GENERATING OPERATORS OF SYMMETRY BREAKING — FROM DISCRETE TO CONTINUOUS

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Abstract. Based on the "generating operator" of the Rankin–Cohen brackets introduced in Kobayashi–Pevzner [arXiv:2306.16800], we present a method to construct various fundamental operators with continuous parameters such as invariant trilinear forms on infinite-dimensional representations, the Fourier and the Poisson transforms on the anti-de Sitter space, and integral symmetry breaking operators for the fusion rules, among others, out of a countable set of differential symmetry breaking operators.

Keywords and phrases: generating operator, symmetry breaking operator, Rankin–Cohen bracket, de Sitter space, branching law, Hardy space, holographic operator.

2020 MSC: Primary 22E45 ; Secondary 30H10, 32A45, 43A85, 47B38.

1 Introduction

This article is a continuation of [15], where we initiated a new line of investigation on branching problems for the restriction of representations through "generating operators" between two manifolds. By definition, "generating operators" are built on discrete data, which are a countable family of SBOs (*symmetry breaking operators*) in the case we consider. On the other hand, general irreducible decompositions such as branching laws or Plancherel-type theorems often involve continuous spectrum. The novel feature of this article is an introduction of a method to *transfer* differential SBOs (**countable data**) into a meromorphic family of non-local intertwining operators (**continuous data**). We illustrate the trick by the "generating operators" of the Rankin–Cohen brackets. In particular, by using the "boundary values" of "generating operators", we are able to treat non-holomorphic representations in different geometric settings from the initial one.

Another novel feature of this article is an application of differential SBOs to construct a geometric embedding of discrete series representations for the de Sitter space dS^2 into principal series representations (Theorem 5.3). This is carried out by proving the analytic continuation of L^2-eigenfunctions to the boundary of the conformal compactification. We remark that such an analytic continuation does NOT exist for eigenfunctions that are not square integrable. The proof uses the theory of discretely decomposable restriction [6].

We illustrate the idea by the generating operator of the Rankin–Cohen brackets. The results suggest that the application of the "generating operators" of SBOs is already rich in the SL_2 case. See also [16]. In a subsequent paper, we plan to discuss a generalization of some of the aspects discovered here to other reductive groups of higher rank.

The article is organized as follows. In Section 2, we explain the geometric setting of branching problems for which the "generating operators" make sense by taking the general fusion rules as an example. Section 3 discusses a trick from "discrete" to "continuous" via the generating operators, for which the representation theoretic setting also changes from "discretely decomposable restrictions" to "branching laws with continuous spectrum". Sections 4 and 5 extend the results beyond symmetry breaking through the conformal compactification.

2 Generalities: generating operators for SBOs

This section reveals a representation-theoretic background of the closed formula of the generating operators [15] in a specific setting, and investigates a more general setting in branching problems for which we could expect a further detailed study of the "generating operators". The key requirements are discrete decomposability and multiplicities of the restriction. For simplicity, we confine ourselves to the tensor product case.

2.1. Generating operators for symmetry breaking operators. Let X and Y be two manifolds. For a family of linear maps $R_\ell \colon \Gamma(X) \to \Gamma(Y)$ between the spaces of functions on X and Y, the "generating operator" T is defined as a $\mathrm{Hom}_{\mathbb{C}}(\Gamma(X), \Gamma(Y))$-valued formal power series of t, see [15]:

$$T = \sum_{\ell=0}^{\infty} \frac{R_\ell}{\ell!} t^\ell \in \mathrm{Hom}_{\mathbb{C}}(\Gamma(X), \Gamma(Y)) \otimes \mathbb{C}[[t]]. \tag{2.1}$$

Very special cases of the generating operators include the classical notion of the generating functions of orthogonal polynomials and the semigroups generated by self-adjoint operators *e.g.,* the Hille–Yosida theory.

Suppose that a pair of groups $\widetilde{G} \supset G$ act on $X \supset Y$, respectively, and that $R_\ell \colon \Gamma(X) \to \Gamma(Y)$ ($\ell \in \mathbb{N}$) are a family of SBOs (*symmetry breaking operators*), *i.e.*, each R_ℓ is a G-intertwining operator into a multiplier representation $(\pi_\ell, \Gamma(Y))$ of G. We are particularly interested in the setting where the $\widetilde{G}$-module $\Gamma(X)$ decomposes **discretely** into irreducible representations π_ℓ of G with **bounded multiplicity**.

In the rest of this section, we summarize briefly some recent developments about when such settings arise, with focus on the case of tensor product representations of G, in other words, the case where $\widetilde{G} = G \times G$.

2.2. Generalities: Multiplicity of the tensor product.
In defining the generating operator of SBOs, it would be natural to impose some control of *multiplicities* in the branching laws.

Let G be a real reductive Lie group, $\mathcal{M}(G)$ the category of finitely generated, smooth admissible representations of G of moderate growth, see [22, Chap. 11], and $\mathrm{Irr}(G)$ the set of irreducible objects in $\mathcal{M}(G)$.

Definition 2.1 (multiplicity)**.** For $\pi_1, \pi_2, \tau \in \mathrm{Irr}(G)$, the multiplicity of τ in the tensor product representation $\pi_1 \otimes \pi_2$ is defined by

$$[\pi_1 \otimes \pi_2 : \tau] := \dim_{\mathbb{C}} \mathrm{Hom}_G(\pi_1 \otimes \pi_2, \tau) \in \mathbb{N} \cup \{\infty\},$$

where $\mathrm{Hom}_G(\, , \,)$ denotes the space of SBOs (*i.e.,* continuous G-homomorphisms) between the Fréchet representations.

The finiteness condition of the multiplicity $[\pi_1 \otimes \pi_2 : \tau]$ gives a strong constraint on the group G:

Fact 2.2 ([7], see also [11])**.** *The following three conditions on a non-compact simple Lie group G are equivalent.*

(i) $[\pi_1 \otimes \pi_2 : \tau] < \infty$ *for any* π_1, π_2, $\tau \in \mathrm{Irr}(G)$.
(ii) *The triple product of real flag varieties* G/P *is real spherical.*
(iii) $\mathfrak{g} \simeq \mathfrak{so}(n, 1)$.

The proof includes that the tensor product $\pi_1 \otimes \pi_2$ is of infinite multiplicity for "generic representations" π_1 and π_2 except when $\mathfrak{g} \simeq \mathfrak{so}(n, 1)$. In contrast, if π_1 and π_2 are "sufficiently small" infinite-dimensional representations of G, the multiplicity in $\pi_1 \otimes \pi_2$ may stay finite, see [8, 9, 10] for precise formulation. In particular, one has:

Fact 2.3 ([10])**.** *For any 1-connected non-compact simple Lie group* G*, there always exist infinite-dimensional irreducible representations* π_1*,* π_2 *of* G *such that* $\pi_1 \otimes \pi_2$ *is of uniformly bounded multiplicity:*

$$\sup_{\tau \in \mathrm{Irr}(G)} [\pi_1 \otimes \pi_2 : \tau] < \infty. \tag{2.2}$$

2.3. Generalities: Discrete decomposability of restriction. Another important requirement in defining the "generating operator" of SBOs is the discrete decomposability of the restriction. Applying the general criterion [4, 5, 6] to the tensor product case, one has

Fact 2.4. *Let* π_1*,* π_2 *be two infinite-dimensional irreducible representations of a simple Lie group* G*.*

(1) ([12, Thm. 6.1]) *The following two conditions on the triple* (G, π_1, π_2) *are equivalent:*
   (i) $\pi_1 \otimes \pi_2$ *is discretely decomposable.*
   (ii) G/K *is a Hermitian symmetric space and* π_1*,* π_2 *are simultaneously highest (or lowest) weight modules.*
(2) ([8]) *If one of (therefore both of) these conditions is satisfied, then the uniformly bounded multiplicity property* (2.2) *holds.*

The representations π_1 and π_2 in Fact 2.4 can be realized in the holomorphic category, see *e.g.,* [8], for which structural results of SBOs are investigated in [13] such as the *localness theorem* and the *extension theorem.*

In [15], the generating operators of SBOs are explored in a special case of the general framework of discretely decomposable restrictions with bounded multiplicities, as discussed in Fact 2.2 and Fact 2.4.

3 From Discrete data to continuous data

This section illustrates by an SL_2 example how the generating operators transfer **discrete data** into **continuous data**. The diagram

$$\{R_\ell\}_{\ell\in\mathbb{N}} \dashrightarrow T \dashrightarrow T^{\pm}_{\mu}, \mathcal{P}^{\pm}_{\lambda}, \mathcal{F}^{\pm}_{\lambda}$$

indicates that the closed formula (3.3) of the *generating operator* T of the Rankin–Cohen brackets $\{R_\ell\}_{\ell\in\mathbb{N}}$ is a key to reproduce explicit formulæ of various families of non-local intertwining operators with continuous parameter such as

- symmetry breaking operators $T^{\pm}_{\mu}$ for the fusion rule of the Hardy spaces (or invariant trilinear forms) (Proposition 3.2);
- Poisson transforms $\mathcal{P}^{\pm}_{\lambda}$ for the de Sitter space (Proposition 4.4);
- Fourier transforms $\mathcal{F}^{\pm}_{\lambda}$ on the de Sitter space (Proposition 4.5).

We note that these intertwining operators are already known, *e.g.,* in [17, 18] in a more general setting by other approaches. The novelty here is that the distribution kernels of these non-local operators are explicitly reconstructed from a countable family of differential operators on a different geometry through the *generating operator* T. It should be noted that this is *opposite* to the usual direction such as taking the residues of the meromorphic family of non-local operators.

In this article, we focus on this new trick, and omit the proof of some standard statements such as the meromorphic continuation or the covariance property, which can be proven by existing techniques, *e.g.,* [1, 17, 18]. Our approach is formulated by viewing elements in principal series representations as local cohomologies, or "boundary values" of holomorphic functions in the spirit of Sato's hyperfunctions [21].

3.1. Preliminaries: Principal series representations of G.

We fix some notation for representations of $G = SL(2,\mathbb{R})$. Take a minimal parabolic subgroup P to be the set of lower triangular matrices, and define characters χ^+_λ and χ^-_λ of P, respectively, by

$$\chi^+_\lambda \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} := |a|^{-\lambda}, \quad \chi^-_\lambda \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} := |a|^{-\lambda} \operatorname{sgn} a.$$

Let $\mathcal{L}_\lambda \equiv \mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$ be the homogeneous line bundles over G/P associated to the characters χ_λ^+ and χ_λ^-, respectively. The natural action of G on $C^\infty(G/P, \mathcal{L}_\lambda^\pm)$ defines the principal series representations. By using the Bruhat decomposition, they are expressed as the multiplier representations: for $\varepsilon \in \{+,-\} \equiv \{1,-1\}$,

$$(\varpi_\lambda^\varepsilon(g)f)(x) = |cx+d|^{-\lambda}\operatorname{sgn}(cx+d)^{\frac{1-\varepsilon}{2}} f(\frac{ax+b}{cx+d}) \tag{3.1}$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

3.2. Generating operator for Rankin–Cohen brackets.

Let $Q(\zeta_1, \zeta_2; z, t)$ be a holomorphic function of four variables given by

$$Q(\zeta_1, \zeta_2; z, t) := (\zeta_1 - z)(\zeta_2 - z) + t(\zeta_1 - \zeta_2). \tag{3.2}$$

In [15], we introduced an integral transform $T\colon \mathcal{O}(\mathbb{C}^2) \to \mathcal{O}(\mathbb{C}^2)$ by

$$(Tf)(z,t) := \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_1} \oint_{C_2} \frac{f(\zeta_1,\zeta_2)}{Q(\zeta_1,\zeta_2;z,t)} d\zeta_1 d\zeta_2, \tag{3.3}$$

where C_j are sufficiently small contours around the point z ($j = 1, 2$). It is proven in [15, Thm. 2.3] that T is the "generating operator" of the family of the Rankin–Cohen brackets $\{R_\ell\}_{\ell \in \mathbb{N}}$, see [19], namely,

$$(Tf)(z,t) = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} R_\ell f(z) \quad \text{for any } f \in \mathcal{O}(\mathbb{C}^2), \tag{3.4}$$

where $R_\ell\colon \mathcal{O}(\mathbb{C}^2) \to \mathcal{O}(\mathbb{C})$, $f(\zeta_1, \zeta_2) \mapsto (R_\ell f)(z)$ is defined by

$$(R_\ell f)(z) := \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j}^2 \left.\frac{\partial^\ell f(\zeta_1,\zeta_2)}{\partial \zeta_1^{\ell-j} \partial \zeta_2^j}\right|_{\zeta_1=\zeta_2=z} \quad \text{for } \ell \in \mathbb{N}. \tag{3.5}$$

The operators $\{R_\ell\}_{\ell \in \mathbb{N}}$ are differential SBO for the fusion rule of the two Hardy spaces, see [2, 15, 19] for instance.

In this case, the formal power series (3.4) converges uniformly on any compact set in $\mathbb{C}^2$. Conversely, every operator R_ℓ ($\ell \in \mathbb{N}$) is recovered readily from the generating operator T by

$$R_\ell = \left.\left(\frac{\partial}{\partial t}\right)^\ell\right|_{t=0} \circ T. \tag{3.6}$$

3.3. From "discrete" to "continuous".

This section defines a "meromorphic extension" of the countable family $\{R_\ell\}$ of operators in the spirit of fractional calculus. We construct operators $T_\mu^{\pm}$ that depend meromorphically on μ and the residue operator is equal to R_ℓ up to scalar multiplication for every $\ell \in \mathbb{N}$, see (3.11).

We begin by recalling a classical fact that

$$t_+^\mu := \begin{cases} t^\mu & (t > 0) \\ 0 & (t \leq 0), \end{cases} \qquad t_-^\mu := (-t)_+^\mu$$

are locally integrable functions on $\mathbb{R}$ for $\operatorname{Re}\mu > -1$, and extend to tempered distributions which depend meromorphically on $\mu \in \mathbb{C}$. Their poles are all simple and have the following residues:

$$\left(\frac{\partial}{\partial t}\right)^\ell \Bigg|_{t=0} = \frac{(-1)^\ell}{\Gamma(\mu+1)} t_+^\mu \Bigg|_{\mu=-\ell-1} = \ell! \operatorname*{res}_{\mu=-\ell-1} t_+^\mu. \tag{3.7}$$

Let $f(\zeta_1, \zeta_2) \in \mathcal{O}(\mathbb{C}^2)$. Inspired by the fomulæ (3.6) and (3.7), we define a "meromorphic continuation" of $(R_\ell f)(z)$ by setting

$$\begin{aligned} (T_\mu^{\pm} f)(z) :=& \langle t_\pm^\mu, Tf(z,t) \rangle \\ =& \frac{1}{(2\pi\sqrt{-1})^2} \int_{\mathbb{R}} t_\pm^\mu \left(\oint_{C_1} \oint_{C_2} \frac{f(\zeta_1, \zeta_2)}{Q(\zeta_1, \zeta_2; z, t)} d\zeta_1 d\zeta_2 \right) dt. \end{aligned} \tag{3.8}$$

Our integral formula (3.3) of the generating operator T is formulated originally in the holomorphic category. We now interpret principal series representations via local cohomologies of holomorphic functions (*e.g.,* "boundary values" in the one variable case). We proceed by changing the order of the integration in (3.8). We set

$$K_\pm^\mu(\zeta_1, \zeta_2; \zeta) := \left(\frac{(\zeta_1 - \zeta)(\zeta_2 - \zeta)}{\zeta_1 - \zeta_2} \right)_\pm^\mu \tag{3.9}$$

as hyperfunctions depending meromorphically on $\mu \in \mathbb{C}$. By Lemma 6.2 in Appendix, we have:

Lemma 3.1. *One has*

$$\langle t_\pm^\mu, \frac{1}{Q(\zeta_1, \zeta_2; \zeta, t)} \rangle = \frac{-2\pi\sqrt{-1}}{\zeta_1 - \zeta_2} K_\mp^\mu(\zeta_1, \zeta_2; \zeta).$$

In what follows, we use the notation $\mathcal{L} = \mathcal{L}_1^-$ and $\mathcal{L}_{-2\mu} = \mathcal{L}_{-2\mu}^+$ for simplicity. By Lemma 3.1, we have:

Proposition 3.2. *For $f \in C^\infty(G/P \times G/P, \mathcal{L} \boxtimes \mathcal{L})$, $(T_\mu^\pm f)$ takes the following form*

$$(T_\mu^\pm f)(\zeta) = \frac{-1}{2\pi\sqrt{-1}} \int_{\mathbb{R}^2} f(\zeta_1, \zeta_2) K_\mp^\mu(\zeta_1, \zeta_2; \zeta) \frac{d\zeta_1 d\zeta_2}{\zeta_1 - \zeta_2}, \tag{3.10}$$

and defines a family of symmetry breaking operators

$$T_\mu^\pm \colon C^\infty(G/P \times G/P, \mathcal{L} \boxtimes \mathcal{L}) \to C^\infty(G/P, \mathcal{L}_{-2\mu})$$

which depend meromorphically on $\mu \in \mathbb{C}$. Moreover, one has

$$\operatorname*{res}_{\mu=-\ell-1} T_\mu^\pm f = \frac{1}{\ell!} R_\ell f. \tag{3.11}$$

We note that R_ℓ extends to $G/P \times G/P$ by the extension theorem on differential SBOs in the general setting, see [13, Thm. B]. The residue formula (3.11) follows directly from (3.7), or alternatively from the lemma below.

Lemma 3.3. *For any $\ell \in \mathbb{N}$ and for any $f \in \mathcal{O}(\mathbb{C}^2)$,*

$$\left.\frac{\partial^{2\ell}}{\partial\zeta_1^\ell \partial\zeta_2^\ell}\right|_{\zeta_1=\zeta_2=\zeta} ((\zeta_1 - \zeta_2)^\ell f) = (-1)^\ell \ell! (R_\ell f)(\zeta).$$

Proposition 3.4 (Holographic operator)**.** *As the dual operator of $T_\mu^\pm$ (up to scalar multiplication by $-2\pi\sqrt{-1}$),*

$$H_\mu^\pm \colon \mathcal{D}'(G/P, \mathcal{L}_{2\mu+2}) \to \mathcal{D}'(G/P \times G/P, \mathcal{L} \boxtimes \mathcal{L})$$

gives a family of G-intertwining operators depending meromorphically on $\mu \in \mathbb{C}$. The operators take the following form:

$$(H_\mu^\pm h)(\zeta_1, \zeta_2) = \frac{1}{\zeta_1 - \zeta_2} \int_{\mathbb{R}} h(\zeta) K_\mp^\mu(\zeta_1, \zeta_2; \zeta) d\zeta. \tag{3.12}$$

4 From Rankin–Cohen brackets to Poisson transforms

This section gives yet another example from "**discrete**" to "**continuous**". We shall see that the Rankin–Cohen brackets $\{R_\ell\}_{\ell \in \mathbb{N}}$ yields a pair of the Poisson transforms $\mathcal{P}_\lambda^\pm$ on the de Sitter space dS^2 via the "generating operator" T in (3.3). Our strategy is to restrict the holographic operators $H_\mu^\pm$ in Proposition 3.4, summarized as

$$\{R_\ell\}_{\ell\in\mathbb{N}} \rightsquigarrow T \rightsquigarrow T_\mu^\pm \rightsquigarrow H_\mu^\pm \rightsquigarrow \mathcal{P}_\lambda^\pm.$$

4.1. Bruhat coordinates of dS^2.

The de Sitter space dS^2 is a Lorentzian manifold with constant curvature $+1$, defined as a surface of the Minkowski space $\mathbb{R}^{2,1}$:

$$\mathrm{dS}^2 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2-z^2=1\}.$$

We may realize dS^2 in the matrix form

$$\{A = \begin{pmatrix} x & y+z \\ y-z & -x\end{pmatrix} : \det A = -1\} \subset \mathfrak{sl}(2,\mathbb{R}),$$

on which $G = SL(2,\mathbb{R})$ acts via the adjoint representation. Let

$$I_{1,1} := \begin{pmatrix} 1 & \\ & -1\end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}), \quad H := \{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix} : a\in\mathbb{R}^\times\} \subset G.$$

Then dS^2 is identified with the homogeneous space G/H by

$$G/H \xrightarrow{\sim} \mathrm{dS}^2, \quad gH \mapsto \mathrm{Ad}(g)I_{1,1} = \begin{pmatrix} ad+bc & -2ab \\ 2cd & -(ad+bc)\end{pmatrix}$$

where $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$. In the coordinates, one has

$$(x,y,z) = (ad+bc, -ab+cd, -ab-cd). \tag{4.1}$$

The third realization of dS^2 is given via the G-orbit decomposition

$$G/P \times G/P = \mathrm{dS}^2 \amalg G/P \qquad \text{(disjoint)} \tag{4.2}$$

under the diagonal action of G. Let $w := \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$. Since $P \cap wPw^{-1} = H$, the G-orbit through (eP, wP) is identified with G/H. Combining this with the Bruhat decomposition $G/P = \mathbb{R}\cup\{\infty\}$, one has the diagram below:

$$\begin{array}{ccccccc} \mathrm{dS}^2 & \simeq & G/H & \hookrightarrow & G/P\times G/P & \hookleftarrow & \mathbb{R}^2 \\ \mathrm{Ad}(g)I_{1,1} & \leftarrow\!\shortmid & gH & \mapsto & (gP, gwP) & \leftarrow\!\shortmid & (\zeta_1,\zeta_2). \end{array} \tag{4.3}$$

Then $(x,y,z) \in \mathrm{dS}^2$ has the following coordinates by (4.1) and (4.3):

$$(\zeta_1, \zeta_2) = (-\frac{y+z}{x+1}, \frac{x+1}{y-z}). \tag{4.4}$$

It is convenient to list some elementary formulæ concerning (4.4):

Lemma 4.1. *Retain the setting as above. One has*

$$\zeta_1 - \zeta_2 = \frac{-2}{y-z}, \tag{4.5}$$

$$(\zeta_1 + \sqrt{-1})(\zeta_2 + \sqrt{-1}) = \frac{2\sqrt{-1}(x+\sqrt{-1}y)}{y-z}, \tag{4.6}$$

$$\frac{\zeta_1 - \zeta_2}{(\zeta_1 - \zeta)(\zeta_2 - \zeta)} = \frac{2(1+x)}{((1+x)\zeta + (y+z))((1+x) - (y-z)\zeta)}.$$

The Minkowski metric $ds^2 = dx^2 + dy^2 - dz^2$ on $\mathbb{R}^{2,1}$ induces an invariant measure on the de Sitter space dS^2 as below.

Lemma 4.2. *In the coordinates* $(x,y,z) = (\cosh t \cos\theta, \cosh t \sin\theta, \sinh t)$ *and* (4.4)*, the invariant measure on* dS^2 *takes the following form:*

$$\frac{dxdy}{2z} = \cosh t dt d\theta = \frac{2}{(\zeta_1 - \zeta_2)^2} d\zeta_1 d\zeta_2.$$

4.2. Tensor product of principal series and $C^\infty(G/H)$.

The open embedding (4.2) of the de Sitter space dS^2 in $G/P \times G/P$ connects the tensor product of two principal series representations of the group G with the harmonic analysis on $\mathrm{dS}^2 \simeq G/H$:

Lemma 4.3. *For any* $\lambda \in \mathbb{C}$ *and* $\varepsilon \in \{+,-\}$*, the line bundle* $\mathcal{L}_\lambda^\varepsilon \boxtimes \mathcal{L}_\lambda^\varepsilon$ *becomes trivial as a* G*-equivariant bundle when restricted to the submanifold* G/H*. Accordingly, the pull-back induces a* G*-homomorphism*

$$\iota_\lambda^* \colon C^\infty(G/P \times G/P, \mathcal{L}_\lambda^\varepsilon \boxtimes \mathcal{L}_\lambda^\varepsilon) \hookrightarrow C^\infty(G/H),$$

$$f(\zeta_1,\zeta_2) \mapsto F(x,y,z) = (\frac{2}{z-y})^\lambda f(-\frac{y+z}{x+1}, \frac{x+1}{y-z}) = (\zeta_1 - \zeta_2)^\lambda f(\zeta_1, \zeta_2).$$

4.3. Poisson transforms on dS^2.

We define the Poisson transforms as the composition $\mathcal{P}_\lambda^\pm := \iota_1^* \circ H^\pm_{\frac{\lambda}{2}-1}$. By Lemma 4.3 and by (3.9), $\mathcal{P}_\lambda^\pm$ takes the form

$$(\mathcal{P}_\lambda^\pm h)(x,y,z) = \int_{\mathbb{R}} \mathcal{K}_\mp^{\frac{\lambda}{2}-1}(x,y,z;\zeta) h(\zeta) d\zeta,$$

where $\mathcal{K}^\mu_\pm$ is the pull-back of $K^\mu_\pm$ in (3.9) from $(G/P)^3$ to $G/H \times G/P$, see (4.3) and (4.4). By Lemma 4.1, $\mathcal{K}^\mu_\pm$ amounts to

$$\mathcal{K}^\mu_\pm(x,y,z;\zeta) = \left(\frac{((1+x)\zeta + (y+z))((1+x) - (y-z)\zeta)}{2(1+x)}\right)^\mu_\pm.$$

Let Δ be the Laplacian on dS^2 with respect to the Lorentzian metric induced from the Minkowski space $\mathbb{R}^{2,1}$. For $\Gamma = C^\infty$, L^2, ..., we set

$$\mathcal{F}(G/H, \mathcal{M}_\lambda) := \{f \in \Gamma(G/H) : \Delta f = -\frac{1}{4}\lambda(\lambda-2)f\}. \tag{4.7}$$

Proposition 4.4 (Poisson transform)**.** *The transform*

$$\mathcal{P}^\pm_\lambda \colon C^\infty(G/P, \mathcal{L}_\lambda) \to C^\infty(G/H, \mathcal{M}_\lambda) \subset C^\infty(G/H)$$

define G-intertwining operators that depend meromorphically on $\lambda \in \mathbb{C}$.

We note that there are two Poisson transforms $\mathcal{P}^+_\lambda$ and $\mathcal{P}^-_\lambda$ in our setting because the H-action on G/P has two open orbits, see [18].

4.4. Fourier transform.

The Plancherel formula for $\mathrm{dS}^2 \simeq G/H$ is known, see [3] for instance, which contains both discrete and continuous spectrum:

$$L^2(G/H) \simeq \sum_{\ell=0}^{\infty}{}^{\oplus} (\pi^+_{2\ell+2} \oplus \pi^-_{2\ell+2}) \oplus 2\int^{\oplus}_{(0,\infty)} \varpi_{1+\sqrt{-1}\nu}\, d\nu. \tag{4.8}$$

In the right-hand side $\pi^+_{2\ell+2}$ is the holomorphic discrete series representation with minimal K-type $\chi_{2\ell+2}$, and $\pi^-_{2\ell+2}$ is its contragredient representation. By an abuse of notation, we write $\varpi_{1+\sqrt{-1}\nu}$ for the spherical unitary principal series representation of G, obtained as the unitarization of $\varpi^+_{1+\sqrt{-1}\nu}$ of G, see (3.1).

In this section, we discuss how the generating operator of the Rankin–Cohen brackets (**discrete data**) is connected with the **continuous spectrum** in the Plancherel formula (4.8) of dS^2.

As the dual of $\mathcal{P}^\pm_{2-\lambda}$, we define the Fourier transform by

$$\mathcal{F}^\pm_\lambda \colon C^\infty_c(G/H) \to C^\infty(G/P, \mathcal{L}_\lambda).$$

Proposition 4.5 (Fourier transform)**.** *$\mathcal{F}^\pm_\lambda$ takes the form*

$$(\mathcal{F}^\pm_\lambda h)(\zeta) = \int_{G/H} \mathcal{K}^{-\frac{\lambda}{2}}_\mp(x,y,z;\zeta) h(x,y,z)\, d\mu_{G/H}, \tag{4.9}$$

and one has $\mathcal{F}_\lambda^\pm \circ \iota_1^* = T^\pm_{-\frac{1}{2}\lambda}$ *(up to non-zero scalar multiple).*

In summary, a countable set of differential SBOs (the Rankin–Cohen brackets $\{R_\ell\}_{\ell\in\mathbb{N}}$) led us to the non-local operators $\mathcal{F}_\lambda^\pm$ (Fourier transforms) in the framework "from discrete to continuous" via the "generating operator" T. The parameters $\lambda \in 1+\sqrt{-1}\mathbb{R}$ contribute to the continuous part of the Plancherel theorem (4.8).

In Section 5.2, we shall see that the Rankin–Cohen brackets again show up in dealing with the discrete part of (4.8).

5 Embedding of discrete series into principal series

Casselman's embedding theorem, see *e.g.,* [22] tells us that every irreducible admissible representation of a real reductive group can be realized as a subrepresentation of some principal series representation. However, this abstract theorem does not provide an explicit intertwining operator from a geometric model of the irreducible representation into a principal series representation.

In this section, we prove that the Rankin–Cohen brackets give geometric embeddings of discrete series representations of the de Sitter space dS^2 into principal series representations. Since the Rankin–Cohen brackets R_ℓ involve the restriction to the diagonal submanifold G/P, R_ℓ is not well defined initially for functions on dS^2 because $G/P \cap \mathrm{dS}^2 = \emptyset$, see (4.2). The key ingredients of the proof are

- $L^2(\mathrm{dS}^2) \simeq \widehat{\pi} \otimes \widehat{\pi}$, see (5.1) below,
- the theory of admissible restrictions [6], and
- the extension theorem of differential SBOs [13].

5.1. Analytic extension from dS^2 to $G/P \times G/P$.

We recall from [3] (*cf.* (4.8)) that the space of L^2-eigenfunctions of the Laplacian splits into the sum of two irreducible representations of G:

$$L^2(G/H, \mathcal{M}_{2\ell+2}) \simeq \pi^+_{2\ell+2} \oplus \pi^-_{2\ell+2} \quad \text{for } \ell \in \mathbb{N}.$$

Let π denote the unitary principal series representation on the Hilbert space $L^2(G/P, \mathcal{L})$ where $\mathcal{L} = \mathcal{L}_1^-$. The pull-back ι_λ^* in Lemma 4.3 with $(\lambda, \varepsilon) = (1, -)$,

$$f(\zeta_1,\zeta_2) \mapsto F(x,y,z) = \frac{2}{z-y} f(-\frac{y+z}{x+1}, -\frac{x+1}{z-y}) = (\zeta_1-\zeta_2) f(\zeta_1,\zeta_2)$$

induces a unitary equivalence up to scaling:

$$\iota_1^*\colon L^2(G/P,\mathcal{L})\widehat{\otimes}L^2(G/P,\mathcal{L}) \xrightarrow{\sim} L^2(G/H) \tag{5.1}$$

because G/H is conull in $G/P \times G/P$.

Thus the fusion rule of the left-hand side (*cf.* Repka [20]) is equivalent to the Plancherel formula of dS^2 given in (4.8).

The following theorem is a key to the proof of Theorem 5.3 for an explicit embedding of discrete series representations. We note that an analogous extension statement is not true if we drop the square integrability assumption of eigenfunctions in Theorem 5.1.

Theorem 5.1. *Any K-finite function of the discrete series for G/H extends to a real analytic section for $\mathcal{L}\boxtimes\mathcal{L}$ over $G/P\times G/P$ via* (5.1).

Proof. By (5.1), the Plancherel formula for G/H may be interpreted as the fusion rule of $\pi\widehat{\otimes}\pi$. Let $\mathbb{H}(\Pi_+)$ and $\mathbb{H}(\Pi_-)$ denote the Hardy space for the upper half plane Π_+ and the lower one Π_-, respectively. Then one has a unitary equivalence $\pi \simeq \mathbb{H}(\Pi_+)\oplus\mathbb{H}(\Pi_-)$, and the discrete part and the continuous part in (4.8) are explained as

$$\begin{aligned} \mathbb{H}(\Pi_\varepsilon)\,\widehat{\otimes}\,\mathbb{H}(\Pi_\varepsilon) \ &\simeq \sum_{\ell=0}^{\infty}{}^{\oplus}\pi^{\varepsilon}_{2\ell+2} \qquad \varepsilon = + \text{ or } -, \\ \mathbb{H}(\Pi_+)\,\widehat{\otimes}\,\mathbb{H}(\Pi_-) \ &\simeq \int_{(0,\infty)}^{\oplus}\varpi_{1+\sqrt{-1}\nu}d\nu. \end{aligned} \tag{5.2}$$

The tensor product $\mathbb{H}(\Pi_+)\,\widehat{\otimes}\,\mathbb{H}(\Pi_-)$ is unitarily isomorphic to $L^2(G/K)$, and does not contain discrete spectrum in the fusion rule. On the other hand, any discrete series for dS^2 arises from the K-admissible tensor product $\mathbb{H}(\Pi_\varepsilon)\,\widehat{\otimes}\,\mathbb{H}(\Pi_\varepsilon)$ ([5]), hence any K-finite vector f becomes $(K\times K)$-finite by [6]. Since the direct product group $K\times K$ acts transitively on $G/P\times G/P$, the function $f\in L^2(G/H)$ extends to a real analytic section $\widetilde{f}$ over $G/P\times G/P$ via (5.1). $\square$

Example 5.2. Let f_ℓ be a function on $\mathrm{dS}^2\simeq G/H$ given by

$$f_\ell^{\pm}(x,y,z) := \left(\frac{\sqrt{-1}}{x\pm\sqrt{-1}y}\right)^{\ell+1}.$$

Then it belongs to a K-finite function in $L^2(G/H, \mathcal{M}_{2\ell+2})$, giving a minimal K-type in $\pi^{\pm}_{2\ell+2}$, and extends to an analytic section

$$\widetilde{f^{\pm}_{\ell}}(\zeta_1, \zeta_2) = (\zeta_1 - \zeta_2)^{\ell}(\zeta_1 \pm \sqrt{-1})^{-\ell-1}(\zeta_2 \pm \sqrt{-1})^{-\ell-1},$$

for the line bundle $\mathcal{L} \boxtimes \mathcal{L}$ over $G/P \times G/P$, by (4.5) and (4.6).

As shown in [14, Prop. 2.28], $\widetilde{f^{+}_{\ell}}$ gives a minimal K-type of $\pi^{+}_{2\ell+2}$ in the decomposition (5.2). Likewise for $\widetilde{f^{-}_{\ell}}$ in $\pi^{-}_{2\ell+2}$.

5.2. Embedding of discrete series for dS^2.

We recall from (4.2) that dS^2 is realized as an open dense subset of $G/P \times G/P$, with the boundary being isomorphic to $\mathrm{diag}(G/P)$.

Theorem 5.3 (embedding of discrete series)**.** *The Rankin–Cohen brackets R_{ℓ} induces an injective $(\mathfrak{g}, K)$-homomorphism from discrete series representations $\pi^{+}_{2\ell+2}$ and $\pi^{-}_{2\ell+2}$ for the de Sitter space dS^2 into the principal series representation $C^{\infty}(G/P, \mathcal{L}_{2\ell+2})$ for every $\ell \in \mathbb{N}$.*

Proof. Any K-finite function f in $L^2(G/H, \mathcal{M}_{2\ell+2})$ extends to a real analytic section $\widetilde{f}$ for the line bundle $\mathcal{L} \boxtimes \mathcal{L} \to G/P \times G/P$ by Theorem 5.1. Therefore $f \mapsto R_{\ell}\widetilde{f}$ is a well-defined $(\mathfrak{g}, K)$-homomorphism from $L^2(G/H, \mathcal{M}_{2\ell+2})_K$ to $C^{\infty}(G/P, \mathcal{L}_{2\ell+2})_K$.

Finally, let us prove that this map is injective. Since $L^2(G/H, \mathcal{M}_{2\ell+2})$ splits into irreducible representations $\pi^{+}_{2\ell+2}$ and $\pi^{-}_{2\ell+2}$, it suffices to show

$$R_{\ell}\widetilde{f^{+}_{\ell}} \neq 0, \quad R_{\ell}\widetilde{f^{-}_{\ell}} \neq 0, \tag{5.3}$$

where $f^{\pm}_{\ell}$ are defined in Example 5.2. Then the assertion (5.3) holds because

$$(R_{\ell}\widetilde{f^{+}_{\ell}})(\zeta) = \frac{(2\ell)!}{\ell!}(\zeta + \sqrt{-1})^{-2\ell-2} \neq 0,$$

see [15, Ex. 3.9], and likewise for $R_{\ell}\widetilde{f^{-}_{\ell}}$. $\square$

6 Appendix: Hyperfunctions and the Riemann–Liouville integral

Our key idea from "discrete" to "continuous" in Section 3 is to use the fractional power of normal derivative (3.6). In order to implement the classical idea of the Riemann–Liouville integral into the "generating operators", we utilize the theory of hyperfunctions.

Lemma 6.1. *The following formulæ hold as a meromorphic continuation of $\lambda \in \mathbb{C}$ and an analytic continuation of $w \in \mathbb{C}$:*

$$\langle t_+^\lambda, \frac{1}{t+w}\rangle = -\frac{\pi w^\lambda}{\sin \pi\lambda} \quad \textit{if } w \notin (-\infty, 0],$$
$$\langle t_-^\lambda, \frac{1}{t+w}\rangle = \frac{\pi (-w)^\lambda}{\sin \pi\lambda} \quad \textit{if } w \notin [0, \infty).$$

Proof. Suppose $-1 < \operatorname{Re}\lambda < 0$. Then the following integral converges to the Beta function:

$$\int_0^\infty \frac{t^\lambda}{t+1} dt = B(\lambda+1, -\lambda) = \frac{-\pi}{\sin \pi\lambda}.$$

Suppose $w \in \mathbb{C}$ with $\operatorname{Re} w > 0$. Then the change of variables yields

$$\int_0^\infty \frac{t^\lambda}{t+w} dt = \int_\gamma \frac{(sw)^\lambda}{s+1} ds,$$

where the path γ is given by $\{\frac{t}{w} : 0 \le t < \infty\}$. By the Cauchy integral formula, one sees readily that the integral does not change if we replace the path γ with $[0, \infty)$. Hence the first equality holds initially defined as the convergent integral for $-1 < \operatorname{Re}\lambda < 0$ and $\operatorname{Re} w > 0$, and extends meromorphically in $w \in \mathbb{C} \setminus (-\infty, 0]$ and $\lambda \in \mathbb{C}$.

The proof of the second statement is similar. □

The sheaf $\mathcal{B}$ of hyperfunctions is defined as local cohomology group. In one dimensional case, for an open set U in $\mathbb{R}$, $\mathcal{B}(U) \simeq \mathcal{O}(\widetilde{U} \setminus U)/\mathcal{O}(\widetilde{U})$ where $\widetilde{U}$ is any open set in $\mathbb{C}$ containing U, and this definition does not depend on the choice of $\widetilde{U}$ [21].

Then $w^\lambda \in \mathcal{O}(\mathbb{C} \setminus (-\infty, 0])$ defines a hyperfunction

$$(e^{\sqrt{-1}\pi\lambda} - e^{-\sqrt{-1}\pi\lambda}) w_-^\lambda = 2\sqrt{-1} \sin \pi\lambda w_-^\lambda$$

as a "boundary value" [21], and $(-w)^\lambda = (e^{-\sqrt{-1}\pi} w)^\lambda \in \mathcal{O}(\mathbb{C} \setminus [0, \infty))$ defines

$$(e^{-\sqrt{-1}\pi\lambda} - e^{\sqrt{-1}\pi\lambda}) w_+^\lambda = -2\sqrt{-1} \sin \pi\lambda w_+^\lambda.$$

Hence Lemma 6.1 may be reinterpreted as below.

Lemma 6.2. *As hyperfunctions that depend meromorphically on $\lambda \in \mathbb{C}$, one has the following equations.*

$$\langle t_\pm^\lambda, \frac{1}{t+w}\rangle = -2\pi\sqrt{-1} w_\mp^\lambda.$$

Acknowledgement.
The article is an outgrowth of the invited address at the Nordic Congress of Mathematicians held on July, 2023 in Denmark. The author expresses his gratitude to J. Frahm, M. Pevzner, and G. Zhang for their warm hospitality. He also thanks Professor Kubo for reading carefully the first draft, and anonymous referees for their comments. This work was partially supported by the JSPS under the Grant-in Aid for Scientific Research (A) (JP18H03669, JP23H00084).

Data availability.
Data sharing not applicable to this work as no datasets were generated or analysed during the current study.

Statement on conflict if interest.
On behalf of all authors, the corresponding author states that there is no conflict of interest.

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