A SHORT PROOF FOR RANKIN–COHEN BRACKETS AND GENERATING OPERATORS

TOSHIYUKI KOBAYASHI AND MICHAEL PEVZNER

Abstract. Motivated by the concept of "generating operators" for a countable family of operators introduced in the recent paper (arXiv:2306.16800), we find a method to reconstruct the Rankin–Cohen brackets from a very simple multivariable contour integral, and obtain a new proof of their covariance. We also establish a closed formula of the "generating operator" for the Rankin–Cohen brackets in full generality.

Keywords and phrases: generating operator, symmetry breaking operator, Rankin–Cohen bracket, Jacobi polynomial, branching law, representation theory, Hardy space.

2020 MSC: Primary 47B38; Secondary 11F11, 22E45, 32A27, 30H10, 33C45, 43A85.

1. Introduction

Given a family of linear maps $R^{(\ell)} \colon \Gamma(X) \to \Gamma(Y)$ $(\ell \in \mathbb{N})$ between the spaces of functions on two manifolds X and Y, the **generating operator** T is an operator-valued formal power series in t defined by

$$T = \sum_{\ell=0}^{\infty} a_\ell R^{(\ell)} t^\ell \in \operatorname{Hom}_{\mathbb{C}}(\Gamma(X), \Gamma(Y)) \otimes \mathbb{C}[[t]], \tag{1}$$

where $a_\ell \in \mathbb{C}$ are normalizing constants. This concept was introduced in [13] when $a_\ell = \frac{1}{\ell!}$.

Special cases of the generating operators include the classical notion of **generating functions** for orthogonal polynomials which are defined in the setting where $X = \{\text{point}\}$, $Y = \mathbb{C}$, and $a_\ell \equiv 1$, see *e.g.,* [1], whereas the **semigroups** generated by self-adjoint operators D as one may recall the Hille–Yosida theory, correspond to the setting where $X = Y$, $R^{(\ell)} =$ the ℓ-th power of D, and $a_\ell = \frac{1}{\ell!}$. In [9, 13] we initiated a new line of investigation in the general setting where $X \neq \{\text{point}\}$ and $X \neq Y$ by taking (X, Y) to be $(\mathbb{C}^2, \mathbb{C})$ as the first test case.

The idea of the generating operator is to capture all the information of a countable family of operators just by a single operator. Its applications, symbolically stated in [9] as "from discrete to continuous", include

- a construction of non-local symmetry breaking operators with continuous parameters out of a countable family of differential operators;
- a realization of an embedding of discrete series representations for the two-dimensional de Sitter space into principal series representations of $SL(2, \mathbb{R})$.

In this article, we find a closed formula for the generating operator of the Rankin–Cohen brackets $\{\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}\}_{\ell\in\mathbb{N}}$ for arbitrary $\lambda_1, \lambda_2 \in \mathbb{N}_+$ by choosing appropriate normalizing constants $\{a_\ell\}_{\ell\in\mathbb{N}}$, and the resulting formula generalizes the $\lambda_1 = \lambda_2 = 1$ case proven in [13].

We also give a new method how to find the Rankin–Cohen brackets by introducing their integral expression. The covariance property (7) of the Rankin–Cohen brackets is immediate from this viewpoint. The whole idea is inspired by the method of finding "generating operators".

Convention. $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\quad \mathbb{N}_+ = \{1, 2, \cdots\}$.

2. A short proof for Rankin–Cohen brackets

This section introduces a complex integral transform (4), which yields a new way to construct the Rankin–Cohen bidifferential operator $\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}$ and a simple proof of its covariance property, see Corollary 1.

The Rankin–Cohen bracket $\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2} \colon \mathcal{O}(\mathbb{C} \times \mathbb{C}) \to \mathcal{O}(\mathbb{C})$ was originally introduced in [3, 18] as a tool for constructing holomorphic modular forms of higher weights from those of lower weights. It is a bidifferential operator defined by

$$(2) \quad \mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2} := \mathrm{Rest} \circ \sum_{j=0}^{\ell} (-1)^j \binom{\lambda_1 + \ell - 1}{j} \binom{\lambda_2 + \ell - 1}{\ell - j} \partial_1^{\ell-j} \partial_2^j,$$

where $\lambda_1, \lambda_2 \in \mathbb{N}_+$ and $\ell \in \mathbb{N}$, $\partial_j = \frac{\partial}{\partial \zeta_j}$ $(j = 1, 2)$, and Rest denotes the restriction of a function in ζ_1 and ζ_2 with respect to the diagonal embedding $\mathbb{C} \hookrightarrow \mathbb{C} \times \mathbb{C}$.

We introduce a multivariable complex integral in (4), with the covariance property (Proposition 1), from which we obtain the Rankin–Cohen brackets in Theorem 1.

Let $\lambda_1, \lambda_2 \in \mathbb{N}_+$ and $\ell \in \mathbb{N}$. We consider a holomorphic function in $\{(\zeta_1, \zeta_2, z) \in \mathbb{C}^3 : \zeta_1 \neq z \neq \zeta_2\}$ defined by

$$A^{(\ell)}_{\lambda_1,\lambda_2}(\zeta_1, \zeta_2; z) := \frac{(\zeta_1 - \zeta_2)^{\lambda_1+\lambda_2+\ell-2}}{(\zeta_1 - z)^{\lambda_2+\ell}(\zeta_2 - z)^{\lambda_1+\ell}}. \tag{3}$$

Let D be an open set in $\mathbb{C}$, $f(\zeta_1, \zeta_2)$ a holomorphic function in $D \times D$, and $z \in D$. Then the integral

$$(T^{(\ell)}_{\lambda_1,\lambda_2} f)(z) := \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_1} \oint_{C_2} A^{(\ell)}_{\lambda_1,\lambda_2}(\zeta_1, \zeta_2; z) f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2 \tag{4}$$

is well-defined, independently of the choice of contours C_j $(j = 1, 2)$ in D around the point z. Hence one has a linear map

$$T^{(\ell)}_{\lambda_1,\lambda_2} \colon \mathcal{O}(D \times D) \to \mathcal{O}(D).$$

The proof of the following integral expression of the Rankin–Cohen bracket actually reconstructs the explicit formula of the equivariant bi-differential operator $\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}$ in (2) up to scalar multiplication.

Theorem 1. *For any $\lambda_1, \lambda_2 \geq 1$ and $\ell \in \mathbb{N}$, one has*

$$T^{(\ell)}_{\lambda_1,\lambda_2} = \frac{(-1)^{\lambda_1+\ell-1}(\lambda_1 + \lambda_2 + \ell - 2)!}{(\lambda_1 + \ell - 1)!\,(\lambda_2 + \ell - 1)!} \mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}.$$

Proof of Theorem 1. We iterate the residue computation for the variables ζ_1 and ζ_2. We begin with the integration over the first variable $\zeta_1 \in C_1$.

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}} \oint_{C_1} A^{(\ell)}_{\lambda_1,\lambda_2}(\zeta_1, \zeta_2; z) f(\zeta_1, \zeta_2) d\zeta_1 \\
&= \frac{\partial_1^{\lambda_2+\ell-1}|_{\zeta_1=z}((\zeta_1 - \zeta_2)^{\lambda_1+\lambda_2+\ell-2} f(\zeta_1, \zeta_2))}{(\lambda_2 + \ell - 1)!\,(\zeta_2 - z)^{\lambda_1+\ell}} \\
&= \sum_{j=0}^{\lambda_2+\ell-1} \frac{(-1)^{\lambda_1+\lambda_2+\ell+j}(\lambda_1 + \lambda_2 + \ell - 2)!\,\partial_1^{\lambda_2+\ell-j-1} f(z, \zeta_2)}{j!(\lambda_2 + \ell - j - 1)!\,(\lambda_1 + \lambda_2 + \ell - j - 2)!\,(\zeta_2 - z)^{j-\lambda_2+2}}.
\end{aligned}$$

In turn, the residue computation for the second variable $\zeta_2 \in C_2$ shows

$$\frac{1}{2\pi\sqrt{-1}} \oint_{C_2} \frac{\partial_1^{\lambda_2+\ell-j-1} f(z,\zeta_2)}{(\zeta_2 - z)^{j-\lambda_2+2}} d\zeta_2$$

$$= \begin{cases} \frac{1}{(j-\lambda_2+1)!} (\partial_1^{\lambda_2+\ell-j-1} \partial_2^{j-\lambda_2+1}) f(z,z) & \text{if } j \geq \lambda_2 - 1, \\ 0 & \text{if } j < \lambda_2 - 1. \end{cases}$$

We set $r := j - \lambda_2 + 1$. Since $\lambda_2 \geq 1$, the conditions $0 \leq j \leq \lambda_1 + \lambda_2 - 2$ and $j \geq \lambda_2 - 1$ are reduced to the inequality $0 \leq r \leq \ell$. Combining the above formulas, one sees from (4) that $(-1)^{\lambda_1+\ell-1}(T_{\lambda_1,\lambda_2}^{(\ell)} f)(z)$ equals

$$(\lambda_1 + \lambda_2 + \ell - 2)! \sum_{r=0}^{\ell} \frac{(-1)^r (\partial_1^{\ell-r} \partial_2^r f)(z,z)}{(\lambda_2 + r - 1)!\,(\ell - r)!\,(\lambda_1 + \ell - r - 1)!\,r!}$$

$$= \frac{(\lambda_1 + \lambda_2 + \ell - 2)!}{(\lambda_1 + \ell - 1)!\,(\lambda_2 + \ell - 1)!} \operatorname{RC}_{\lambda_1,\lambda_2}^{(\ell)} f(z).$$

Hence Theorem 1 is proved. $\square$

Next, we examine the covariance property of the kernel function $A_{\lambda_1,\lambda_2}^{(\ell)}(\zeta_1, \zeta_2; z)$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$, we write $g \cdot z := \frac{az+b}{cz+d}$. Then one has

$$g \cdot \zeta - g \cdot z = \frac{\zeta - z}{(c\zeta + d)(cz + d)}, \tag{5}$$

hence the following lemma.

Lemma 1. *For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$, the function $A_{\lambda_1,\lambda_2}^{(\ell)}$ in (3) satisfies the following covariance property:*

$$\frac{A_{\lambda_1,\lambda_2}^{(\ell)}(g \cdot \zeta_1, g \cdot \zeta_2; g \cdot z)}{A_{\lambda_1,\lambda_2}^{(\ell)}(\zeta_1, \zeta_2; z)} = \frac{(cz + d)^{\lambda_1+\lambda_2+2\ell}}{(c\zeta_1 + d)^{\lambda_1-2}(c\zeta_2 + d)^{\lambda_1-2}}.$$

For $g \in SL(2,\mathbb{C})$ such that $g \cdot D \subset D$, one defines a linear map $\varpi_\lambda(g^{-1})\colon \mathcal{O}(D) \to \mathcal{O}(g \cdot D)$ by

$$(\varpi_\lambda(g^{-1})f)(z) := (cz + d)^{-\lambda} f(\frac{az + b}{cz + d}). \tag{6}$$

Then Lemma 1 yields the following:

Proposition 1. *For any $h \in SL(2,\mathbb{C})$ such that $D \subset h \cdot D$, one has*

$$(\varpi_{\lambda_1+\lambda_2+2\ell}(h) T^{(\ell)}_{\lambda_1,\lambda_2} f)(z) = T^{(\ell)}_{\lambda_1,\lambda_2}((\varpi_{\lambda_1}(h) \boxtimes \varpi_{\lambda_2}(h)) f)(z).$$

Proof. We set $g := h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We note $d(g\cdot\zeta) = (c\zeta+d)^{-2}d\zeta$. By definition (6) and by Lemma 1, $T^{(\ell)}_{\lambda_1,\lambda_2}((\varpi_{\lambda_1}(h) \boxtimes \varpi_{\lambda_2}(h)) f)(z)$ is equal to

$$\begin{aligned}
&\frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_1} \oint_{C_2} \frac{A^{(\ell)}_{\lambda_1,\lambda_2}(\zeta_1,\zeta_2;z) f(g\cdot\zeta_1, g\cdot\zeta_2)}{(c\zeta_1+d)^{\lambda_1}(c\zeta_2+d)^{\lambda_2}} d\zeta_1 d\zeta_2 \\
=&\frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_1} \oint_{C_2} \frac{A^{(\ell)}_{\lambda_1,\lambda_2}(g\cdot\zeta_1, g\cdot\zeta_2; g\cdot z) f(g\cdot\zeta_1, g\cdot\zeta_2)}{(cz+d)^{\lambda_1+\lambda_2+2\ell}} d(g\cdot\zeta_1) d(g\cdot\zeta_2) \\
=&\frac{(cz+d)^{-\lambda_1-\lambda_2-2\ell}}{(2\pi\sqrt{-1})^2} \oint_{g\cdot C_1} \oint_{g\cdot C_2} A^{(\ell)}_{\lambda_1,\lambda_2}(\xi_1,\xi_2; g\cdot z) f(\xi_1,\xi_2) d\xi_1 d\xi_2 \\
=&(\varpi_{\lambda_1+\lambda_1+2\ell}(h) T^{(\ell)}_{\lambda_1,\lambda_2} f)(z).
\end{aligned}$$

Thus the proposition is proved. □

Since $T^{(\ell)}_{\lambda_1,\lambda_2}$ is a non-zero multiple of the Rankin–Cohen bracket $\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}$ by Theorem 1, Proposition 1 implies the covariance property of the bi-differential operator $\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}$:

Corollary 1. *For any $g \in SL(2,\mathbb{C})$, one has*

$$\varpi_{\lambda_1+\lambda_2+2\ell}(g) \circ \mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2} = \mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2} \circ (\varpi_{\lambda_1}(g) \boxtimes \varpi_{\lambda_2}(g)), \tag{7}$$

or in other words,

$$d\varpi_{\lambda_1+\lambda_2+2\ell}(Z) \circ \mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2} = \mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2} \circ (d\varpi_{\lambda_1}(Z) \boxtimes \mathrm{id} + \mathrm{id} \boxtimes d\varpi_{\lambda_2}(Z))$$

for any $Z = \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})$, where

$$d\varpi_\lambda(Z) = -\lambda(p - rz) - (2pz + q - rz^2)\frac{d}{dz}.$$

Remark 1. The complete classification of the operators satisfying the covariance property (7) has been recently accomplished in [11, Thm. 9.1] by using the F-method, which is based on an 'algebraic Fourier transform' of Verma modules. One sees from the classification that such an operator is proportional to the Rankin–Cohen bracket for 'generic'

λ_1 and λ_2, however, there exist some other bi-differential operators satisfying the same property (7) for 'very singular' pairs (λ_1, λ_2).

Remark 2. Various approaches have been known for the proof of the covariance property (7) for the Rankin–Cohen brackets since the original proof by H. Cohen [3] and N.V. Kuznecov [14] based on the idea of Jacobi-like forms.

D. Zagier [20] proposed an insightful proof involving theta series. P. Olver et al. [15, 16] made an observation that the Rankin–Cohen brackets can be interpreted as the projectivization of the Transvectants (Überschienbung) from the classical invariant theory given by iterated powers of Cayley's Ω-process, see Gordan and Gundelfinger [4, 5].

Other approaches include a recursion relation (*e.g.,* [17]) to find singular vectors (highest weight vectors) of the tensor produce of two $\mathfrak{sl}_2$-modules, and a residue formula of a meromorphic continuation of the integral symmetry breaking operators (*e.g.,* [8]).

A recent approach, referred to as the F-method ([11, Sect. 7]), clarifies an intrinsic reason why the coefficients of the Rankin–Cohen brackets coincide with those of the Jacobi polynomials, see (12) below. J.-L. Clerc has proposed yet another proof in [2] using the Bernstein–Sato identity for the power of the determinant function and the intertwining property of the Knapp–Stein operator.

3. Generating operators for the Rankin–Cohen Brackets

This section provides a closed formula of the generating operator for the Rankin–Cohen brackets $\{\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}\}_{\ell\in\mathbb{N}}$. The main result of this section is Theorem 2, which generalizes [13, Thm. 2.3] proven in the $\lambda_1 = \lambda_2 = 1$ case.

For $\lambda_1, \lambda_2 \in \mathbb{N}_+$, we set

$$A_{\lambda_1,\lambda_2}(\zeta_1, \zeta_2; z, t) := \frac{(\zeta_1 - \zeta_2)^{\lambda_1+\lambda_2-2}(\zeta_1 - z)^{1-\lambda_2}(z - \zeta_2)^{1-\lambda_1}}{(\zeta_1 - z)(\zeta_2 - z) + t(\zeta_1 - \zeta_2)}. \tag{8}$$

Lemma 2. *If $|t(\zeta_1 - \zeta_2)| < |\zeta_1 - z|\,|\zeta_2 - z|$, then $A_{\lambda_1,\lambda_2}(\zeta_1, \zeta_2; z, t)$ is a holomorphic function of four variables ζ_1, ζ_2, z and t, and has a* *convergent power series expansion:*

$$A_{\lambda_1,\lambda_2}(\zeta_1,\zeta_2;z,t) = \sum_{\ell=0}^{\infty}(-1)^{\lambda_1+\ell-1}A^{(\ell)}_{\lambda_1,\lambda_2}(\zeta_1,\zeta_2;z)t^{\ell}.$$

Proof. In light of the Taylor series expansion of

$$\frac{1}{(\zeta_1-z)(\zeta_2-z)+t(\zeta_1-\zeta_2)} = \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(\zeta_1-\zeta_2)^{\ell}t^{\ell}}{(\zeta_1-z)^{\ell+1}(\zeta_2-z)^{\ell+1}},$$

the assertion follows from the definition (8). □

For a domain D in $\mathbb{C}$, we set

$$U_D := \{(z,t) \in D \times \mathbb{C} : 2|t| < d(z,\partial D)\}, \tag{9}$$

where $d(z) \equiv d(z,\partial D)$ is the distance from $z \in D$ to the boundary ∂D. We put $d(z) := \infty$ if $D = \mathbb{C}$. If D is the Poincaré upper half plane $\Pi := \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$, then $\partial D = \mathbb{R}$ and $d(z,\partial D) = \operatorname{Im} z$.

Example 1. (1) $U_D = \mathbb{C} \times \mathbb{C}$ if $D = \mathbb{C}$.
(2) $U_D = \{(z,t) \in \mathbb{C}^2 : 2|t| < \operatorname{Im} z\}$ if $D = \Pi$.

As in (4), the integral transform

(10)
$$(T_{\lambda_1,\lambda_2}f)(z,t) := \frac{1}{(2\pi\sqrt{-1})^2}\oint_{C_1}\oint_{C_2} A_{\lambda_1,\lambda_2}(\zeta_1,\zeta_2;z,t)f(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2$$

defines a linear map

$$T_{\lambda_1,\lambda_2}\colon \mathcal{O}(D\times D) \to \mathcal{O}(U_D).$$

The integral (10) yields a generating operator for the Rankin–Cohen brackets $\{\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}\}_{\ell\in\mathbb{N}}$ as below.

Theorem 2. *The integral operator T_{λ_1,λ_2} in* (10) *is expressed as*

$$(T_{\lambda_1,\lambda_2}f)(z,t) = \sum_{\ell=0}^{\infty}\frac{(\lambda_1+\lambda_2+\ell-2)!\,t^{\ell}}{(\lambda_1+\ell-1)!\,(\lambda_2+\ell-1)!}(\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}f)(z). \tag{11}$$

The following corollary is derived from Theorem 2 as in [13].

Corollary 2. $T_{\lambda_1,\lambda_2}\colon \mathcal{O}(D\times D) \to \mathcal{O}(D)$ *is injective for any positive integers λ_1, λ_2.*

Proof of Theorem 2. Accordingly to Lemma 2, we expand $T_{\lambda_1,\lambda_2} f(z,t)$ into the Taylor series of t:

$$T_{\lambda_1,\lambda_2} f(z,t) = \sum_{\ell=0}^{\infty} (-1)^{\lambda_1+\ell-1} t^\ell (T^{(\ell)}_{\lambda_1,\lambda_2} f)(z)$$

with coefficients $T^{(\ell)}_{\lambda_1,\lambda_2} f(z) \in \mathcal{O}(D)$. Now the assertion follows from Theorem 1. $\square$

Example 2. The formula (11) generalizes [13, Thm. 2.3] which treated the case $\lambda_1 = \lambda_2 = 1$. In this case, $T_{1,1}$ is an integral operator against the kernel

$$A_{1,1}(\zeta_1, \zeta_2; z, t) = \frac{1}{(\zeta_1 - z)(\zeta_2 - z) + t(\zeta_1 - \zeta_2)},$$

and Theorem 2 reduces to

$$(T_{1,1} f)(z,t) = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} (R^{(\ell)}_{1,1} f)(z).$$

Remark 3. We remind a remarkable relationship between the Rankin–Cohen brackets and the Jacobi polynomials. The classical Jacobi polynomial $P_\ell^{(\alpha,\beta)}(x)$ is a polynomial of degree ℓ given by

$$P_\ell^{(\alpha,\beta)}(x) = \sum_{j=0}^{\ell} \frac{(\alpha + j + 1)_{\ell-j} (\alpha + \beta + \ell + 1)_j}{j!(\ell - j)!} \left(\frac{x-1}{2}\right)^j.$$

Here the Pochhammer symbol $(x)_n$ is defined as the rising factorial $x(x+1)\cdots(x+n-1)$. We inflate the Jacobi polynomial into a homogeneous polynomial in two variables x and y of degree ℓ by

$$\widetilde{P}_\ell^{(\alpha,\beta)}(x,y) := y^\ell P_\ell^{(\alpha,\beta)}(1 + \frac{2x}{y}).$$

Then the F-method [11, Lem. 9.4] establishes the correspondence:

$$\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2} = \mathrm{Rest} \circ \widetilde{P}_\ell^{(\lambda_1-1, 1-\lambda_1-\lambda_2-2\ell)}(\frac{\partial}{\partial \zeta_1}, \frac{\partial}{\partial \zeta_2}) \tag{12}$$

by showing that the 'symbol' of $\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}$ satisfies the Jacobi differential equation

$$((1-x^2)\frac{d^2}{dx^2} + (\beta - \alpha - (\alpha + \beta + 2)x)\frac{d}{dx} + \ell(\ell + \alpha + \beta + 1))f(x) = 0$$

where $\alpha = \lambda_1 - 1$ and $\beta = 1 - \lambda_1 - \lambda_2 - 2\ell$.

On the other hand, the generating function for the Jacobi polynomials $P_\ell^{(\alpha,\beta)}(x)$ is given by

$$\sum_{\ell=0}^{\infty} P_\ell^{(\alpha,\beta)}(x)t^\ell = \frac{2^{\alpha+\beta}}{R(1-t+R)^\alpha(1+t+R)^\beta} \tag{13}$$

where $R = (1-2xt+t^2)^{\frac{1}{2}}$, see *e.g.,* [1, Thm. 6.4.2]. For $\alpha = \beta = 0$, the Jacobi polynomial reduces to the Legendre polynomial $P_\ell(x)$, of which the generating function is given by $R^{-1} = (1-2xt+t^2)^{-\frac{1}{2}}$.

However, the generating function (13) with the normalizing constants $a_\ell \equiv 1$, see (1), is not directly related to the generating operator T_{λ_1,λ_2} defined in (11) with a_ℓ decreasing rapidly as $\ell \to \infty$, see (23).

4. Generating operators for symmetry breaking

This section explains our results from the viewpoint of the representation theory.

In general, the generating operator T is a single operator that should contain all the information of a countable family operators $R^{(\ell)}$ ($\ell \in \mathbb{N}$). In our setting, the family $\{\mathrm{RC}_{\lambda_1,\lambda_2}^{(\ell)}\}_{\ell\in\mathbb{N}}$ of the Rankin–Cohen brackets arises as *symmetry breaking operators* of the fusion rule of two irreducible unitary representations of $SL(2,\mathbb{R})$. We formulate this property in terms of a generating operator with appropriate normalizing constants $\{a_\ell\}_{\ell\in\mathbb{N}}$ in (1). The main result of this section is Theorem 3.

To be precise, we define the following Hilbert spaces:

$$\begin{aligned}
\mathcal{H}^2(\Pi) :=& \{F \in \mathcal{O}(\Pi) : \|F\|_{\mathrm{Hardy}} < \infty\} \quad &&\text{(Hardy space)},\\
\mathcal{H}^2(\Pi)_\lambda :=& \{F \in \mathcal{O}(\Pi) : \|F\|_\lambda < \infty\} \quad &&\text{(weighted Bergman spaces)},
\end{aligned}$$

where the norms are given by

$$\begin{aligned}
\|F\|_{\mathrm{Hardy}}^2 :=& \sup_{y>0} \int_{-\infty}^{\infty} |F(x+\sqrt{-1}y)|^2 dx,\\
\|F\|_\lambda^2 :=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(x+\sqrt{-1}y)|^2 y^{\lambda-2} dx dy.
\end{aligned}$$

Then $\mathcal{H}^2(\Pi)$ and $\mathcal{H}^2(\Pi)_\lambda$ are invariant subspaces of the representations $(\varpi_\lambda, \mathcal{O}(\Pi))$ of $SL(2,\mathbb{R})$, see (6), for $\lambda = 1$ and $\lambda \geq 2$, respectively, yielding irreducible unitary representations. By an abuse of notation, we shall use the same letter ϖ_λ to denote these unitary representations.

Then the fusion rule (abstract irreducible decomposition) of the tensor product representation $\varpi_{\lambda_1}\widehat{\otimes}\varpi_{\lambda_2}$ is known by Repka [19] as follows.

$$\varpi_{\lambda_1}\widehat{\otimes}\varpi_{\lambda_2} \simeq {\sum_{\ell=0}^{\infty}}^{\oplus}\varpi_{\lambda_1+\lambda_2+2\ell} \quad \text{(Hilbert direct sum)}, \tag{14}$$

where $\widehat{\otimes}$ and $\sum^{\oplus}$ stand for the Hilbert space completion of an algebraic tensor product and that of an algebraic direct sum. A remarkable feature in the fusion rule (14) is that it has no continuous spectrum, see [6] for the general theory of discrete decomposability and [7] for that of multiplicity-free decompositions.

The Rankin–Cohen bracket $\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}\colon \mathcal{O}(\Pi\times\Pi)\to\mathcal{O}(\Pi)$ induces a projection map

$$\mathcal{H}^2(\Pi)_{\lambda_1}\otimes\mathcal{H}^2(\Pi)_{\lambda_2}\to\mathcal{H}^2(\Pi)_{\lambda_1+\lambda_2+2\ell}$$

for all $\lambda_1,\lambda_2\in\mathbb{N}_+$ and $\ell\in\mathbb{N}$.

We now collect these data for $\{\mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}\}_{\ell\in\mathbb{N}}$ in a single operator, namely, the generating operator. Let ${\sum\limits_{\ell=0}^{\infty}}^{\oplus}\mathcal{H}^2(\Pi)_{\lambda_1+\lambda_2+2\ell}\otimes\mathbb{C}t^\ell$ denote the Hilbert completion of the algebraic direct sum

$$\bigoplus_{\ell=0}^{\infty}\mathcal{H}^2(\Pi)_{\lambda_1+\lambda_2+2\ell}\otimes\mathbb{C}t^\ell$$

equipped with the pre-Hilbert structure given by

$$(u\otimes t^\ell, v\otimes t^{\ell'}) := \delta_{\ell\ell'}(u,v)_{\lambda_1+\lambda_2+2\ell}.$$

For $\lambda_1,\lambda_2>1$, we set

$$a_\ell(\lambda_1,\lambda_2) := (c_\ell(\lambda_1,\lambda_2)r_\ell(\lambda_1,\lambda_2))^{-\frac{1}{2}}, \tag{15}$$

where we follow [12, (2.8) and (2.9)] for the notations of positive constants $c_\ell(\lambda_1,\lambda_2)$ and $r_\ell(\lambda_1,\lambda_2)$ as below.

$$\begin{aligned} c_\ell(\lambda_1,\lambda_2) :=& \frac{\Gamma(\lambda_1+\ell)\Gamma(\lambda_2+\ell)}{(\lambda_1+\lambda_2+2\ell-1)\Gamma(\lambda_1+\lambda_2+\ell-1)\ell!},\\ r_\ell(\lambda_1,\lambda_2) :=& \frac{\Gamma(\lambda_1+\lambda_2+2\ell-1)}{2^{2\ell+2}\pi\Gamma(\lambda_1-1)\Gamma(\lambda_2-1)}. \end{aligned}$$

We also set

$$a_\ell(1,1) := \lim_{\lambda_1 \downarrow 1} \lim_{\lambda_2 \downarrow 1} (\lambda_1 - 1)^{\frac{1}{2}} (\lambda_2 - 1)^{\frac{1}{2}} a_\ell(\lambda_1, \lambda_2) = \left(\frac{\ell!(2\ell-1)!!}{2^{\ell+2}\pi(2\ell+1)} \right)^{-\frac{1}{2}}. \tag{16}$$

Theorem 3. (1) *If $\lambda_1, \lambda_2 > 1$, then the generating operator*

$$T = \sum_{\ell=0}^{\infty} a_\ell(\lambda_1, \lambda_2)\, \mathrm{RC}^{(\ell)}_{\lambda_1,\lambda_2}\, t^\ell \tag{17}$$

is a unitary map that yields the decomposition

$$\mathcal{H}^2(\Pi)_{\lambda_1} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda_2} \xrightarrow{\sim} \sum_{\ell=0}^{\infty}{}^{\oplus} \mathcal{H}^2(\Pi)_{\lambda_1+\lambda_2+2\ell} \otimes \mathbb{C}t^\ell. \tag{18}$$

(2) *Similarly, the generating operator* (17) *with $\lambda_1 = \lambda_2 = 1$ gives a unitary map*

$$\mathcal{H}^2(\Pi) \widehat{\otimes} \mathcal{H}^2(\Pi) \xrightarrow{\sim} \sum_{\ell=0}^{\infty}{}^{\oplus} \mathcal{H}^2(\Pi)_{2\ell+2} \otimes \mathbb{C}t^\ell.$$

Remark 4. The normalizing constants $\{a_\ell(\lambda_1,\lambda_2)\}_{\ell\in\mathbb{N}}$ defined in (15) are different from those in (11). However, they have the same asymptotic behavior as ℓ tends to infinity, that is,

$$\lim_{\ell\to\infty} (a_\ell(\lambda_1,\lambda_2)\ell!)^{\frac{1}{\ell}} = 1. \tag{19}$$

As we will see in Theorem 4 below, the formal power series (17) converges owing to (19).

Proof. Theorem 3 is derived from the formula of the operator norm of the Rankin–Cohen brackets proven in [12, Thm. 2.7] for $\lambda_1, \lambda_2 > 1$ and in [13, Thm. 5.1] for $\lambda_1 = \lambda_2 = 1$. □

5. Freedom of normalizing constants

The definition of the generating operator in (1) allows us the freedom to choose normalizing constants $\{a_\ell\}_{\ell\in\mathbb{N}}$. In fact, the closed formula in Theorem 2 is obtained by taking a_ℓ to be $\frac{(\lambda_1+\lambda_2+\ell-2)!}{(\lambda_1+\ell-1)!\,(\lambda_2+\ell-1)!}$ rather than $a_\ell = \frac{1}{\ell!}$ or $a_\ell \equiv 1$. This section explores how the choice of $\{a_\ell\}_{\ell\in\mathbb{N}}$

affects the generating operator in terms of its kernel function by (20) and (22).

Let $h(s)$ be a holomorphic function of one variable s near the origin, and set

$$\varphi(\zeta_1, \zeta_2; z) := \frac{\zeta_1 - \zeta_2}{(\zeta_1 - z)(\zeta_2 - z)},$$

$$A^{(h)}(\zeta_1, \zeta_2; z, t) := \frac{(\zeta_1 - \zeta_2)^{\lambda_1+\lambda_2-2}}{(\zeta_1 - z)^{\lambda_2}(\zeta_2 - z)^{\lambda_1}} h(t\varphi(\zeta_1, \zeta_2; z)). \tag{20}$$

If $h(s) = (-1)^{\lambda_1-1}(1+s)^{-1}$, then $A^{(h)}(\zeta_1, \zeta_2; z, t)$ in (20) coincides with $A_{\lambda_1,\lambda_2}(\zeta_1, \zeta_2; z, t)$, see (8).

In the generality of (20), an analogous covariance property to Lemma 1 still holds:

Proposition 2. *For any holomorphic function $h(s)$ near the origin and for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$, one has*

$$A^{(h)}(g \cdot \zeta_1, g \cdot \zeta_2; g \cdot z, \frac{t}{(cz+d)^2}) = \frac{(cz+d)^{\lambda_1+\lambda_2}}{(c\zeta_1 + d)^{\lambda_1-2}(c\zeta_2+d)^{\lambda_2-2}} A^{(h)}(\zeta_1, \zeta_2; z, t),$$

whenever the formula makes sense.

Proof. In view of the formula (5), one has

$$\varphi(g \cdot \zeta_1, g \cdot \zeta_2; g \cdot z) = (cz+d)^2 \varphi(\zeta_1, \zeta_2; z),$$

the proof goes in parallel to that of Lemma 1. □

Suppose that we are given a sequence $\{a_\ell\}_{\ell \in \mathbb{N}}$ of complex numbers. In order to apply the above framework, we define $h(s)$ by $\{a_\ell\}_{\ell \in \mathbb{N}}$ as follows. We set for fixed $\lambda_1, \lambda_2 \in \mathbb{N}_+$

$$h_\ell := \frac{(\lambda_1 + \ell - 1)!(\lambda_2 + \ell - 1)!}{(\lambda_1 + \lambda_2 + \ell - 2)!} a_\ell \quad \text{for } \ell \in \mathbb{N}, \tag{21}$$

and define $h(s)$ by

$$h(s) := \sum_{\ell=0}^{\infty} h_\ell s^\ell. \tag{22}$$

The power series (22) converges, if $\limsup_{\ell\to\infty}|h_\ell|^{\frac{1}{\ell}} < \infty$, or equivalently, if

$$\frac{1}{\rho} := \limsup_{\ell\to\infty}(|a_\ell|\ell!)^{\frac{1}{\ell}} < \infty. \tag{23}$$

Then $h(s)$ is a holomorphic function in $\{s \in \mathbb{C} : |s| < \rho\}$.

The integral transform

$$(T^{(h)}f) := \frac{1}{(2\pi\sqrt{-1})^2}\oint_{C_1}\oint_{C_2} A^{(h)}(\zeta_1,\zeta_2;z,t)f(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2 \tag{24}$$

is a generating operator of the Rankin–Cohen brackets $\{\mathrm{RC}_{\lambda_1,\lambda_2}^{(\ell)}\}_{\ell\in\mathbb{N}}$ with normalizing constants $\{a_\ell\}_{\ell\in\mathbb{N}}$.

Theorem 4 (Generating operator of the Rankin–Cohen brackets)**.** *Let $h(s)$ be defined by $\{a_\ell\}_{\ell\in\mathbb{N}}$ as in* (21) *and* (22)*. The integral operator $T^{(h)}$ in* (24) *is expressed as*

$$(T^{(h)}f)(z,t) = \sum_{\ell=0}^{\infty} a_\ell(\mathrm{RC}_{\lambda_1,\lambda_2}^{(\ell)} f)(z)t^\ell. \tag{25}$$

In particular, the operator-valued formal series

$$\sum_{\ell=0}^{\infty} a_\ell \,\mathrm{RC}_{\lambda_1,\lambda_2}^{(\ell)}\, t^\ell$$

converges for $|t| \ll 1$ if $\{a_\ell\}_{\ell\in\mathbb{N}}$ satisfies (23)*.*

Theorem 2 corresponds to the case $h(s) = (-1)^{\lambda_1-1}(1+s)^{-1}$.

Remark 5. The radius of convergence of the power series (25) is zero if we take $a_\ell \equiv 1$.

Acknowledgement.
The first author warmly thanks Professor Vladimir Dobrev for his hospitality during the 15th International Workshop: Lie Theory and its Applications in Physics, held in Varna, Bulgaria 19–25, June 2023. The authors were partially supported by the JSPS under the Grant-in-Aid for Scientific Research (A) (JP23H00084) and by the JSPS Fellowship L23505.

References

[1] G. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge, 1999.

[2] J.-L. Clerc, *Covariant bi-differential operators on matrix space*, *Ann. Inst. Fourier*, (2017), **67**, pp.1427–1455.

[3] H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters*, Math. Ann. **217** (1975), 271–285.

[4] P. Gordan, Invariantentheorie, Teubner, Leipzig, 1887.

[5] S. Gundelfinger, *Zur der binären Formen*, J. Reine Angew. Math., **100** (1886), pp. 413–424.

[6] T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups* II*—micro-local analysis and asymptotic K-support*, Ann. of Math. (2), **147** (1998), 709–729.

[7] T. Kobayashi, *Multiplicity-free representations and visible actions on complex manifolds*, Publ. Res. Inst. Math. Sci. **41** (2005), pp. 497–549, special issue commemorating the fortieth anniversary of the founding of RIMS.

[8] T. Kobayashi, *Residue formula for regular symmetry breaking operators*, Contemporary Mathematics, **714**, (2018), 175–197, Amer. Math. Soc.

[9] T. Kobayashi, *Generating operators of symmetry breaking – from discrete to continuous*, To appear in Indag. Math. Available also at ArXiv:2307.16587.

[10] T. Kobayashi, M. Pevzner, *Differential symmetry breaking operators.* I. *General theory and F-method*, Selecta Math. (N.S.) **22** (2016), 801–845.

[11] T. Kobayashi, M. Pevzner, *Differential symmetry breaking operators.* II. *Rankin–Cohen operators for symmetric pairs*, Selecta Math. (N.S.) **22** (2016), 847–911.

[12] T. Kobayashi, M. Pevzner, *Inversion of Rankin–Cohen operators via holographic transform*, Ann. Inst. Fourier **70** (2020), 2131–2190.

[13] T. Kobayashi, M. Pevzner, *A generating operator for Rankin–Cohen brackets*, arXiv: 2306.16800.

[14] N. V. Kuznecov, *A new class of identities for the Fourier coefficients of modular forms.* (Russian), Acta Arith. **27** (1975), pp.505–519.

[15] P. J. Olver, Classical Invariant Theory, London Math. Society Student Texts **44**, Cambridge University Press, 1999.

[16] P. J. Olver, J.A. Sanders, *Transvectants, modular forms, and the Heisenberg algebra*, Adv. in Appl. Math., **25** (2000), 252–283.

[17] M. Pevzner, *Rankin–Cohen brackets and representations of conformal Lie groups.* Annales Math. B. Pascal, **19** (2012) 455-484.

[18] R. A. Rankin, *The construction of automorphic forms from the derivatives of a given form*, J. Indian Math. Soc. **20** (1956), 103–116.

[19] J. Repka, *Tensor products of unitary representations of* $\mathrm{SL}_2(\mathbb{R})$, Amer. J. Math. **100** (1978), 747–774.

[20] D. Zagier, *Modular forms and differential operators*, K. G. Ramanathan memorial issue, Proc. Indian Acad. Sci. Math. Sci., Indian Academy of Sciences. Proceedings. Mathematical Sciences, **104**, (1994), 57–75.

Toshiyuki KOBAYASHI
Graduate School of Mathematical Sciences,
The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo, 153-8914, Japan.
&
French-Japanese Laboratory in Mathematics and its Interactions,
FJ-LMI CNRS IRL2025, Tokyo, Japan
E-mail: `toshi@ms.u-tokyo.ac.jp`

Michael PEVZNER
LMR, Université de Reims-Champagne-Ardenne, CNRS UMR 9008, F-51687,
Reims, France.
&
French-Japanese Laboratory in Mathematics and its Interactions,
FJ-LMI CNRS IRL2025, Tokyo, Japan
E-mail: `pevzner@math.cnrs.fr`