GENERATING OPERATORS AND BRANCHING PROBLEMS

TOSHIYUKI KOBAYASHI, MICHAEL PEVZNER

"*Mathématique est l'art de donner le même nom à des choses différentes*" [1] was saying Henri Poincaré.

Many, at a first glance, unrelated mathematical phenomena can be understood through the paradigm of branching problems in the realm of Representation Theory. Such intricate combinatorial results as the Littlewood–Richardson rules, the Clebsch–Gordon coefficients, the Cauchy identities and other fine properties of the Schur functions, number-theoretic issues summarized in the Gross–Prassad–Gan conjectures, the theta correspondence or analytic problems of Plancherel type theorems for symmetric spaces can be formulated and addressed using this framework.

More precisely, *branching problems* ask how a given irreducible representation π of a group G behaves when restricted to a given subgroup $G' \subset G$. The decomposition of the tensor product of two irreducible representations (*fusion rule*) is a special case of this problem, where (G, G') is of the form $(G_1 \times G_1, \Delta(G_1))$ for some G_1. In the setting where (G, G') are pairs of reductive groups and π is an infinite dimensional representation of G, branching problems usually have a 'wild' behavior in the sense that such decompositions may run over continuous set of parameters (continuous spectrum) and irreducible summands therein may occur infinitely many times (infinite multiplicities). Therefore, the use of analytic methods becomes necessary and the choice of appropriate geometric models for such representations turns out to be important. Once such a model being fixed the analysis of explicit operators from $\mathrm{Hom}_{\mathrm{G}'}(\pi_{|\mathrm{G}'}, \tau)$ for a given irreducible representation τ of G' becomes possible and particularly meaningful when the dimension of the latter space equals one. Such (continuous) operators are referred to as *Symmetry Breaking Operators* (SBO for short) and have been intensively studied in the last decade (see [5, 6]). This study is much more involved than the analysis of branching laws, as the latter treats only the decomposition of representations, whereas the former considers the decomposition of vectors. For instance, the branching law for a unitary principal series representation of the Lie group $SL(2, \mathbb{R})$ to its maximal compact subgroup $SO(2)$ amounts to the classical theorem on the decomposition of periodic square integrable functions into Fourier series whereas the single SBO between the restricted representation and a given irreducible representation of the commutative

[1] 数学とは、異なるものを同じものとみなす芸術・技術です。

group $SO(2)$, that is a character, boil down in this case to the individual Fourier coefficient of a given function.

When both representations are infinite dimensional interesting phenomena occur. For instance, the branching law for the tensor product of two holomorphic discrete series representations π_{λ_1} and π_{λ_2} of $SL(2,\mathbb{R})$ is well known [9, 11]. It is given by the following multiplicity free direct Hilbert sum of holomorphic discrete series representations:

$$\pi_{\lambda_1} \otimes \pi_{\lambda_2} \simeq \sum_{a\in\mathbb{N}}^{\oplus} \pi_{\lambda_1+\lambda_2+2a}. \tag{0.1}$$

What is even more surprising is the fact that in the case where the representations π_λ are realized in weighted Bergman spaces on the Poincaré upper half-plane, the explicit intertwining operator from $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ to the irreducible summand $\pi_{\lambda_1+\lambda_2+2a}$ with a fixed parameter a is given by the celebrated bidifferential operator, referred to as the *Rankin–Cohen bracket* of degree a and defined by

(0.2)

$$\mathcal{RC}_{\lambda_1,\lambda_2}^{\lambda_3}(f_1,f_2)(z) := \sum_{\ell=0}^{a}(-1)^\ell \begin{pmatrix} \lambda_1+a-1 \\ \ell \end{pmatrix} \begin{pmatrix} \lambda_2+a-1 \\ a-\ell \end{pmatrix} f_1^{(a-\ell)}(z) f_2^{(\ell)}(z),$$

where $\lambda_3 = \lambda_1+\lambda_2+2a$ and $f^{(n)}(z) = \frac{d^n f}{dz^n}(z)$.

The detailed account on the analysis and explicit exhaustive description of such symmetry breaking operators as well as the method of constructing them in six different parabolic geometries can be found in [5, 6].

It turns out that the whole family of such operators $\left\{\mathcal{RC}_{\lambda_1,\lambda_2}^{\lambda_1+\lambda_2+2a}\right\}_{a\in\mathbb{N}}$ has a rich internal structure (*e.g.* [12]). Combination of these ideas led the notion of the Rankin–Cohen Transform and its inversion through holographic operators [7]. Going even further in this direction brought to light a new line of investigation on "*generating operators*" for a family of differential operators between two manifolds [8] which we present here.

1 General Framework

To any sequence $\{a_\ell\}_{\ell\in\mathbb{N}}$ one may associate a formal power series such as $\sum_{\ell=0}^{\infty} a_\ell t^\ell$ or $\sum_{\ell=0}^{\infty} a_\ell \frac{t^\ell}{\ell!}$. The resulting *generating functions* are fascinating objects providing powerful tools for studying various combinatorial problems when a_ℓ are integers or, more generally, polynomials. One may quantize this construction by considering differential operators as non-commutative analogues of polynomials and may study the resulting "generating operators". Dealing with the sequence of differential operators given by iterated powers of some remarkable operator yields the notion of an operator semigroup which is nowadays a classical tool for the spectral theory of unbounded operators (*e.g.* the Hille–Yosida theory). Inspired by the ideas of deformation quantization we explore yet another direction by introducing a sequence of differential operators with a different algebraic structure which is not defined by one single operator anymore.

Let us start with our general setting. Suppose that $\Gamma(X)$ and $\Gamma(Y)$ are the spaces of functions on X and Y, respectively. Given a family of linear operators $R_\ell \colon \Gamma(X) \to \Gamma(Y)$, we consider a formal power series

$$T \equiv T(\{R_\ell\}; t) := \sum_{\ell=0}^{\infty} \frac{R_\ell}{\ell!} t^\ell \in \operatorname{Hom}(\Gamma(X), \Gamma(Y)) \otimes \mathbb{C}[[t]]. \tag{1.1}$$

When $X = \{\text{point}\}$, R_ℓ is identified with an element of $\Gamma(Y)$, and such a formal power series is called a *generating function*, which has been particularly prominent in the classical study of orthogonal polynomials for $\Gamma(Y) = \mathbb{C}[y]$.

When $X = Y$, $\operatorname{Hom}(\Gamma(X), \Gamma(Y)) \simeq \operatorname{End}(\Gamma(X))$ has a ring structure and one may take R_ℓ to be the ℓ-th power of a *single* operator R on X. In this case, the operator T in (1.1) may be written as e^{tR} if the summation converges. We note that even if R is a differential operator on a manifold X, the resulting operator $T = e^{tR}$ is not a differential operator any more in general. For example, if $R = \frac{d}{dz}$ acting on $\mathcal{O}(\mathbb{C})$, then $T = e^{t\frac{d}{dz}}$ is the shift operator $f(z) \mapsto f(z+t)$. For a self-adjoint operator R with bounded eigenvalues from the above, the operator T has been intensively studied as the *semigroup* e^{tR} generated by R for $\operatorname{Re} t > 0$: typical examples include

- the heat kernel for $R = \Delta$,
- the Hermite semigroup for $R = \frac{1}{4}(\Delta - |x|^2)$ on $L^2(\mathbb{R}^n)$,
- the Laguerre semigroup for $R = |x|(\frac{\Delta}{4} - 1)$ on $L^2(\mathbb{R}^n, \frac{1}{|x|}dx)$.

Let us consider a more general setting where we allow $X \neq \{\text{point}\}$ and $X \neq Y$. In this generality, we refer to T in (1.1) as the *generating operator* for a family of operators $R_\ell \colon \Gamma(X) \to \Gamma(Y)$.

In this work we initiate a new line of investigation of "generating operators" in the setting that $(X, Y) = (\mathbb{C}^2, \mathbb{C})$ and that $\{R_\ell\}$ are the Rankin–Cohen brackets (0.2). We present a closed formula of the generating operator T as an integral operator, through which we explore its basic properties and various aspects.

2 Basic properties of the integral operator T

Let D be an open set in $\mathbb{C}$. For a holomorphic function $f(\zeta_1, \zeta_2)$ in $D \times D$, we introduce an integral transform by

$$(Tf)(z, t) := \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_1} \oint_{C_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z)(\zeta_2 - z) + t(\zeta_1 - \zeta_2)} d\zeta_1 d\zeta_2, \tag{2.1}$$

where C_j are contours in D around the point z $(j = 1, 2)$. The denominator will be denoted by

$$Q \equiv Q(\zeta_1, \zeta_2; z, t) := (\zeta_1 - z)(\zeta_2 - z) + t(\zeta_1 - \zeta_2). \tag{2.2}$$

We note that the denominator is an irreducible polynomial of ζ_1 and ζ_2 when $t \neq 0$. We shall give closed formulas of $Tf(z,t)$ in Example 3.5 for certain specific family of meromorphic functions $f(\zeta_1, \zeta_2)$.

We begin with general properties of the operator T.

Theorem 2.1.
(1) *There exists an open neighbourhood U of $D \times \{0\}$ in $\mathbb{C}^2$ such that $T\colon \mathcal{O}(D \times D) \to \mathcal{O}(U)$ is well-defined.*
(2) *$Tf(z,0) = f(z,z)$ for any $z \in D$.*
(3) *For any neighbourhood U of $D \times \{0\}$ in $\mathbb{C}^2$, T is injective.*

Example 2.2.
(1) $U_D = \mathbb{C} \times \mathbb{C}$ if $D = \mathbb{C}$.
(2) $U_D = \{(z,t) \in \mathbb{C}^2 : 2|t| < \operatorname{Im} z\}$ if D is the upper half plane $\Pi := \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$.

We show that T is a "generating operator" for the family of the Rankin–Cohen brackets. For $\ell \in \mathbb{N}$ we define $R_\ell\colon \mathcal{O}(D \times D) \to \mathcal{O}(D)$, $f(\zeta_1,\zeta_2) \mapsto (R_\ell f)(z)$ by

$$R_\ell f(z) := \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j}^2 \left. \frac{\partial^\ell f(\zeta_1,\zeta_2)}{\partial \zeta_1^{\ell-j} \partial \zeta_2^j} \right|_{\zeta_1=\zeta_2=z}. \tag{2.3}$$

Theorem 2.3 (generating operator of Rankin–Cohen brackets)**.** *The integral operator T in (2.1) is expressed as*

$$Tf(z,t) = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} R_\ell f(z) \quad \text{for any } f \in \mathcal{O}(D \times D).$$

Remark 2.4. *For $f(\zeta_1,\zeta_2) = f_1(\zeta_1) f_2(\zeta_2)$ with some $f_1, f_2 \in \mathcal{O}(D)$, $(R_\ell f)(z)$ takes the form $\sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j}^2 \frac{\partial^{\ell-j} f_1(z)}{\partial z^{\ell-j}} \frac{\partial^j f_2(z)}{\partial z^j}$, which is the Rankin–Cohen bidifferential operator $R_{\lambda',\lambda''}^{\lambda'''}(f_1,f_2)$ at $(\lambda',\lambda'',\lambda''') = (1,1,2+2\ell)$, see (0.2).*

3 Differential operator P and the generating operator

The following differential operator on $\mathbb{C}^2$ plays a key role in the analysis of the generating operator T.

$$P := (\zeta_1 - \zeta_2)^2 \frac{\partial^2}{\partial \zeta_1 \partial \zeta_2} - (\zeta_1 - \zeta_2)\left(\frac{\partial}{\partial \zeta_1} - \frac{\partial}{\partial \zeta_2}\right). \tag{3.1}$$

The following result holds.

Theorem 3.1. *Let D be an open set in $\mathbb{C}$. For any $f \in \mathcal{O}(D \times D)$,*

$$T(Pf)(z,t) = -\left(t\frac{\partial}{\partial t}\right)\left(t\frac{\partial}{\partial t} + 1\right) Tf(z,t).$$

One derives from Theorem 3.1 that the set of eigenvalues of P is discrete:

Corollary 3.2 (Eigenvalues of P)**.** *Let D be a connected open set in $\mathbb{C}$. If there is a non-zero function $f \in \mathcal{O}(D \times D)$ satisfying $Pf = \lambda f$ for some $\lambda \in \mathbb{C}$, then λ is of the form $-\ell(\ell+1)$ for some $\ell \in \mathbb{N}$.*

For $\ell \in \mathbb{N}$, we consider the space of all eigenfunctions:

$$\mathcal{S}ol(D \times D)_\ell := \{f \in \mathcal{O}(D \times D) : Pf = -\ell(\ell+1)f\}. \tag{3.2}$$

We shall see in Corollary 4.2 that $\mathcal{S}ol(D \times D)_\ell$ is infinite-dimensional for any $\ell \in \mathbb{N}$ and any non-empty open subset D.

Remark 3.3. *In Theorem 5.1, we shall prove that P defines a self-adjoint operator on the tensor product of two Hardy spaces.*

Corollary 3.4. *Let $\ell \in \mathbb{N}$. Then the following two conditions on $f \in \mathcal{O}(D \times D)$ are equivalent:*
(i) *$f \in \mathcal{S}ol(D \times D)_\ell$,*
(ii) *$Tf(z,t)$ is of the form $t^\ell \varphi(z)$ for some $\varphi \in \mathcal{O}(D)$.*

We end this section with an example of closed formulæ for $Tf(z,t)$ for a specific family of functions $f \in \mathcal{O}(D \times D)$:

Example 3.5. For $\ell \in \mathbb{N}$, we set

$$f_\ell(\zeta_1, \zeta_2) := (\zeta_1 - \zeta_2)^\ell (\zeta_1 + \sqrt{-1})^{-\ell-1} (\zeta_2 + \sqrt{-1})^{-\ell-1}. \tag{3.3}$$

Then one has the following:
(1) $Pf_\ell = -\ell(\ell+1) f_\ell$.
(2) $(Tf_\ell)(z,t) = \binom{2\ell}{\ell} t^\ell (z + \sqrt{-1})^{-2\ell-2}$.

4 Generating operators and holographic operators

Throughout this section, we assume that D is a convex domain in $\mathbb{C}$. Then any two elements $\zeta_1, \zeta_2 \in D$ can be joined by a line segment contained in D. For $\ell \in \mathbb{N}$, we consider a weighted average of $g \in \mathcal{O}(D)$ along the line segment between ζ_1 and ζ_2 given by

$$(\Psi_\ell g)(\zeta_1, \zeta_2) := (\zeta_1 - \zeta_2)^\ell \int_{-1}^{1} g\left(\frac{(\zeta_2 - \zeta_1)v + (\zeta_1 + \zeta_2)}{2}\right) (1 - v^2)^\ell dv.$$

We investigate the "generating operator" T in connection with Ψ_ℓ. Recall from Corollary 3.4 that if $f \in \mathcal{S}ol(D \times D)_\ell$, namely, if $Pf = -\ell(\ell+1)f$, then $t^{-\ell}(Tf)(z,t)$ is independent of t, which we shall simply denote by $(t^{-\ell}Tf)(z)$.

Theorem 4.1. *Let $\ell \in \mathbb{N}$.*
(1) *$t^{-\ell}T \colon \mathcal{S}ol(D \times D)_\ell \xrightarrow{\sim} \mathcal{O}(D)$ is a bijection.*
(2) *The inverse of $t^{-\ell}T$ is given by the integral operator Ψ_ℓ, namely, $\Psi_\ell \colon \mathcal{O}(D) \xrightarrow{\sim} \mathcal{S}ol(D \times D)_\ell$ is a bijection and $t^{-\ell}T \circ \Psi_\ell = \frac{2^{2\ell+1}}{2\ell+1} \,\mathrm{id}$.*

As an immediate consequence of Theorem 4.1 (2), one has the following:

Corollary 4.2. *For any $\ell \in \mathbb{N}$, $\mathcal{S}ol(\mathbb{C}\times\mathbb{C})_\ell$ is infinite-dimensional.*

Remark 4.3. *When D is the upper half plane Π, the integral operator Ψ_ℓ appeared in the study of the holographic transforms for the branching problem of infinite-dimensional representations of $SL(2,\mathbb{R})$. In this case, the bijectivity of Ψ_ℓ was shown in* [7] *by a different approach based on the representation theory (see Section 6).*

5 The generating operator T and the Hardy space

Let Π be the upper half plane. As we have seen in Example 2.2, the "generating operator" $T\colon \mathcal{O}(\Pi\times\Pi)\to\mathcal{O}(U_\Pi)$ is well-defined where $U_\Pi=\{(z,t)\in\mathbb{C}^2 : 2|t|<\operatorname{Im} z\}$. This section discusses how the generating operator T acts on the tensor product of two Hardy spaces.

We recall that the Hardy space on Π is a Hilbert space defined by

$$\mathbf{H}(\Pi)=\{h\in\mathcal{O}(\Pi) : \|h\|^2_{\mathbf{H}(\Pi)}:=\sup_{y>0}\int_{-\infty}^{\infty}|h(x+\sqrt{-1}y)|^2dx<\infty\}.$$

Let $\mathbf{H}(\Pi\times\Pi)$ be the Hilbert completion $\mathbf{H}(\Pi)\widehat{\otimes}\mathbf{H}(\Pi)$ of the tensor product of two Hardy spaces $\mathbf{H}(\Pi)$. Any holomorphic differential operator P acting on $\mathcal{O}(\Pi\times\Pi)$ induces a continuous operator on $\mathbf{H}(\Pi\times\Pi)$. In turn, the eigenspace $\mathbf{H}(\Pi\times\Pi)_\ell:=\mathcal{S}ol(\Pi\times\Pi)_\ell\cap\mathbf{H}(\Pi\times\Pi)$ is a Hilbert subspace for every $\ell\in\mathbb{N}$.

Theorem 5.1. *Let P be the differential operator given in* (3.1).
(1) *The differential operator P defines a self-adjoint operator on the Hilbert space $\mathbf{H}(\Pi\times\Pi)$.*
(2) (Eigenspace decomposition) *$\mathbf{H}(\Pi\times\Pi)$ decomposes into the discrete Hilbert sum of eigenspaces $\mathbf{H}(\Pi\times\Pi)_\ell$ of P where ℓ runs over $\mathbb{N}$.*
(3) *The generating operator T induces a family of linear operators*

$$t^{-\ell}T\colon \mathbf{H}(\Pi\times\Pi)_\ell \xrightarrow{\sim} \mathcal{O}(\Pi)\cap L^2(\Pi, y^{2\ell}dxdy)$$

which are unitary up to rescaling:

$$\|t^{-\ell}Tf\|^2_{L^2(\Pi,y^{2\ell+2}dxdy)}=b_\ell\|f\|^2_{\mathbf{H}(\Pi\times\Pi)} \quad \textit{for any } f\in\mathbf{H}(\Pi\times\Pi)_\ell \tag{5.1}$$

where we set

$$b_\ell:=\frac{(2\ell)!}{2^{2\ell+2}\pi(2\ell+1)(\ell!)^2}=\frac{(2\ell-1)!!}{4\pi(2\ell+1)(2\ell)!!}. \tag{5.2}$$

The proof of Theorem 5.1 uses the double Fourier–Laplace transform $\mathcal{F}$ defined by

$$F(x,y)\mapsto(\mathcal{F}F)(\zeta_1,\zeta_2):=\int_0^\infty\int_0^\infty F(x,y)e^{\sqrt{-1}(x\zeta_1+y\zeta_2)}dxdy.$$

According to the Payley–Wiener theorem, the Fourier–Laplace transform $\mathcal{F}$ establishes a bijection from $L^2(\mathbb{R}_+\times\mathbb{R}_+)$ onto $\mathbf{H}(\Pi\times\Pi)$, and satisfies $\|\mathcal{F}F\|^2_{\mathbf{H}(\Pi\times\Pi)}=$

$(2\pi)^2\|F\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}_+)}$ for all $F \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$. The inverse $\mathcal{F}^{-1}\colon \mathbf{H}(\Pi\times\Pi) \to L^2(\mathbb{R}_+\times\mathbb{R}_+)$ is given by

$$(\mathcal{F}^{-1}f)(x,y) = \lim_{\eta_1\downarrow 0}\lim_{\eta_2\downarrow 0}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} f(\zeta_1,\zeta_2)e^{-\sqrt{-1}(\zeta_1 x+\zeta_2 y)}d\xi_1 d\xi_2,$$

where we write $\zeta_j = \xi_j + \sqrt{-1}\eta_j$.

The change of variables $(x,y) = (\frac{s}{2}(1-v), \frac{s}{2}(1+v))$ yields a unitary map $L^2(\mathbb{R}_+\times(-1,1), sdsdv) \xrightarrow{\sim} L^2(\mathbb{R}_+\times\mathbb{R}_+, 2dxdy)$. We denote its composition with $\mathcal{F}$ by

$$\widetilde{\mathcal{F}}\colon L^2(\mathbb{R}_+\times(-1,1), sdsdv) \to \mathbf{H}(\Pi\times\Pi).$$

The inverse is given by $(\widetilde{\mathcal{F}}^{-1}f)(s,v) = (\mathcal{F}^{-1}f)(\frac{s}{2}(1-v), \frac{s}{2}(1-v))$.

Proposition 5.2. (1) *$\widetilde{\mathcal{F}}\colon L^2(\mathbb{R}_+\times(-1,1), sdsdv) \xrightarrow{\sim} \mathbf{H}(\Pi\times\Pi)$ is a unitary map up to a scalar multiplication, namely,*

$$\|f\|^2_{\mathbf{H}^2(\Pi\times\Pi)} = 2\pi^2\|(\widetilde{\mathcal{F}}^{-1}f)(s,v)\|^2_{L^2(\mathbb{R}_+\times(-1,1),sdsdv)} \quad \text{for } f\in\mathbf{H}(\Pi\times\Pi).$$

(2) *The operator $\widetilde{P} := \widetilde{\mathcal{F}}^{-1}\circ P\circ\mathcal{F}$ takes the following form:*

$$\widetilde{P} = (1-v^2)\partial_v^2 - 2v\partial_v. \tag{5.3}$$

Proof of (1) and (2) in Theorem 5.1. (1) By Proposition 5.2, the differential operator P is equivalent via $\widetilde{\mathcal{F}}$ to the Legendre differential operator $\widetilde{P}$ which does not involve the variable s. Since $\widetilde{P}$ defines a self-adjoint operator on $L^2(\mathbb{R}_+\times(-1,1), sdsdv)$, so does P on $\mathbf{H}(\Pi\times\Pi)$ via $\widetilde{\mathcal{F}}$.
(2) By (5.3), $Pf = \lambda f$ if and only if $\widetilde{P}(\widetilde{\mathcal{F}}^{-1}f) = \lambda(\widetilde{\mathcal{F}}^{-1}f)$. Hence $\widetilde{\mathcal{F}}$ induces an isomorphism $L^2(\mathbb{R}_+, sds)\otimes\mathbb{C}P_\ell(v) \xrightarrow{\sim} \mathbf{H}(\Pi\times\Pi)_\ell$ for every $\ell\in\mathbb{N}$, where $P_\ell(v)$ is the ℓ-th Legendre polynomial. Therefore the proof of the second statement is reduced to the classical theorem that $\{P_\ell\}_{\ell\in\mathbb{N}}$ forms an orthogonal basis in $L^2((-1,1), dv)$. $\square$

To prove the third statement of Theorem 5.1, we apply the "generating operator" T to the diagram below:

$$\begin{array}{ccccc}
\mathcal{O}(\Pi\times\Pi) & \supset \mathbf{H}(\Pi\times\Pi) & \underset{\widetilde{\mathcal{F}}}{\overset{\sim}{\leftarrow}} & L^2(\mathbb{R}_+, sds)\widehat{\otimes}L^2(-1,1) \\
\cup & \cup & & \cup \\
\mathcal{S}ol(\Pi\times\Pi)_\ell & \supset \mathbf{H}(\Pi\times\Pi)_\ell & \underset{\widetilde{\mathcal{F}}}{\overset{\sim}{\leftarrow}} & L^2(\mathbb{R}_+, sds)\otimes\mathbb{C}P_\ell(v).
\end{array} \tag{5.4}$$

We recall that the weighted Bergman space is defined by

$$\mathbf{H}^2(\Pi)_\lambda := \mathcal{O}(\Pi)\cap L^2(\Pi, y^{\lambda-2}dxdy)$$

for $\lambda > 1$. We also recall some basic properties of the Fourier–Laplace transform of one variable $\varphi(\xi)\mapsto(\mathcal{F}_\mathbb{R}\varphi)(z) := \int_0^\infty \varphi(\xi)e^{\sqrt{-1}z\xi}d\xi$. By the Plancherel formula, one has

$$\int_\mathbb{R}|\mathcal{F}_\mathbb{R}\varphi(x+\sqrt{-1}y)|^2dx = 2\pi\int_0^\infty|\varphi(\xi)|^2e^{-2y\xi}d\xi.$$

Integrating the both-hand sides against the measure $y^{\lambda-2}dy$, one obtains

$$\|\mathcal{F}_{\mathbb{R}}\varphi\|^2_{\mathbf{H}^2(\Pi)_\lambda} = 2^{2-\lambda}\pi\Gamma(\lambda-1)\|\varphi\|^2_{L^2(\mathbb{R}_+,\xi^{1-\lambda}d\xi)}. \tag{5.5}$$

Thus $\mathcal{F}_{\mathbb{R}}$ gives a bijection from $L^2(\mathbb{R}_+,\xi^{1-\lambda}d\xi)$ onto $\mathbf{H}^2(\Pi)_\lambda$.

Proposition 5.3. *Let* $c_\ell := \frac{(-1)^{\frac{3}{2}\ell}}{(2\ell+1)\ell!}$ *and*

$$T^{\mathcal{F}}(h(z)P_\ell(v)) := c_\ell\, h(\xi)\,\xi^{\ell+1}\,t^\ell.$$

Then the following diagram commutes.

$$\begin{tikzcd}
\mathbf{H}(\Pi\times\Pi)_\ell \arrow[d, "T", "\sim"'] & L^2(\mathbb{R}_+, sds)\otimes\mathbb{C}P_\ell(v) \arrow[l, "\sim", "\widetilde{\mathcal{F}}"'] \arrow[d, "T^{\mathcal{F}}", "\sim"'] \\
\mathbf{H}^2(\Pi)_{2+2\ell}\otimes\mathbb{C}t^\ell & L^2(\mathbb{R}_+,\xi^{-1-2\ell}d\xi)\otimes\mathbb{C}t^\ell \arrow[l, "\sim", "\mathcal{F}_{\mathbb{R}}\otimes\mathrm{id}"']
\end{tikzcd}$$

Proof of (3) in Theorem 5.1. In light of the isomorphism

$$\widetilde{\mathcal{F}}\colon L^2(\mathbb{R}_+,sds)\otimes\mathbb{C}P_\ell(v)\xrightarrow{\sim}\mathbf{H}(\Pi\times\Pi)_\ell,$$

we take $h\in L^2(\mathbb{R}_+,sds)$ and set $f:=\widetilde{\mathcal{F}}(hP_\ell)\in\mathbf{H}(\Pi\times\Pi)_\ell$. By Proposition 5.2, one has

$$\|f\|^2_{\mathbf{H}(\Pi\times\Pi)} = 2\pi^2\|h\|^2_{L^2(\mathbb{R}_+,sds)}\|P_\ell\|^2_{L^2(-1,1)} = \frac{4\pi^2}{2\ell+1}\|h\|^2_{L^2(\mathbb{R}_+,sds)}. \tag{5.6}$$

Applying (5.5) to $\varphi:=t^{-\ell}T^{\mathcal{F}}(hP_\ell)$ with $\lambda=2\ell+2$, one has from Proposition 5.3 that

$$\begin{aligned}\|t^{-\ell}Tf\|^2_{\mathbf{H}^2(\Pi)_{2+2\ell}} &= 2^{-2\ell}\pi(2\ell)!\|t^{-\ell}T^{\mathcal{F}}(hP_\ell)\|^2_{L^2(\mathbb{R}_+,\xi^{-1-2\ell}d\xi)}\\ &= \frac{\pi(2\ell)!}{2^{2\ell}(2\ell+1)^2(\ell!)^2}\|h\|^2_{L^2(\mathbb{R}_+,sds)}.\end{aligned} \tag{5.7}$$

It follows from (5.2), (5.6) and (5.7) that

$$\|t^{-\ell}Tf\|^2_{\mathbf{H}^2(\Pi)_{2+2\ell}} = \frac{(2\ell-1)!!}{4\pi(2\ell+1)(2\ell)!!}\|f\|^2_{\mathbf{H}(\Pi\times\Pi)} = b_\ell\|f\|^2_{\mathbf{H}(\Pi\times\Pi)}.$$

Hence the third statement of Theorem 5.1 is proved. □

6 Representation theory and the generating operator T

If D is simply connected, then the group $\mathrm{Aut}(D)$ of biholomorphic diffeomorphisms acts transitively on D. This section discusses different perspectives of our generating operator T from the viewpoint of the automorphism group of the domain, in particular, from the (infinite-dimensional) representation theory of real reductive groups. Lie theory reveals structures of the generating operator T that are not otherwise evident.

6.1. Normal derivatives and the generating operator T.

Let π be an irreducible representation of a group G, and G' a subgroup. The G-module π may be seen as a G'-module by restriction, for which we write $\pi|_{G'}$. For an irreducible representation ρ of the subgroup G', a *symmetry breaking operator* (SBO for short) is an intertwining operator from $\pi|_{G'}$ to ρ, whereas a *holographic operator* is an intertwining operator from ρ to $\pi|_{G'}$. Suppose that the representations π and ρ are geometrically defined, *e.g.,* they are realized in the spaces $\Gamma(X)$ and $\Gamma(Y)$ of functions on a G-manifold X and its G'-submanifold Y, respectively, or more generally, in the spaces of sections for some equivariant vector bundles.

When the restriction $\pi|_{G'}$ is discretely decomposable [3], one may expect that taking "normal derivatives" with respect to the submanifold $Y \hookrightarrow X$ would yield SBOs. However, this is not the case even for the irreducible decomposition (*fusion rule*) of the tensor product of two representations of $SL(2,\mathbb{R})$. See [6, Thm. 5.3] for more general cases. The underlying geometry for the fusion rule of the Hardy spaces $\mathbf{H}(\Pi)$ is given by a diagonal embedding of $Y = \Pi$ into $X := Y \times Y$. Instead of using $X = \Pi \times \Pi$, we consider $\widetilde{X} := U_\Pi$ as in Example 2.2. In this case the "normal derivative" of ℓ-th order with respect to $Y \hookrightarrow \widetilde{X}$ is given simply by

$$N_\ell := \mathrm{Rest}_{t=0} \circ (\frac{\partial}{\partial t})^\ell.$$

A distinguishing feature of the generating operator T is that all the normal derivatives N_ℓ give rise to symmetry breaking operators after the transformation by T, symbolically written in the following diagram (see (6.2) for the notation π_λ):

$$\begin{array}{ccc} \mathcal{O}(X) & \overset{T}{\hookrightarrow} & \mathcal{O}(\widetilde{X}) \\ \text{SBO} \searrow & \circlearrowleft & \swarrow \ell\text{-th normal derivative } N_\ell \\ & (\pi_{2+2\ell}, \mathcal{O}(Y)) & \end{array}$$

6.2. Modular forms and the generating operator T.

The Rankin–Cohen brackets were introduced in [2, 10] to construct holomorphic modular forms of higher weight from those of lower weight. This section highlights the relationship of our generating operator T in (2.1) and modular forms.

By Theorem 2.3, one has

$$N_\ell \circ T = R_\ell, \tag{6.1}$$

where R_ℓ are the Rankin–Cohen brackets (2.3). Then by a direct computation, one sees the following covariance property:

Proposition 6.1. *For all $\ell \in \mathbb{N}$, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and for any $f \in \mathcal{O}(\Pi \times \Pi)$, one has*

$$N_\ell \circ (Tf^g)(z) = (cz+d)^{-2\ell-2}((N_\ell \circ T)f)(\frac{az+b}{cz+d})$$

where $f^g(\zeta_1,\zeta_2) := (c\zeta_1+d)^{-1}(c\zeta_2+d)^{-1} f(\frac{a\zeta_1+b}{c\zeta_1+d}, \frac{a\zeta_2+b}{c\zeta_2+d})$.

To clarify its representation-theoretic meaning, we write π_λ ($\lambda \in \mathbb{Z}$) for a representation of $SL(2,\mathbb{R})$ on $\mathcal{O}(\Pi)$ given by

$$\pi_\lambda(g)h(z) = (cz+d)^{-\lambda} h(\frac{az+b}{cz+d}) \qquad \text{for } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.2}$$

Then Proposition 6.1 tells us that

$$(N_\ell \circ T) \circ (\pi_1(g) \boxtimes \pi_1(g)) = \pi_{2\ell+2}(g) \circ (N_\ell \circ T) \tag{6.3}$$

for any $g \in SL(2,\mathbb{R})$. Therefore, for a subgroup Γ, $N \circ T(f)$ is Γ-invariant whenever f is $(\Gamma \times \Gamma)$-invariant.

Suppose that Γ is a congruence subgroup of $SL(2,\mathbb{Z})$. For any modular form h of level Γ and weight 1, we set

$$H(z,t) := \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_1} \oint_{C_2} \frac{h(\zeta_1)h(\zeta_2)}{(\zeta_1 - z)(\zeta_2 - z) + t(\zeta_1 - \zeta_2)} d\zeta_1 d\zeta_2.$$

It follows from (6.1) that $(N_\ell H)(z) = (\frac{\partial}{\partial t})^\ell\big|_{t=0} H(z,t) = R_\ell(h(\zeta_1)h(\zeta_2))(z)$ is a modular form of level Γ and weight $2\ell+2$ for all $\ell \in \mathbb{N}$.

6.3. Unitary representations and the generating operator T. Viewed as a representation of the universal covering group $SL(2,\mathbb{R})\tilde{}$, the representation π_λ is well-defined for all $\lambda \in \mathbb{C}$. For $\lambda > 1$, π_λ leaves the weighted Bergman space $\mathbf{H}^2(\Pi)_\lambda = \mathcal{O}(\Pi) \cap L^2(\Pi, y^{\lambda-2}dxdy)$ invariant, and $SL(2,\mathbb{R})\tilde{}$ acts as an irreducible unitary representation on the Hilbert space $\mathbf{H}^2(\Pi)_\lambda$. These unitary representations $(\pi_\lambda, \mathbf{H}^2(\Pi)_\lambda)$ are referred to as (relative) *holomorphic discrete series representations* of $SL(2,\mathbb{R})\tilde{}$. In particular, the set of holomorphic discrete series representations of the group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I_2\} \simeq \mathrm{Aut}(\Pi)$ is given by $\{\pi_\lambda : \lambda = 2,4,6,\dots\}$.

If $\lambda = 1$ then $\mathbf{H}^2(\Pi)_\lambda = \{0\}$, however, the Hardy space $\mathbf{H}(\Pi)$ is an invariant subspace of $(\pi_\lambda, \mathcal{O}(\Pi))$ with $\lambda = 1$, and $SL(2,\mathbb{R})$ acts on $\mathbf{H}(\Pi)$ as an irreducible unitary representation, too.

With these notations, one may interpret Theorem 5.1 as a decomposition of the tensor product of two copies of the unitary representation $(\pi_1, \mathbf{H}(\Pi))$ on the Hardy space into a multiplicity-free discrete sum of irreducible unitary representations:

$$\mathbf{H}(\Pi)\widehat{\otimes}\mathbf{H}(\Pi) \simeq {\sum_{\ell=0}^{\infty}}^{\oplus} \mathbf{H}^2(\Pi)_{2+2\ell} \qquad \text{(Hilbert direct sum)}.$$

The right-hand side may be seen as a “model” of holomorphic discrete series representations of $PSL(2,\mathbb{R})$ in the sense that all such representations occur exactly once.

Moreover, the Hardy norm $\|\cdot\|_{\mathbf{H}(\Pi)}$ may be regarded as the residue of the analytic continuation of the norm of the weighted Bergman space $\mathbf{H}^2(\Pi)_\lambda$ which is originally defined for real $\lambda > 1$:

$$\| \cdot \|^2_{\mathbf{H}(\Pi)} = \lim_{\lambda \downarrow 1} (\lambda - 1) \| \cdot \|^2_{\mathbf{H}^2(\Pi)_\lambda}.$$

Then the exact formula (5.1) in Theorem 5.1 may be thought of as the limit of [7, Thm. 2.7] which dealt with the weighted Bergman spaces, namely, our b_ℓ in Theorem 5.1 may be rediscovered by the following limit procedure with the notation as in [7, (2.3) and (2.4)]:

$$\begin{aligned}
&\frac{1}{(\ell!)^2} \lim_{\lambda' \downarrow 1} \lim_{\lambda'' \downarrow 1} \frac{c_\ell(\lambda', \lambda'') r_\ell(\lambda', \lambda'')}{(\lambda' - 1)(\lambda'' - 1)} \\
=&\frac{1}{(\ell!)^2} \lim_{\lambda' \downarrow 1} \lim_{\lambda'' \downarrow 1} \frac{\Gamma(\lambda' + \ell)\Gamma(\lambda'' + \ell)}{(\lambda' + \lambda'' + 2\ell - 1)\Gamma(\lambda' + \lambda'' + \ell - 1)\ell!} \cdot \frac{\Gamma(\lambda' + \lambda'' + 2\ell - 1)}{2^{2\ell+2}\pi\Gamma(\lambda')\Gamma(\lambda'')} \\
=&\frac{(2\ell)!}{(2\ell + 1)\pi(\ell!)^2 2^{2\ell+2}} = \frac{(2\ell - 1)!!}{4\pi(2\ell + 1)(2\ell)!!} = b_\ell.
\end{aligned}$$

6.4. From discrete to continuous - some further developments. It is known that covariant differential operators are often obtained as residues of a meromorphic family of integral transformations. For instance, the iterated powers of the Dirac operator are the residues of the meromorphic family of the Knapp–Stein intertwining operators, see *e.g.,* a recent paper [1].

The inverse direction is more involved. In fact, some covariant differential operators cannot be obtained as residues, which are referred to as *sporadic operators*. One of the important applications of the *generating operator* introduced in this article provides us a method to go in the inverse direction, namely, to construct a meromorphic family of non-local symmetry breaking operators out of discrete data. In the subsequent paper [4], we give a toy model which constructs various fundamental operators such as invariant trilinear forms on infinite-dimensional representations, the Fourier and the Poisson transforms on the anti-de Sitter space, and non-local symmetry breaking operators for the fusion rules among others, out of just countable data of the Rankin–Cohen brackets, for which the key of the proof is the explicit formula (2.1) of the *generating operator* proved in this article.

References

[1] J.-L. Clerc, B. Ørsted, Conformal covariance for the powers of the Dirac operator. J. Lie Theory **30** (2020), 345–360.

[2] H. Cohen, Sums involving the values at negative integers of L-functions of quadratic characters, Math. Ann. **217** (1975), 271–285.

[3] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications, Invent. Math., **117** (1994), 181–205; Part II, Ann. of Math., **147** (1998), 709–729; Part III, Invent. Math., **131** (1998), 229–256.

[4] T. Kobayashi, Generating operators of symmetry breaking – from discrete to continuous, ArXiv:2307.16587.

[5] T. Kobayashi, M. Pevzner, *Differential symmetry breaking operators.* I. *General theory and F-method.* Selecta Math. (N.S.) **22** (2016), 801–845.

[6] T. Kobayashi, M. Pevzner, *Differential symmetry breaking operators.* II. *Rankin–Cohen operators for symmetric pairs.* Selecta Math. (N.S.) **22** (2016), 847–911.

[7] T. Kobayashi and M. Pevzner, Inversion of Rankin–Cohen operators via holographic transform, Ann. Inst. Fourier (Grenoble), **70**, (2020), 2131–2190.

[8] T. Kobayashi and M. Pevzner, A generating operator for Rankin–Cohen brackets, ArXiv: 2306.16800.

[9] V. F. Molchanov, Tensor products of unitary representations of the three-dimensional Lorentz group. *Math. USSR, Izv.* **15** (1980), pp. 113–143.

[10] R. A. Rankin, The construction of automorphic forms from the derivatives of a given form, J. Indian Math. Soc. **20** (1956), 103–116.

[11] J. Repka, Tensor products of holomorphic discrete series representations. *Can. J. Math.* **31** (1979), pp. 836–844.

[12] D. Zagier, Modular forms and differential operators. K. G. Ramanathan memorial issue. *Proc. Indian Acad. Sci. Math. Sci.* **104** (1994), pp. 57–75.

Toshiyuki KOBAYASHI (`toshi@ms.u-tokyo.ac.jp`)
Graduate School of Mathematical Sciences,
The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo, 153-8914, Japan.

Michael PEVZNER (`pevzner@math.cnrs.fr`)
French-Japanese Laboratory in Mathematics and its Interactions,
FJ-LMI CNRS IRL2025, Tokyo, Japan