and coisotropic actions on symplectic manifolds [Ki21], etc.

Some of these methods have broader applications, though upper estimates of the multiplicities are not necessarily as sharp as those in [KOt13].

The proof of the converse statement employs integral transforms [Ko14], which provide lower bounds for the multiplicities in Theorems 3.9 and 3.12.

4 Bounded Multiplicity Triple for Restriction

In search of a natural framework for a detailed and potentially fruitful analysis of branching problems (for example, Stage C in the ABC program [Ko15], which studies the restriction of representations), we focus on a specific family of "small" representations $\Pi \in \mathrm{Irr}(G)$ for which the restriction has the *uniformly bounded* property: By *uniformity*, we consider not only representations $\pi \in \mathrm{Irr}(G')$ but also the family of representations $\Pi \in \mathrm{Irr}(G)$ as described below.

Question 4.1. Find a triple (Ω, G, G') where $\Omega \subset \mathrm{Irr}(G)$ and $G \supset G'$ which satisfies the following uniform boundedness property:

$$\sup_{\Pi \in \Omega} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty. \tag{4.1}$$

This question further explores Question 3.13. We have seen several affirmative results in the previous sections, such as

$$\Omega = \mathrm{Irr}(G) \qquad \text{(Theorem 3.12)}$$

$$\Omega = \{\text{minimal representations}\} \qquad \text{(Theorem 3.16)}$$

In the next section, we will discuss Question 4.1, focusing on $\Omega := \{H\text{-distinguished representations}\}$.

4.1. Restriction of H-distinguished representations $H \nearrow G$. We set up some notation. Let H be a closed subgroup of a Lie group G.

Definition 4.2. We say $\Pi \in \mathrm{Irr}(G)$ is an *H-distinguished representation* of G, if $(\Pi^{-\infty})^H \neq \{0\}$, or equivalently if

$$\mathrm{Hom}_G(\Pi, C^\infty(G/H)) \neq \{0\}$$

by the Frobenius reciprocity. Let $\mathrm{Irr}(G)_H$ denote the subset of $\mathrm{Irr}(G)$ consisting of H-distinguished irreducible admissible representations.

4.2. Bounded multiplicity triple for $H \nearrow G \searrow G'$.

Definition 4.3. A triple $H \subset G \supset G'$ of real reductive Lie groups is said to be a *bounded multiplicity triple* if (4.1) holds for $\Omega = \mathrm{Irr}(G)_H$.

This means that for every representation $\Pi \in \mathrm{Irr}(G)$ that can be realized in $C^\infty(G/H)$, the multiplicity of the restriction $\Pi|_{G'}$ is uniformly bounded:

$$\sup_{\Pi \in \mathrm{Irr}(G)_H} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty.$$

We shall discuss an aspect of the classification theory of bounded multiplicity triples established in [Ko22a] below.

We begin with some naïve considerations for Definition 4.3 including:

(1) If the subgroup H is "large" in G, then we may expect that the representation $\Pi \in \mathrm{Irr}(G)_H$ will be "small".

(2) If the subgroup G' is "large" in G and if the representation Π is "small", then we may expect that the subgroup G' will have a strong degree of the control, in the sense that the restriction $\Pi|_{G'}$ will have a bounded multiplicity property.

The "largeness" of the subgroups depends on the properties we are examining. The conditions for H are formulated in Theorem 4.9, while those for G' and H are presented in Theorem 4.6 (ii) below.

4.3. Relative Borel subalgebra $\mathfrak{b}_{G/H}$ and relative parabolic subgroup $P_{G/H}$.

In non-commutative harmonic analysis of real reductive Lie groups G, the notion of minimal parabolic subgroups of G and Borel subalgebras in the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$ plays a fundamental role. We consider the generalization of this notion to reductive symmetric spaces G/H associated to an involutive automorphism σ of G.

We take a maximal compact subgroup $G_U \subset G_{\mathbb{C}}$ such that $G_U \cap G$ and $G_U \cap H$ are also maximal compact subgroups of G and H, respectively. We fix an $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ which is also non-degenerate on the subalgebra $\mathfrak{h}$. We write $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^{\perp}$ for the direct sum decomposition and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \mathfrak{h}_{\mathbb{C}}^{\perp}$ for its complexification. Recall that to a given hyperbolic element Y in $\mathfrak{g}$, one associates a parabolic subalgebra of $\mathfrak{g}$, defined as the sum of eigenspaces of $\mathrm{ad}(Y)$ with non-negative eigenvalues.

Definition 4.4. (Relative Borel subalgebra $\mathfrak{b}_{G/H}$ and parabolic subalgebra $\mathfrak{p}_{G/H}$ [Ko22a]). Let (G, H) be a reductive symmetric pair.

(1) *A Borel subalgebra* $\mathfrak{b}_{G/H}$ for G/H is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. It is defined by a generic element of $\mathfrak{h}_{\mathbb{C}}^{\perp} \cap \sqrt{-1}\mathfrak{g}_U$ or its conjugate by an inner automorphism of $G_{\mathbb{C}}$.

(2) *A minimal parabolic subalgebra* $\mathfrak{p}_{G/H}$ for G/H is a real parabolic

subalgebra of $\mathfrak{g}$. It is defined by a generic element of $\mathfrak{h}^\perp \cap \sqrt{-1}\mathfrak{g}_U$ or its conjugate by an inner automorphism of G.

According to the definition, $\mathfrak{b}_{G/H}$ is determined solely from the complexified symmetric pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.

Remark 4.5. In contrast to the usual definition of a Borel subalgebra, the relative Borel subalgebra $\mathfrak{b}_{G/H}$ is not necessarily solvable.

4.4. Bounded multiplicity theorem for $H \nearrow G \searrow G'$.

In Theorems 4.6 and 4.8 below, we set the following conditions:

$$\begin{aligned} &(G,H) && : \text{a reductive symmetric pair,} \\ &G' && : \text{a reductive subgroup of } G. \end{aligned}$$

We do not need to assume that (G, G') is a symmetric pair.

Theorem 4.6 (Bounded Multiplicity Criterion, [Ko22a, Thm. 1.2])**.** *The following two conditions* (i) *and* (ii) *on a triple $H \subset G \supset G'$ are equivalent.*
(i) (Representation Theory) *$H \subset G \supset G'$ is a bounded multiplicity triple; that is,*

$$\sup_{\Pi \in \mathrm{Irr}(G)_H} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty.$$

(ii) (Complex Geometry) *The generalized flag variety $G_\mathbb{C}/B_{G/H}$ is $G'_\mathbb{C}$-spherical.*

Example 4.7 ($\operatorname{diag} G \nearrow G \times G \searrow G' \times G'$)**.** In view of the natural bijection,

$$\mathrm{Irr}\, G \simeq \mathrm{Irr}(G \times G)_{\operatorname{diag} G}, \quad \pi \leftrightarrow \pi \boxtimes \pi^\vee, \tag{4.2}$$

where $\pi^\vee$ is the contragredient representation, Theorem 4.6 in this special case implies the bounded multiplicity theorem [KOt13, Thm. D], as stated in the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 3.12.

The following theorem, when applied to the triple $\operatorname{diag} G \subset G \times G \supset G' \times G'$, recovers the implication (ii) $\Rightarrow$ (i) in Theorem 3.10 (*cf.* [KOt13, Thm. C]).

Theorem 4.8 (Finite Multiplicity Triple $H \nearrow G \searrow G'$ [Ko22a])**.** *Retain the setting for $G' \subset G \supset H$. Then the following implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *hold:*
(i) (Complex geometry) $\sharp(P'_{\mathbb{C}} \backslash G_{\mathbb{C}}/(P_{G/H})_{\mathbb{C}}) < \infty$, *where P' is a minimal parabolic subgroup of G'.*
(ii) (Representation Theory) *The multiplicity $[\Pi|_{G'} : \pi] < \infty$ for every $\Pi \in \mathrm{Irr}(G)_H$ and for every $\pi \in \mathrm{Irr}(G')$.*
(iii) (Real Geometry) *The generalized real flag variety $G/P_{G/H}$ is G'-real spherical, that is, P' has an open orbit in $G/P_{G/H}$ (Definitions 3.5 and 4.4).*

For the proof of Theorems 4.6 and 4.8, we prove a "QP estimate" [Ko22a, Thm. 3.1] for the uniform bounded multiplicity of the restriction along the same line in [KOt13, Tu22] and use a uniform upper estimate of the "largeness" for all H-distinguished representations. The latter can be formulated via the following generalization of Harish-Chandra's subquotient theorem as outlined below.

4.5. Generalizing Harish-Chandra's subquotient theorem.

Harish-Chandra's subquotient theorem [HC54b] has been strengthened as the *subrepresentation theorem* by Casselman [CM82] among others: it asserts that any $\pi \in \mathrm{Irr}(G)$ can be realized as a subrepresentation (also as a quotient) of some principal series representation.

We present a theorem that sharpen the subrepresentation theorem for $\pi \in \mathrm{Irr}(G)_H$, by replacing principal series representations with induced representations from finite-dimensional representations of the parabolic subgroup $P_{G/H}$.

In the special case of $(G \times G, \mathrm{diag}\, G)$, Theorem 4.9 recovers the subrepresentation theorem through the isomorphism (4.2).

Theorem 4.9 (Subrepresentation Theorem for G/H)**.** *Let (G, H) be a reductive symmetric pair. For any $\Pi \in \mathrm{Irr}(G)_H$, there exists a finite-dimensional irreducible representation V of $P_{G/H}$ such that*

$$\mathrm{Hom}_G(\mathrm{Ind}_{P_{G/H}}^G(V), \Pi) \neq \{0\}.$$

Moreover, the representation V in Theorem 4.9 has specific constraints that can be formulated using the relative Borel subalgebra $\mathfrak{b}_{G/H}$: see [Ko22a, Thm. 1.8] for details. The proof of the uniform boundedness property in Theorem 4.6 makes use of these constraints.

4.6. Classification: Bounded multiplicity triples.

A complete classification of bounded multiplicity triples $H \subset G \supset G'$ has been achieved in [Ko21a, Ko22a]; this classification is based on the criterion in Theorem 4.6. It has an unexpected relationship with the new spectral theory of locally symmetric spaces beyond the classical Riemannian setting, which will be discussed in the next section.

To conclude this section, we provide examples of such bounded multiplicity triples.

Example 4.10 ($H \nearrow G \searrow G'$)**.** For any p_1, q_1, p_2, q_2 with $p_1+p_2 = p$, $q_1 + q_2 = q$, the triple

$$(G, G', H) = (O(p, q), O(p_1, q_1) \times O(p_2, q_2), O(p - 1, q))$$

is a bounded multiplicity triple, that is,

$$\sup_{\Pi \in \mathrm{Irr}(G)_H} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty.$$

5 Application of Branching Problem $G \downarrow G'$ in Geometry

In this section, we discuss a seemingly unrelated topic, specifically the spectral analysis of pseudo-Riemannian locally symmetric spaces $\Gamma\backslash G/H$, beyond the classical Riemannian setting.

We shall see that the general theory of the restriction $G \downarrow G'$, as discussed in Sections 2 to 4, provides a new tool for the study of $L^2(\Gamma\backslash G/H)$, where H is a *non-compact* subgroup.

We begin with the general setup. Let G be a Lie group, H be a closed subgroup, and Γ be a *discontinuous group* for $X = G/H$. This means that Γ is a discrete subgroup of G acting properly discontinuously and freely on G/H. Consequently, the double coset space $\Gamma\backslash G/H$, with the quotient topology, is Hausdorff and admits the unique C^∞ manifold structure for which the covering map $p_\Gamma \colon G/H \to \Gamma\backslash G/H$ is a local diffeomorphism.

$$\begin{array}{ccccc} & & G & & \\ & \swarrow & & \searrow & \\ \Gamma\backslash G & & & & G/H \\ & \searrow & & \swarrow & p_\Gamma \\ & & \Gamma\backslash G/H & & \end{array}$$

Via the covering map p_Γ, any G-invariant local geometric object can be pushed forward to the quotient manifold $X_\Gamma := \Gamma\backslash G/H$.

Let $\mathbb{D}_G(X)$ denote the algebra of G-invariant differential operators on $X = G/H$. Then any $D \in \mathbb{D}_G(X)$ induces a differential operator D_Γ on the quotient X_Γ via the covering map $X \to X_\Gamma$. We consider the set

$$\mathbb{D}(X_\Gamma) := \{D_\Gamma : D \in \mathbb{D}(G/H)\}$$

as the algebra of *intrinsic differential operators* on the locally homogeneous space X_Γ.

Example 5.1. If $G \supset H$ is a pair of real reductive Lie groups, then X_Γ inherits a pseudo-Riemannian structure from a G-invariant pseudo-Riemannian structure on $X = G/H$. The Laplacian Δ_{X_Γ} is an element of $\mathbb{D}(X_\Gamma)$.

We remark that for non-compact H, the pseudo-Riemannian structure is not necessarily positive definite; consequently, the Laplacian is generally not an elliptic differential operator. For instance, if the quotient space G/H is Lorentzian, the following equation on the space X_Γ:

$$\Delta_{X_\Gamma} f = \lambda f \tag{5.1}$$

is a hyperbolic equation.

Suppose now that $X = G/H$ is a reductive symmetric space. Then the algebra $\mathbb{D}(X_\Gamma) \simeq \mathbb{D}_G(X)$ is a commutative ring. Hence, it is natural to consider joint eigenfunctions of $\mathbb{D}(X_\Gamma)$ on $C^\infty(X_\Gamma)$ rather than focusing on the single equation (5.1).

For any $\mathbb{C}$-algebra homomorphism $\lambda\colon \mathbb{D}(X_\Gamma) \to \mathbb{C}$, we denote by $C^\infty(X_\Gamma, \mathcal{M}_\lambda)$ the space of smooth functions defined on X_Γ that satisfy the system of equations

$$(\mathcal{M}_\lambda) \qquad D_\Gamma f = \lambda(D) f \quad \text{for any } D \in \mathbb{D}_G(X).$$

Let $\mathbb{D}_G(X)^\wedge$ be the set of all $\mathbb{C}$-algebra homomorphisms $\lambda\colon \mathbb{D}(X_\Gamma) \to \mathbb{C}$, which can be identified with the quotient $\mathfrak{j}_{\mathbb{C}}^\vee/W$ of the dual $\mathfrak{j}_{\mathbb{C}}^\vee$ of a Cartan subspace $\mathfrak{j}$ for (G, H) by the Weyl group W for the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$, as shown in the Harish-Chandra isomorphism [HC58].

Not much attention has been paid to the spectral theory on $X_\Gamma = \Gamma\backslash G/H$ in the general setting where H is *non-compact* and Γ is an infinite discontinuous group. For instance, the following questions regarding the spectral theory remain open in this generality.

Problem 5.2.
(1) The expansion of arbitrary functions defined on X_Γ in terms of joint eigenfunctions of the algebra $\mathbb{D}(X_\Gamma)$ of intrinsic differential operators.
(2) Understanding the distributions of L^2-eigenvalues.

5.1. Spectral analysis on $\Gamma\backslash G/H$ in the classical case.

These problems for spectral analysis on $X_\Gamma = \Gamma\backslash G/H$ are formulated from a broader perspective, building on the rich and deep body of classical results that have been extensively studied. Special (classical) cases that have been particularly fruitful include:

(1) Let $H = K$. When H is a maximal compact subgroup K of G, X_Γ becomes a *Riemannian* locally symmetric space. A vast theory has been developed over several decades. Problem 5.2 is particularly enriched in connection with the local theory of automorphic forms when Γ is an arithmetic subgroup.
(2) Let $\Gamma = \{e\}$. When $\Gamma = \{e\}$, X_Γ reduces to the homogeneous space G/H. Problem 5.2 (1) has been extensively studied in the case where G/H is a reductive symmetric space and $\Gamma = \{e\}$, with significant contributions from Gelfand, Harish-Chandra [HC76] for group manifolds $(G \times G)/\operatorname{diag} G$; Helgason, Flensted-Jensen, T. Oshima, Delorme, and others for symmetric spaces G/H.
(3) Let $G = \mathbb{R}^{p,q}$, $\Gamma = \mathbb{Z}^{p+q}$, and $H = \{0\}$. In this case, $\operatorname{Spec}_d(X_\Gamma)$ consists of values of indefinite quadratic forms of signature (p, q)

at the dual lattice $\Gamma^\vee$. See [Ko16] for a discussion on Problem 5.2 (2), which highlights a relation to the Oppenheim conjecture (see *e.g.*, Margulis [Mg00] and references therein) in Diophantine approximation.

5.2. Spectral analysis on $\Gamma\backslash G/H$ beyond the classical Riemannian setting.

The situation changes drastically beyond the classical setting, that is, when H is no longer compact and $\Gamma \neq \{e\}$. New difficulties arise, including:

(1) (Representation theory) The space $L^2(X_\Gamma)$ is no longer a subspace of $L^2(\Gamma\backslash G)$, where the group G acts as a unitary representation. Moreover, we cannot expect the regular representation of G on $L^2(\Gamma\backslash G)$ to have finite multiplicities, contrary to the classical theorem of Gelfand–Piatetski–Shapiro.

(2) (Analysis) In contrast to the Riemannian case, where $H = K$, the pseudo-Riemannian Laplacian Δ_{X_Γ} is no longer an elliptic differential operator. Moreover, it is unclear whether Δ_{X_Γ} is essentially self-adjoint, due to the absence of a general theory.

(3) (Geometry) If H is not compact, then not all homogeneous spaces G/H can admit discontinuous groups of infinite order (*e.g.,* the Calabi Markus phenomenon [CM62, Ko89]).

5.3. Standard locally homogeneous spaces $\Gamma\backslash G/H$.

The geometric issue (3) in Section 5.2 raises a "local to global" problem beyond the Riemannian setting, which can be formulated by group-theoretic terms as the following fundamental question.

Question 5.3. How can we find a discrete subgroup Γ that acts properly discontinuously on G/H?

Here are some elementary observations regarding two extreme cases.

Observation 5.4. (1) Any discrete subgroup Γ suffices if H is compact.
(2) However, any lattice Γ of the whole group G does not suffice if H is non-compact, because $\Gamma\backslash G/H$ cannot be Hausdorff in this case.

A straightforward method to find discrete subgroups Γ that answer Question 5.3 is to utilize "standard quotients": we recall that G' acts *properly* if the map $G' \times X \to X \times X$, given by $(g, x) \mapsto (x, gx)$, is proper.

Definition 5.5. Let G' be a reductive subgroup, which acts properly on X, and let Γ be a torsion-free discrete subgroup of G'. Then Γ acts properly discontinuously on X, and we say the quotient manifold $X_\Gamma = \Gamma\backslash G/H$ is a *standard quotient* of X.

By using standard quotients, it turns out that there exist several families of reductive symmetric spaces G/H that admit "large" discontinuous groups Γ, *e.g.,* such that $X_\Gamma = \Gamma\backslash G/H$ is compact or has finite volume [Ku81, Ko89].

5.4. Spectral theory of standard locally symmetric space $\Gamma\backslash G/H$.

For the study of spectral analysis of pseudo-Riemannian locally symmetric spaces, we focus on *standard* quotients X_Γ (Definition 5.5) of a reductive symmetric space $X = G/H$, where Γ is a discrete subgroup of a reductive subgroup G' that acts properly on X.

Theorem 5.6 (Expansion into Joint Eigenfunctions, [KK20, KK24])**.** *Assume that $G'_{\mathbb{C}}$ acts spherically on $X_{\mathbb{C}}$ (Definition 3.3). Then any $f \in C_c^\infty(\Gamma\backslash X)$ can be expanded into joint eigenfunctions of $\mathbb{D}(X_\Gamma)$ on $\Gamma\backslash X$. More precisely, there exist a measure μ on $\mathbb{D}_G(X)^\wedge(\simeq \mathfrak{j}_{\mathbb{C}}^\vee/W)$ and a family of maps*

$$\mathcal{F}_\lambda \colon C_c^\infty(X_\Gamma) \to C^\infty(X_\Gamma; \mathcal{M}_\lambda)$$

such that

$$f = \int_{\mathbb{D}_G(X)^\wedge} \mathcal{F}_\lambda f \, d\mu(\lambda),$$

$$\|f\|^2_{L^2(X_\Gamma)} = \int_{\mathbb{D}_G(X)^\wedge} \|\mathcal{F}_\lambda f\|^2_{L^2(X_\Gamma)} d\mu(\lambda)$$

for any $f \in C_c^\infty(X_\Gamma)$.

Remark 5.7. In Theorem 5.6, we do not assume that $\mathrm{vol}(\Gamma\backslash G') < \infty$. In particular, Theorem 5.6 holds even when $\Gamma = \{e\}$.

The following corollary answers an analytic issue (2) in Section 5.2 in the setting of Theorem 5.6.

Corollary 5.8. *In the setting of Theorem 5.6, the pseudo-Riemannian Laplacian Δ_{X_Γ} is essentially self-adjoint on $L^2(X_\Gamma)$.*

5.5. Examples.

The properness criterion for the triple (G', G, H) was established in [Ko89], and is both explicit and computable. Below are some examples to which Theorem 5.6 is applied.

Example 5.9 (Riemannian locally symmetric space)**.** Let G/K be a Riemannian symmetric space. Set $G' := G$. Then the triple (G', G, K) clearly satisfies the assumption of Theorem 5.6. Consequently, the conclusion of Theorem 5.6 holds for any Riemannian locally symmetric space $\Gamma\backslash G/K$, including the case of infinite volume.

Example 5.10 (standard anti-de Sitter manifolds)**.** Let X be an odd-dimensional anti-de Sitter space, that is, $X = G/H = SO(2n, 2)/SO(2n, 1)$. The subgroup $G' := U(n, 1)$ acts properly on X, and $G'_{\mathbb{C}} = GL(n+1, \mathbb{C})$ acts spherically on $X_{\mathbb{C}} = SO(2n + 2, \mathbb{C})/SO(2n + 1, \mathbb{C})$. Therefore, Theorem 5.6 holds for any standard anti-de Sitter manifold X_Γ with $\Gamma \subset U(n, 1)$.

Example 5.11 (indefinite Kähler manifolds)**.** The homogeneous space $X = G/H = SO(2n, 2)/U(n, 1)$ has a natural indefinite-Kähler structure, and the subgroup $G' = SO(2n, 1)$ acts properly. Moreover, $G'_{\mathbb{C}} = SO(2n + 1, \mathbb{C})$ acts spherically on $X_{\mathbb{C}} = SO(2n + 2, \mathbb{C})/GL(n + 1, \mathbb{C})$. Thus, Theorem 5.6 holds for any standard indefinite-Kähler manifold X_Γ with $\Gamma \subset SO(2n, 1)$.

Example 5.12 (15-dimensional space form of signature $(8, 7)$)**.** Let $X = G/H = SO(8, 8)/SO(8, 7)$. Then X is a pseudo-Riemannian manifold of signature $(8, 7)$ with constant negative sectional curvature. The subgroup $G' = Spin(7, 1)$ of G acts properly on X, and $G'_{\mathbb{C}} = Spin(8, \mathbb{C})$ acts spherically on $X_{\mathbb{C}}$. Thus, Theorem 5.6 holds for any standard 15-dimensional space from X_Γ of signature (8,7) with $\Gamma \subset Spin(7, 1)$.

A classification of the triples $G' \subset G \supset H$ that satisfy the assumptions in Theorem 5.13—specifically, the subgroup G' acts properly on $X = G/H$ and $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical– can be found in [KK20, KK24]. As will be observed in Section 5.7, such triples are *bounded multiplicity triples* (Definition 4.3) discussed in the previous section.

5.6. Key step: Admissible restriction $H \nearrow G \searrow G' \supset \Gamma$.

The new approach for proving the spectral theory presented in Theorem 5.6 involves utilizing the global analysis of $G'_{\mathbb{C}}$-spherical spaces and the restriction of irreducible representations Π of G to the subgroup G' that acts properly on X. Specifically, we demonstrate that this restriction $\Pi|_{G'}$ is always discretely decomposable with uniformly bounded multiplicities.

The following theorem bridges the spectral analysis on $X_\Gamma = \Gamma\backslash G/H$ and recent progress of branching problems that we outlined in Sections 2 to 4.

Theorem 5.13. *Let $X = G/H$ be a reductive symmetric space. Suppose that a reductive subgroup G' of G acts properly on X and that the complexification $G'_{\mathbb{C}}$ acts spherically on $X_{\mathbb{C}}$. Then the restriction $\Pi|_{G'}$ is G'-admissible (Definition 2.1) for any H-distinguished $\Pi \in \widehat{G}$ (Definition 4.2). Moreover, the multiplicities are uniformly bounded:*

$$\sup_{\Pi \in \mathrm{Irr}(G)_H} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty.$$

5.7. $H \subset G \supset G' \quad \rightsquigarrow \quad H \overset{\mathrm{ind}}{\nearrow} G \overset{\mathrm{rest}}{\searrow} G'$.

The conclusion of Theorem 5.13 follows from two key aspects of branching problems: G'-admissibility, as discussed in Section 2 and the uniformly bounded multiplicities outlined in Sections 3 and 4.

The following diagram summarizes the related results:

$$
\begin{array}{ccc}
G'_{\mathbb{C}} \underset{\text{spherical}}{\curvearrowright} G_{\mathbb{C}}/B_{G_{\mathbb{C}}/H_{\mathbb{C}}} & \overset{\text{Theorem 4.6}}{\Longleftrightarrow} & \sup\limits_{\Pi \in \mathrm{Irr}(G)_H} \sup\limits_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty \\
\Longrightarrow & & \Longrightarrow \\
\begin{cases} G'_{\mathbb{C}} \underset{\text{spherical}}{\curvearrowright} G_{\mathbb{C}}/H_{\mathbb{C}} \\ G' \underset{\text{proper}}{\curvearrowright} G/H \end{cases} & \underset{\text{Theorem 5.13}}{\Longrightarrow} & \begin{array}{l} \Pi|_{G'} \text{ is discretely decomposable} \\ \text{with uniformly bounded multiplicities} \\ \text{for all } \Pi \in \mathrm{Irr}(G)_H. \end{array}
\end{array}
$$

The proof of Theorem 5.13 ([Ko17], *cf.* [KK20, KK24]) does not rely on the G'-admissibility criterion (Theorem 2.3), which is formulated purely in terms of representation theory. It would be interesting to explore a direct connection between the geometric assumption in Theorem 5.13 and the transversality condition of the two cones (condition (ii) in Theorem 2.3).

5.8. Strategy of the proof for Theorem 5.6.

The proof of Theorem 5.6 in [KK20, KK24] is quite comprehensive. Here, we outline the key ingredients. Recall that we consider the standard quotient:

$$\Gamma \subset G' \subset G \curvearrowright X \quad \rightsquigarrow \quad X_\Gamma = \Gamma\backslash X$$

1. (Hidden symmetry) If the action of $G'_{\mathbb{C}}$ on $X_{\mathbb{C}}$ is spherical (see Theorem 3.9), it can be shown that the algebra $\mathbb{D}_{G'}(X)$ leaves the space $C^\infty(X; \mathcal{M}_\lambda)$ invariant, see [Ko17, KK19]. In other words, $\mathbb{D}_{G'}(X)$ acts as a hidden symmetry of the joint eigenspace $C^\infty(X; \mathcal{M}_\lambda)$:

$$\mathbb{D}_G(X) \subset \mathbb{D}_{G'}(X) \curvearrowright C^\infty(X; \mathcal{M}_\lambda).$$

2. (Branching law $G \downarrow G'$) If the G'-action on X is also proper, then any $\pi \in \mathrm{Irr}(G)$ realized in $\mathcal{D}'(X)$ is G'-admissible (Theorem 5.13). This property is particularly clear in the special case where $H = K$ and $G' = G$, as it corresponds to Harish-Chandra's admissibility.

References

[AV99] D. N. Akhiezer, E. B. Vinberg, *Weakly symmetric spaces and spherical varieties*, Transform. Groups **4** (1999), 3–24.

[CM62] E. Calabi, L. Markus, *Relativistic space forms*, Ann. of Math. (2), **75** (1962), 63–76.

[CM82] W. Casselman, D. Miličić, *Asymptotic behavior of matrix coefficients of admissible representations*, Duke Math. J. **49** (1982), no. 4, 869–930.

[D77] J. Dixmier, C^*-Algebras, North-Holland Math. Library, **15**, 1977, xiii+492 pp.

[DGV17] M. Duflo, E. Galina, J. A. Vargas, *Square integrable representations of reductive Lie groups with admissible restriction to* $SL_2(\mathbb{R})$, J. Lie Theory, **27**, (2017), 1033–1056.

[E87] P. Enflo, *On the invariant subspace problem for Banach spaces*, Acta. Math. (158) 1987, 213–313.

[GW00] B. Gross, N. Wallach, *Restriction of small discrete series representations to symmetric subgroups*, The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), Proc. Sympos. Pure Math., **68**, Amer. Math. Soc., 2000, 255–272.

[HC53] Harish-Chandra, *Representations of a semisimple Lie group on a Banach space. I*, Trans. Amer. Math. Soc. **75** (1953), 185–243.

[HC54a] Harish-Chandra, *Representations of semisimple Lie groups. II*, Trans. Amer. Math. Soc. **76** (1954), 26–65.

[HC54b] Harish-Chandra, *Representations of semisimple Lie groups. III*, Trans. Amer. Math. Soc. **76** (1954), 234–253.

[HC58] Harish-Chandra, *Spherical functions on a semisimple Lie group. I*, Amer. J. Math. **80** (1958), 241–310.

[HC76] Harish-Chandra, *Harmonic analysis on real reductive groups. III*, The Maass-Selberg relations and the Plancherel formula, Ann. of Math. (2) **104** (1976), 117–201.

[H79] R. Howe, *θ-series and invariant theory*, Proc. Sympos. Pure Math., **33** (1979), Amer. Math. Soc. 275–285.

[KK16] F. Kassel, T. Kobayashi, *Poincaré series for non-Riemannian locally symmetric spaces*, Adv. Math. **287** (2016), 123–236.

[KK19] F. Kassel, T. Kobayashi, *Invariant differential operators on spherical homogeneous spaces with overgroups*, J. Lie Theory, **29** (2019), 663–775.

[KK20] F. Kassel, T. Kobayashi, *Spectral analysis on pseudo-Riemannian locally symmetric spaces*, Proc. Japan Acad. Ser, A, Math. Sci. **96** (2020), no. 8, 69–74, DOI 10.3792/pjaa.96.013.

[KK24] F. Kassel, T. Kobayashi, Spectral Analysis on Standard Locally Homogeneous Spaces, Lecture Notes in Mathematics, **2367**, Springer-Nature. xi+116pp. Available also at ArXiv:1912.12601v2.

[Ki21] M. Kitagawa, *Uniformly bounded multiplicities, polynomial identities and coisotropic actions*, arXiv:2109.05555v2.

[Ki24] M. Kitagawa, *Kobayashi's conjectures on the discrete decomposability*. In: Festschrift in Honor of Toshiyuki Kobayashi, Volume 3, Progr. Math. **359**, Birkhäuser/Springer, 2025, 31–99.

[Kn01] A. Knapp. Representation Theory of Semisimple Groups, Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001.

[Kn94] F. Knop, *A Harish-Chandra homomorphism for reductive group actions*, Ann. Math. **140**, (1994), 253–288.

[Ko89] T. Kobayashi, *Proper action on a homogeneous space of reductive type*, Math. Ann. **285** (1989), pp. 249–263

[Ko94] T. Kobayashi, *Discrete decomposability of the restriction of* $A_{\mathfrak{q}}(\lambda)$ *with respect to reductive subgroups and its applications*, Invent. Math., **117** (1994), 181–205.

[Ko95] T. Kobayashi, *Introduction to harmonic analysis on real spherical homogeneous spaces*, In: Proceedings of the 3rd Summer School on Number Theory "Homogeneous Spaces and Automorphic Forms" in Nagano, (ed. F. Sato), 1995, 22–41.

[Ko98a] T. Kobayashi, *Discrete decomposability of the restriction of* $A_{\mathfrak{q}}(\lambda)$ *with respect to reductive subgroups* II*—micro-local analysis and asymptotic* K*-support*, Ann. of Math., **147** (1998), 709–729.

[Ko98b] T. Kobayashi, *Discrete decomposability of the restriction of* $A_{\mathfrak{q}}(\lambda)$ *with respect to reductive subgroups* III*—restriction of Harish-Chandra modules and associated varieties*, Invent. Math., **131** (1998), 229–256.

[Ko02] T. Kobayashi, *Branching problems of unitary representations*, In Proc. of ICM 2002, Beijing, volume 2, 615–627, 2002.

[Ko05a] T. Kobayashi, *Multiplicity-free representations and visible actions on complex manifolds*, Publ. Res. Inst. Math. Sci., **41** (2005), 497–549, special issue commemorating the fortieth anniversary of the founding of RIMS.

[Ko05b] T. Kobayashi, *Restrictions of unitary representations of real reductive groups*, Lie theory, Progr. Math., **229**, Birkhäuser, 2005, 139–207.

[Ko07] T. Kobayashi, *Multiplicity-free theorems of the restrictions of unitary highest weight modules with respect to reductive symmetric pairs*, Progr. Math., **255**, Birkhäuser, 2007, 45–109.

[Ko14] T. Kobayashi, *Shintani functions, real spherical manifolds, and symmetry breaking operators*, Dev. Math., **37** (2014), 127–159.

[Ko15] T. Kobayashi, *A program for branching problems in the representation theory of real reductive groups*, In: Representations of Reductive Groups—In Honor of the 60th Birthday of David A. Vogan, Jr., Progr. Math., **312**, Birkhäuser, 2015, 277–322.

[Ko16] T. Kobayashi, *Intrinsic sound of anti-de Sitter manifolds*, Springer Proc. Math. Stat. **191** (2016), 83–99.

[Ko17] T. Kobayashi, *Global analysis by hidden symmetry*, Progr. Math., **323**, Birkhäuser, 2017, 359–397. In Honor of Roger Howe on the Occasion of His 70th Birthday.

[Ko21a] T. Kobayashi, *Branching laws of unitary representations associated to minimal elliptic orbits for indefinite orthogonal group* $O(p,q)$, Adv. Math., **388** (2021), Paper No. 107862, 38pp.

[Ko21b] T. Kobayashi, *Admissible restrictions of irreducible representations of reductive Lie groups: Symplectic geometry and discrete decomposability*, Pure Appl. Math. Q. **17** (2021), 1321–1343, Special volume in memory of Bertram Kostant.

[Ko22a] T. Kobayashi, *Bounded multiplicity theorems for induction and restriction*. J. Lie Theory **32** (2022), 197–238.

[Ko22b] T. Kobayashi, *Multiplicity in restricting minimal representations*, Springer Proc. Math. Stat. **396** (2022), 1–18.

[Ko24] T. Kobayashi, *Recent advances in branching problems of representations*, Sugaku Expositions **37** (2024), 129–177; Translated by T. Kubo from the original article in Japanese, Sugaku **71** (2019), 388–416.

[KM14] T. Kobayashi, T. Matsuki, *Classification of finite-multiplicity symmetric pairs*, Transform. Groups, **19** (2014), 457–493, Special issue in honor of Dynkin for his 90th birthday.

[KOt13] T. Kobayashi, T. Oshima, *Finite multiplicity theorems for induction and restriction*, Adv. Math., **248** (2013), 921–944.

[KOy12] T. Kobayashi, Y. Oshima, *Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$ with respect to reductive symmetric pairs*, Adv. Math., **231** (2012), 2013–2047.

[KOy15] T. Kobayashi, Y. Oshima, *Classification of symmetric pairs with discretely decomposable restrictions of $(\mathfrak{g}, K)$-modules*. J. Reine Angew. Math. **703** (2015), 201–223.

[Kr76] M. Krämer, *Multiplicity free subgroups of compact connected Lie groups*, Arch. Math. (Basel) **27** (1976), no.1 28–36.

[Ku81] R. S. Kulkarni, *Proper actions and pseudo-Riemannian space forms*, Adv. Math., **40** (1981), 10–51.

[Mg00] G. A. Margulis, *Problems and conjectures in rigidity theory*, in Mathematics: Frontiers and Perspectives, Amer. Math. Soc., Providence, RI, 2000, 161–174.

[Mt50] F. I. Mautner, *Unitary representations of locally compact groups I, II*, Ann. of Math. (2) **51** (1950), 1–25. **52** (1950), 528–556.

[O24] Y. Oshima, Discrete branching laws of derived functor modules. In: Festschrift in Honor of Toshiyuki Kobayashi, Volume 3, Progr. Math., **359**, Birkhäuser/Springer, 2025, 127–636.

[R79] J. Repka, *Tensor products of holomorphic discrete series representations*, Canadian J. Math., **31** (1979), no. 4, 836–844.

[Se52] I. E. Segal, *Hypermaximality of certain operators on Lie groups*, Proc. Amer. Math. Soc. **3** (1952), 13–15.

[SZ12] B. Sun, C.-B. Zhu, *Multiplicity one theorems: the Archimedean case*, Ann. of Math., **175** (2012), 23–44.

[Tn12] Y. Tanaka, *Classification of visible actions on flag varieties*, Proc. Japan Acad. Ser. A Math. Sci., **88** (2012), 91–96.

[Tn22] Y. Tanaka, *Visible actions of compact Lie groups on complex spherical varieties*, J. Differential Geom. **120** (2022), no. 2, 375–388.

[Tn24] Y. Tanaka, *A note on multiplicity-freeness property of cohomology spaces*. In: Festschrift in Honor of Toshiyuki Kobayashi, Volume 1, Progr. Math. **357**, Birkhäuser/Springer, 2025, 533–564.

[Tu22] T. Tauchi, *A generalization of the Kobayashi–Oshima uniformly bounded multiplicity theorem*, Internat. J. Math., **32** (2021), 2150107.

[Vi01] È. B. Vinberg, *Commutative homogeneous spaces and co-isotropic symplectic action*, Russian Math. Surveys **56** (2001) 1–60.

[Wa88] N. R. Wallach, *Real reductive groups*. I, II, Pure Appl. Math. **132** Academic Press, Inc., Boston, MA, 1988; **132**-II, ibid, 1992.

[Wo69] J. Wolf, *The action of a real semi-simple group on a complex flag manifold, I: Orbit structure and holomorphic arc components*, Bull. Amer. Math. Soc. **75** (1969), 1121–1237.