CLASSIFICATION OF SYMMETRIC PAIRS WITH DISCRETELY DECOMPOSABLE RESTRICTIONS OF $(\mathfrak{g}, K)$-MODULES

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Abstract. We give a complete classification of reductive symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with the following property: there exists at least one infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X that is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module.

We investigate further if such X can be taken to be a minimal representation, a Zuckerman derived functor module $A_{\mathfrak{q}}(\lambda)$, or some other unitarizable $(\mathfrak{g}, K)$-module. The tensor product $\pi_1 \otimes \pi_2$ of two infinite-dimensional irreducible $(\mathfrak{g}, K)$-modules arises as a very special case of our setting. In this case, we prove that $\pi_1 \otimes \pi_2$ is discretely decomposable if and only if they are simultaneously highest weight modules.

1. Introduction

The subject of this article is discretely decomposable restrictions of irreducible representations with respect to symmetric pairs.

In order to explain our motivation, we begin by confining ourselves to unitary representations. Let π be an irreducible unitary representation of a Lie group G, and H a subgroup in G. We may think of π as a representation of the subgroup H, denoted simply by $\pi|_H$. Then the restriction $\pi|_H$ is no longer irreducible in general, but is unitarily equivalent to a direct integral of irreducible unitary representations of H, possibly with continuous spectrum. Branching problems ask how the restriction $\pi|_H$ decomposes.

In the case where (G, H) is a pair of real reductive Lie groups, we can take K and $H \cap K$ to be maximal compact subgroups of G and H, respectively. Then, as an algebraic analogue of branching problems of unitary representations, we may consider how the underlying $(\mathfrak{g}, K)$-module X of π behaves as an $(\mathfrak{h}, H \cap K)$-module in the category of Harish-Chandra modules. We proved in [12] that either (1) occurs or (2) occurs:

(1) X is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module.
(2) $\mathrm{Hom}_{\mathfrak{h}, H\cap K}(Y, X) = 0$ for any irreducible $(\mathfrak{h}, H \cap K)$-module Y.

The case (1) fits well into algebraic approach to branching problems. In this case, the branching laws of the restrictions of π and X coincide in the following sense:

$$\begin{aligned} \pi|_H &\simeq \sum_{\tau \in \widehat{H}}^{\oplus} m_\pi(\tau)\tau && \text{(Hilbert direct sum)}, \\ X|_{(\mathfrak{h}, H\cap K)} &\simeq \bigoplus_{\tau \in \widehat{H}} m_\pi(\tau)\tau_{H\cap K} && \text{(algebraic direct sum)}, \end{aligned}$$

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where $\widehat{H}$ denotes the set of equivalence classes of irreducible unitary representations of H, $\tau_{H\cap K}$ is the underlying $(\mathfrak{h}, H\cap K)$-module of τ, and the multiplicity $m_\pi(\tau)$ is the same in both the analytic and algebraic branching laws [14]. On the other hand, the case (2) occurs if the irreducible decomposition of the restriction $\pi|_H$ involves continuous spectrum.

More generally, the feature of the restriction of irreducible $(\mathfrak{g}, K)$-modules X remains essentially the same as $(\mathfrak{h}, H\cap K)$-modules without assuming unitarizability. Namely, either (1) or (2) occurs for any irreducible $(\mathfrak{g}, K)$-module X even if X does not come from a unitary representation of the group G (we note that 'discrete decomposability' in Definition 2.1 is slightly weaker than 'complete reducibility').

Obviously (1) always holds for the pairs (G, H) with $H = K$ because K is compact, whereas (1) never holds for the pair $(G, H) = (SL(n, \mathbb{C}), SL(n, \mathbb{R}))$ if $\dim X = \infty$ ([12, Theorem 8.1]). Such pairs are examples of so-called *symmetric pairs* (G, G^σ), where G^σ is the fixed point group of an involutive automorphism σ of G. The classification of reductive symmetric pairs was accomplished by M. Berger [1] on the Lie algebra level $(\mathfrak{g}, \mathfrak{g}^\sigma)$.

In this paper we highlight the restriction of representations with respect to symmetric pairs (G, G^σ). The tensor product of two representations can be treated as a special case of this framework. Indeed, the 'group case' $(G' \times G', \operatorname{diag} G')$ is a symmetric pair as $\operatorname{diag} G'$ is the fixed point group of the involution σ given by $\sigma(x, y) = (y, x)$. Thus branching laws with respect to symmetric pairs are thought of as a natural generalization of the irreducible decomposition of the tensor product representations.

We consider the following.

Problem A. *Classify all the reductive symmetric pairs (G, G^σ) for which there exists at least one infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X satisfying the property (1).*

The problem reduces to the following two cases:

- $\mathfrak{g}$ is a simple Lie algebra;
- (G, G^σ) is the 'group case' $(G' \times G', \operatorname{diag} G')$ with $\mathfrak{g}'$ simple.

Our main result of this paper is a complete solution to Problem A on the Lie algebra level. The classification is given in Theorem 5.2 for simple $\mathfrak{g}$, and in Theorem 6.1 for the 'group case'. For simple $\mathfrak{g}$, we shall see that there is quite a rich family of such symmetric pairs (G, G^σ) in addition to the obvious case where $G^\sigma = K$ or where $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of holomorphic type (Definition 5.1). See Table 1. Our list contains even the case where $\mathfrak{g}$ is complex and $\mathfrak{g}^\sigma$ is its real form (Corollary 5.9).

In the course of the proof, we need a finer understanding of the $(\mathfrak{g}, K)$-modules X that are discretely decomposable as $(\mathfrak{g}^\sigma, K^\sigma)$-modules (cf. [10, 11, 12]):

Problem B. *Let $(\mathfrak{g}, \mathfrak{g}^\sigma)$ be a reductive symmetric pair. Which infinite-dimensional irreducible $(\mathfrak{g}, K)$-modules X are discretely decomposable as $(\mathfrak{g}^\sigma, K^\sigma)$-modules?*

Problem B was solved previously when X is any Zuckerman derived functor module $A_\mathfrak{q}(\lambda)$, which is cohomologically induced from a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_\mathbb{C}$: we gave a necessary and sufficient condition for discrete decomposability in [10, 12], and a complete classification of the triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ such that $A_\mathfrak{q}(\lambda)$ is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module in a recent paper [20]. We then observed that there exist symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ for which none of $A_\mathfrak{q}(\lambda)$ is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module except for $\mathfrak{q} = \mathfrak{g}_\mathbb{C}$. Even so, however, some other $(\mathfrak{g}, K)$-modules X might satisfy the property (1). This happens, for example, when $\mathfrak{g}$ is a split real form of $\mathfrak{e}_6^\mathbb{C}$, $\mathfrak{e}_7^\mathbb{C}$, and $\mathfrak{e}_8^\mathbb{C}$. This observation has brought us to focus on Problem B for some other 'small' representations X as well. In particular, we prove an easy-to-check criterion in Theorem 4.14 for discrete decomposability of a minimal representation X. These results serve as a part of the proof of our main results.

One might wonder in Problem B whether or not it is possible to find such a unitarizable $(\mathfrak{g}, K)$-module X if there exists at least one such (possibly, non-unitarizable) X. We shall show in Corollary 5.8 that this is always possible. (Notice that the classification of unitarizable irreducible $(\mathfrak{g}, K)$-modules is a long-standing problem in representation theory. Fortunately, it turns out that previous achievements on this unsolved problem suffice to obtain Corollary 5.8.)

Finally, we prove in Theorem 6.1 that the tensor product of two infinite-dimensional irreducible $(\mathfrak{g}, K)$-modules is discretely decomposable if and only if G/K is a Hermitian symmetric space and these modules are simultaneously highest weight modules or they are simultaneously lowest weight modules. This is in sharp contrast to the solution to Problem A for symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ with $\mathfrak{g}$ simple: in this case there exist quite often a family of irreducible $(\mathfrak{g}, K)$-modules X that are discretely decomposable as $(\mathfrak{g}^\sigma, K^\sigma)$-modules but that are neither highest weight modules nor lowest weight modules.

In unitary representation theory of real reductive groups, it is in general a hard problem to find branching laws. Even the tensor product of two irreducible unitary representations, which is a special case of the restriction to symmetric pairs, seems to be too wide for detailed analysis because of its wild feature like infinite multiplicities in its irreducible decomposition for both discrete and continuous spectrum, and the branching problem has been far from being solved. Our motivation to highlight discretely decomposable restrictions is to single out a very nice framework among general branching problems for unitary representations, in which we could expect a detailed study, and in which even a purely algebraic approach would make sense. See [2, 4, 9, 10, 15, 18, 19, 24] for explicit branching laws in various settings in the discretely decomposable case.

The significance of discretely decomposable branching laws is not limited inside representation theory in the narrow sense. Restrictions of representations arise naturally from geometric settings. In particular, *discretely decomposable* restrictions have found their applications in the following different areas of mathematics in recent years:

- modular varieties — vanishing theorems of cohomologies [17],
- construction of new discrete series for *non-symmetric* spaces [10],
- spectral analysis on *non-Riemannian* locally symmetric spaces [8],
- parabolic geometry — construction of equivariant differential operators [16].

The $(\mathfrak{g}, K)$-modules with the smallest Gelfand–Kirillov dimension (e.g. minimal representations) are especially interesting in the context of discretely decomposable restrictions. As we will formulate precisely in this article, minimal representations are the most likely to be discretely decomposable. Moreover, we can observe from some examples that the branching laws of minimal representations become fairly simple. The Segal–Shale–Weil representation is a minimal representation of the metaplectic group and its restrictions to subgroups have been particularly well-studied in the connection of Howe's dual pair correspondence. The restrictions of the minimal representation of indefinite orthogonal group $O(p, q)$ to symmetric subgroups $O(p', q') \times O(p'', q'')$ were studied in [18]. If one of p', q', p'' or q'' is zero, the minimal representation is discretely decomposable and is a direct sum of Zuckerman derived functor modules. Moreover, the Parseval–Plancherel type theorem explains the unitary structure for the minimal representations in terms of those for Zuckerman derived functor modules. The classification theorem of this article would suggest a further study of both algebraic and analytic nature of branching laws for minimal representations in more general cases.

Here is a brief outline of the article. Loosely speaking, the 'smaller' X is, the more likely the restriction $X|_{(\mathfrak{h},H\cap K)}$ is discretely decomposable. We formulate this feature by using associated varieties of $(\mathfrak{g},K)$-modules. For this purpose, some basic properties of associated varieties are given in Section 3. We review a general necessary condition (Fact 4.3) and a general sufficient condition (Fact 4.4) for the discrete decomposability of restrictions in Section 4. We apply them to the case that X attains the minimum of the Gelfand–Kirillov dimension, and obtain a simple criterion for discrete decomposability in this case (Theorem 4.10). The main theorem (classification) is given in Section 5. Concerning the tensor product of two irreducible representations, Problems A and B are solved completely in Section 6.

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2. Preliminaries

Let G be a connected real reductive Lie group with Lie algebra $\mathfrak{g}$. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, write $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$ for its complexification, $\mathfrak{g}_{\mathbb{C}}^* = \mathfrak{k}_{\mathbb{C}}^* + \mathfrak{p}_{\mathbb{C}}^*$ for the dual space, and K for the connected subgroup of G with Lie algebra $\mathfrak{k}$. We denote by $K_{\mathbb{C}}$ the subgroup of the inner automorphism group $\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ generated by $\exp(\mathrm{ad}(\mathfrak{k}_{\mathbb{C}}))$. Notice K is not necessarily a subgroup of $K_{\mathbb{C}}$, but there is a natural morphism $K \to K_{\mathbb{C}}$. The adjoint group $K_{\mathbb{C}}$ acts canonically on $\mathfrak{p}_{\mathbb{C}}$ and on the dual space $\mathfrak{p}_{\mathbb{C}}^*$. We take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and choose a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Let B_K be the Borel subgroup of $K_{\mathbb{C}}$ corresponding to the positive roots.

Let $\mathcal{N}(\mathfrak{g}_{\mathbb{C}}^*)$ be the nilpotent variety of the dual space $\mathfrak{g}_{\mathbb{C}}^*$, and set $\mathcal{N}(\mathfrak{p}_{\mathbb{C}}^*) := \mathcal{N}(\mathfrak{g}_{\mathbb{C}}^*) \cap \mathfrak{p}_{\mathbb{C}}^*$. By Kostant–Rallis [21], there are only finitely many $K_{\mathbb{C}}$-orbits in $\mathcal{N}(\mathfrak{p}_{\mathbb{C}}^*)$ and each orbit is stable under multiplication by $\mathbb{C}^\times$. Write the orbit decomposition as $\mathcal{N}(\mathfrak{p}_{\mathbb{C}}^*) = \{0\} \sqcup \mathbb{O}_1 \sqcup \cdots \sqcup \mathbb{O}_n$. We say that $\mathbb{O}_i$ is *minimal* if it is minimal among $\mathbb{O}_1, \ldots, \mathbb{O}_n$ with respect to the closure relation, or equivalently, if the closure of $\mathbb{O}_i$ is $\mathbb{O}_i \sqcup \{0\}$.

A simple Lie group G or its Lie algebra $\mathfrak{g}$ is said to be of *Hermitian type* if the associated Riemannian symmetric space G/K is a Hermitian symmetric space, or equivalently, if the center $\mathfrak{z}_K$ of $\mathfrak{k}$ is one-dimensional. If G is of Hermitian type, the $K_{\mathbb{C}}$-module $\mathfrak{p}_{\mathbb{C}}$ decomposes into two irreducible $K_{\mathbb{C}}$-modules: $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$, and $\mathfrak{p}_-$ can be identified with the holomorphic tangent space at the base point in G/K. The decomposition $\mathfrak{p}_{\mathbb{C}}^* = \mathfrak{p}_+^* + \mathfrak{p}_-^*$ for the dual space is again an irreducible decomposition as $K_{\mathbb{C}}$-modules. The following notation will be used throughout the paper.

Definition 2.1 (highest non-compact root)**.** Let G be a non-compact connected simple Lie group.

(1) If G is not of Hermitian type, then $K_{\mathbb{C}}$ acts irreducibly on $\mathfrak{p}_{\mathbb{C}}^*$ and we denote by $\beta \in \sqrt{-1}\mathfrak{t}^*$ the highest weight in $\mathfrak{p}_{\mathbb{C}}^*$.
(2) If G is of Hermitian type, we label the $K_{\mathbb{C}}$-irreducible decomposition as $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$ and denote by $\beta \in \sqrt{-1}\mathfrak{t}^*$ the highest weight in $\mathfrak{p}_+^*$.

Since $\mathfrak{p}_{\mathbb{C}}^*$ is isomorphic to $\mathfrak{p}_{\mathbb{C}}$ as a $K_{\mathbb{C}}$-module by the Killing form, $-\beta$ is also a weight in $\mathfrak{p}_{\mathbb{C}}^*$. For $\mathfrak{g}$ of Hermitian type, $\mathfrak{p}_-^*$ is dual to $\mathfrak{p}_+^*$ and therefore $-\beta$ occurs in

$\mathfrak{p}^*_-$. In either case, the weight spaces $\mathfrak{p}^*_\beta$ and $\mathfrak{p}^*_{-\beta}$ in $\mathfrak{p}^*_\mathbb{C}$ are one-dimensional. Notice that β depends on the labeling $\mathfrak{p}_\pm$ in Definition 2.1 (2).

Here is a description of minimal $K_\mathbb{C}$-orbits in $\mathcal{N}(\mathfrak{p}^*_\mathbb{C})$.

Proposition 2.2. *Let G be a non-compact connected simple Lie group.*

(1) *If G is not of Hermitian type, then there is a unique minimal $K_\mathbb{C}$-orbit in $\mathcal{N}(\mathfrak{p}^*_\mathbb{C})$, which is given by $K_\mathbb{C}\cdot(\mathfrak{p}^*_\beta\setminus\{0\})$.*
(2) *If G is of Hermitian type, then there are two minimal $K_\mathbb{C}$-orbits in $\mathcal{N}(\mathfrak{p}^*_\mathbb{C})$, which are given by $K_\mathbb{C}\cdot(\mathfrak{p}^*_\beta\setminus\{0\})$ and $K_\mathbb{C}\cdot(\mathfrak{p}^*_{-\beta}\setminus\{0\})$. They have the same dimension.*

Proof. (1) Suppose that G is not of Hermitian type. Then $\mathfrak{p}^*_\mathbb{C}$ is an irreducible $K_\mathbb{C}$-module with highest weight β. Let Z be a non-zero $K_\mathbb{C}$-stable subset of $\mathcal{N}(\mathfrak{p}^*_\mathbb{C})$. To prove (1), it is enough to show that the closure $\overline{Z}$ of Z contains $\mathfrak{p}^*_\beta$. Take a non-zero element $x\in Z$ and write x as the sum of $\mathfrak{t}$-weight vectors in $\mathfrak{p}^*_\mathbb{C}$: $x=\sum_{\alpha\in\Delta(\mathfrak{p}^*_\mathbb{C},\mathfrak{t}_\mathbb{C})}x_\alpha$. Since any non-zero B_K-submodule of $\mathfrak{p}^*_\mathbb{C}$ contains $\mathfrak{p}^*_\beta$, we may assume that $x_\beta\neq 0$ by replacing x by bx ($b\in B_K$). We take an element $a\in\sqrt{-1}\mathfrak{t}$ that is regular dominant with respect to $\Delta^+(\mathfrak{k}_\mathbb{C},\mathfrak{t}_\mathbb{C})$. Then $\alpha(a)\in\mathbb{R}$ for any $\alpha\in\Delta(\mathfrak{p}^*_\mathbb{C},\mathfrak{t}_\mathbb{C})$ and $\beta(a)>\alpha(a)$ if $\alpha\in\Delta(\mathfrak{p}^*_\mathbb{C},\mathfrak{t}_\mathbb{C})\setminus\{\beta\}$. Define $x(s):=\exp(\mathrm{ad}(sa))(x)\in\mathfrak{p}^*_\mathbb{C}$ for $s\in\mathbb{R}$. Then

$$e^{-s\beta(a)}x(s)=x_\beta+\sum_{\alpha\in\Delta(\mathfrak{p}^*_\mathbb{C},\mathfrak{t}_\mathbb{C})\setminus\{\beta\}}e^{s\alpha(a)-s\beta(a)}x_\alpha$$

and hence

$$\lim_{s\to\infty}e^{-s\beta(a)}x(s)=x_\beta.$$

Since any $K_\mathbb{C}$-stable subset of $\mathcal{N}(\mathfrak{p}^*_\mathbb{C})$ is stable under the multiplication by $\mathbb{C}^\times$, the vector $e^{-s\beta(a)}x(s)$ is contained in Z for all $s\in\mathbb{R}$. As a consequence, $\overline{Z}\ni x_\beta$ and therefore $\overline{Z}\supset\mathfrak{p}^*_\beta$.

(2) Suppose that G is of Hermitian type. Then $\mathfrak{p}^*_+$ is an irreducible $K_\mathbb{C}$-module with highest weight β. Since $\mathfrak{p}^*_-$ is its contragredient representation, it is an irreducible $K_\mathbb{C}$-module with lowest weight $-\beta$. Let Z be a non-zero $K_\mathbb{C}$-stable subset of $\mathcal{N}(\mathfrak{p}^*_\mathbb{C})$. It is enough to show that $\overline{Z}\supset\mathfrak{p}^*_\beta$ or $\overline{Z}\supset\mathfrak{p}^*_{-\beta}$. Take a non-zero element $x=x_++x_-\in Z$ where $x_+\in\mathfrak{p}^*_+$ and $x_-\in\mathfrak{p}^*_-$. We assume that $x_+\neq 0$. Let $\mathfrak{z}_K$ be the center of $\mathfrak{k}$ and take an element $z\in\sqrt{-1}\mathfrak{z}_K$ such that $\mathrm{ad}(z)=1$ on $\mathfrak{p}^*_+$ and $\mathrm{ad}(z)=-1$ on $\mathfrak{p}^*_-$. Then by an argument similar to the case (1) we have

$$\lim_{s\to\infty}e^{-s}\exp(\mathrm{ad}(sz))x=x_+$$

and therefore $\overline{Z}\ni x_+$. By using a similar argument again, we see that the closure of $K_\mathbb{C}\cdot x$ contains x_β and hence $\overline{Z}\supset\mathfrak{p}^*_\beta$. If $x_+=0$, then $x_-\neq 0$ and we can prove similarly that $\overline{Z}\supset\mathfrak{p}^*_{-\beta}$.

We can switch the two orbits $K_\mathbb{C}\cdot(\mathfrak{p}^*_\beta\setminus\{0\})$ and $K_\mathbb{C}\cdot(\mathfrak{p}^*_{-\beta}\setminus\{0\})$ by taking the complex conjugates with respect to the real form $\mathfrak{g}$. In particular they have the same dimension. $\square$

Proposition 2.2 justifies the following notation:

$$\begin{aligned}\mathbb{O}_{\min}:=&K_\mathbb{C}\cdot(\mathfrak{p}^*_\beta\setminus\{0\})\qquad(\mathfrak{g}\text{: not of Hermitian type}),\\ \mathbb{O}_{\min,\pm}:=&K_\mathbb{C}\cdot(\mathfrak{p}^*_{\pm\beta}\setminus\{0\})\qquad(\mathfrak{g}\text{: Hermitian type}).\end{aligned}$$

Their closures in $\mathfrak{p}^*_\mathbb{C}$ are given by

$$\overline{\mathbb{O}_{\min}}=K_\mathbb{C}\cdot\mathfrak{p}^*_\beta,\qquad\overline{\mathbb{O}_{\min,\pm}}=K_\mathbb{C}\cdot\mathfrak{p}^*_{\pm\beta}.$$

They are related to the minimal nilpotent coadjoint orbit in the following way. Suppose that $\mathfrak{g}_\mathbb{C}$ is a complex simple Lie algebra. This is equivalent to assuming that $\mathfrak{g}$ is a simple real Lie algebra without complex structure. Then there exists a unique non-zero minimal nilpotent $(\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}})$-orbit in $\mathfrak{g}_{\mathbb{C}}^*$, which we denote by $\mathbb{O}_{\min,\mathbb{C}}$.

Lemma 2.3. *In the setting above, exactly one of the following cases occurs.*

(1) $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}^* = \emptyset$.
(2) $\mathfrak{g}$ *is not of Hermitian type and* $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}^* = \mathbb{O}_{\min}$.
(3) $\mathfrak{g}$ *is of Hermitian type and* $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}^* = \mathbb{O}_{\min,+} \cup \mathbb{O}_{\min,-}$.

This follows from the fact [21]: for any $(\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}})$-orbit $\mathbb{O}_{\mathbb{C}}$ in the nilpotent variety $\mathcal{N}(\mathfrak{g}_{\mathbb{C}}^*)$, the intersection $\mathbb{O}_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}^*$ is either empty or the union of a finite number of equi-dimensional $K_{\mathbb{C}}$-orbits $\mathbb{O}_1, \ldots, \mathbb{O}_m$, and the dimension of $\mathbb{O}_j$ $(1 \le j \le m)$ is equal to half the dimension of $\mathbb{O}_{\mathbb{C}}$. We note

$$
\begin{aligned}
&\mathbb{O}_{\min,\mathbb{C}} \neq (\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}) \cdot \mathbb{O}_{\min} && \text{in Case (1)},\\
&\mathbb{O}_{\min,\mathbb{C}} = (\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}) \cdot \mathbb{O}_{\min} && \text{in Case (2)},\\
&\mathbb{O}_{\min,\mathbb{C}} = (\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}) \cdot \mathbb{O}_{\min,+} = (\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}) \cdot \mathbb{O}_{\min,-} && \text{in Case (3)}.
\end{aligned}
$$

In Corollary 5.9, we provide six equivalent conditions to Case (1) including a classification of such $\mathfrak{g}$. (Notice that the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g})$ in Lemma 2.3 corresponds to $(\mathfrak{g}, \mathfrak{g}^\sigma)$ in the notation there.)

We define

$$
m(\mathfrak{g}) := \begin{cases} \dim_{\mathbb{C}} \mathbb{O}_{\min} & (\mathfrak{g}\text{: not of Hermitian type}),\\ \dim_{\mathbb{C}} \mathbb{O}_{\min,\pm} & (\mathfrak{g}\text{: of Hermitian type}). \end{cases}
$$

For the reader's convenience, we list explicit values of $m(\mathfrak{g})$. By the Kostant–Sekiguchi correspondence, $m(\mathfrak{g})$ coincides with half the dimension of the (real) minimal nilpotent coadjoint orbit(s) in $\mathfrak{g}^*$. In Case (1) this is given in [22] as follows:

$\mathfrak{g}$	$\mathfrak{su}^*(2n)$	$\mathfrak{so}(n-1,1)$	$\mathfrak{sp}(m,n)$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-26)}$
$m(\mathfrak{g})$	$4n-4$	$n-2$	$2(m+n)-1$	11	16

In Case (2) and Case (3), $m(\mathfrak{g})$ is determined only by the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$: for $\mathfrak{g}$ without complex structure we have $m(\mathfrak{g}) = \frac{1}{2}\dim_{\mathbb{C}} \mathbb{O}_{\min,\mathbb{C}}$. The dimension of the (complex) minimal nilpotent orbit $\dim_{\mathbb{C}} \mathbb{O}_{\min,\mathbb{C}}$ is well-known. Thus we have:

$\mathfrak{g}_{\mathbb{C}}$	A_n	$B_n (n \ge 2)$	C_n	D_n	$\mathfrak{g}_2^{\mathbb{C}}$	$\mathfrak{f}_4^{\mathbb{C}}$	$\mathfrak{e}_6^{\mathbb{C}}$	$\mathfrak{e}_7^{\mathbb{C}}$	$\mathfrak{e}_8^{\mathbb{C}}$
$m(\mathfrak{g})$	n	$2n-2$	n	$2n-3$	3	8	11	17	29

If $\mathfrak{g}$ is a complex Lie algebra, $m(\mathfrak{g})$ is twice the number above (e.g. $m(\mathfrak{e}_8^{\mathbb{C}}) = 58$).

3. Associated Varieties of $\mathfrak{g}$-modules

The associated varieties $\mathcal{V}_{\mathfrak{g}}(X)$ are a coarse approximation of $\mathfrak{g}$-modules X, which we brought in [12] into the study of discretely decomposable restrictions of Harish-Chandra modules. In this paper, we further develop its idea. For this, we collect some important properties of associated varieties that will be used in the later sections.

Let $\{U_j(\mathfrak{g}_{\mathbb{C}})\}_{j\in\mathbb{N}}$ be the standard increasing filtration of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$. Suppose X is a finitely generated $\mathfrak{g}$-module X. A filtration $\{X_i\}_{i\in\mathbb{N}}$ is called a *good filtration* if it satisfies the following conditions.

- $\bigcup_{i\in\mathbb{N}} X_i = X$.
- X_i is finite-dimensional for any $i \in \mathbb{N}$.
- $U_j(\mathfrak{g}_{\mathbb{C}})X_i \subset X_{i+j}$ for any $i, j \in \mathbb{N}$.
- There exists n such that $U_j(\mathfrak{g}_{\mathbb{C}})X_i = X_{i+j}$ for any $i \ge n$ and $j \in \mathbb{N}$.

The graded algebra $\operatorname{gr} U(\mathfrak{g}_{\mathbb{C}}) := \bigoplus_{j\in\mathbb{N}} U_j(\mathfrak{g}_{\mathbb{C}})/U_{j-1}(\mathfrak{g}_{\mathbb{C}})$ is isomorphic to the symmetric algebra $S(\mathfrak{g}_{\mathbb{C}})$ by the Poincaré–Birkhoff–Witt theorem and we regard the graded module $\operatorname{gr} X := \bigoplus_{i\in\mathbb{N}} X_i/X_{i-1}$ as an $S(\mathfrak{g}_{\mathbb{C}})$-module. Define

$$\begin{aligned}\operatorname{Ann}_{S(\mathfrak{g}_{\mathbb{C}})}(\operatorname{gr} X) &:= \{f \in S(\mathfrak{g}_{\mathbb{C}}) : fv = 0 \text{ for any } v \in \operatorname{gr} X\},\\ \mathcal{V}_{\mathfrak{g}}(X) &:= \{x \in \mathfrak{g}_{\mathbb{C}}^* : f(x) = 0 \text{ for any } f \in \operatorname{Ann}_{S(\mathfrak{g}_{\mathbb{C}})}(\operatorname{gr} X)\}.\end{aligned}$$

Then $\mathcal{V}_{\mathfrak{g}}(X)$ does not depend on the choice of good filtration and is called the *associated variety* of X.

The following basic properties on the associated variety are well-known.

Lemma 3.1. *Let X be a finitely generated $\mathfrak{g}$-module.*

(1) *If X is of finite length, then $\mathcal{V}_{\mathfrak{g}}(X) \subset \mathcal{N}(\mathfrak{g}_{\mathbb{C}}^*)$.*
(2) *$\mathcal{V}_{\mathfrak{g}}(X) = 0$ if and only if X is finite-dimensional.*
(3) *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Then $\mathcal{V}_{\mathfrak{g}}(X) \subset \mathfrak{h}^{\perp}$ if $\mathfrak{h}$ acts locally finitely on X, where $\mathfrak{h}^{\perp} := \{x \in \mathfrak{g}_{\mathbb{C}}^* : x|_{\mathfrak{h}} = 0\}$.*

(1) and (3) imply that if X is a $(\mathfrak{g}, K)$-module of finite length, then $\mathcal{V}_{\mathfrak{g}}(X)$ is a $K_{\mathbb{C}}$-stable closed subvariety of $\mathcal{N}(\mathfrak{p}_{\mathbb{C}}^*)$ because $\mathfrak{k}_{\mathbb{C}}^{\perp} = \mathfrak{p}_{\mathbb{C}}^*$.

We may recall that there is another well-known variety in $\mathfrak{g}_{\mathbb{C}}^*$ attached to a $\mathfrak{g}$-module X by using the annihilator ideal of X in $U(\mathfrak{g}_{\mathbb{C}})$. Define the two-sided ideal

$$\operatorname{Ann} X := \{f \in U(\mathfrak{g}_{\mathbb{C}}) : fv = 0 \text{ for any } v \in X\}$$

and view the quotient $U(\mathfrak{g}_{\mathbb{C}})/\operatorname{Ann}(X)$ as a $\mathfrak{g}$-module by the product from left. Then its associated variety $\mathcal{V}_{\mathfrak{g}}(U(\mathfrak{g}_{\mathbb{C}})/\operatorname{Ann} X)$ is an $(\operatorname{Int}\mathfrak{g}_{\mathbb{C}})$-stable closed subvariety of $\mathfrak{g}_{\mathbb{C}}^*$. If X is irreducible, it is known that there is a unique nilpotent $(\operatorname{Int}\mathfrak{g}_{\mathbb{C}})$-orbit $\mathbb{O}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}^*$ such that $\overline{\mathbb{O}_{\mathbb{C}}} = \mathcal{V}_{\mathfrak{g}}(U(\mathfrak{g}_{\mathbb{C}})/\operatorname{Ann} X)$. It should be noted that $\mathcal{V}_{\mathfrak{g}}(X)$ has more information about the original $(\mathfrak{g}, K)$-module X than $\mathcal{V}_{\mathfrak{g}}(U(\mathfrak{g})/\operatorname{Ann} X)$, and we shall use $\mathcal{V}_{\mathfrak{g}}(X)$ for the study of branching problems. A relation between $\mathcal{V}_{\mathfrak{g}}(X)$ and $\mathcal{V}_{\mathfrak{g}}(U(\mathfrak{g}_{\mathbb{C}})/\operatorname{Ann} X)$ for a $(\mathfrak{g}, K)$-module X is summarized as follows:

Fact 3.2 ([27, Theorem 8.4])**.** *Let X be an irreducible $(\mathfrak{g}, K)$-module. Let $\mathbb{O}_{\mathbb{C}}$ be as above. Then we have:*

(1) *$\mathcal{V}_{\mathfrak{g}}(X) \subset \mathcal{V}_{\mathfrak{g}}(U(\mathfrak{g}_{\mathbb{C}})/\operatorname{Ann} X) \cap \mathfrak{p}_{\mathbb{C}}^*$.*
(2) *$\mathbb{O}_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}^*$ is the union of a finite number of $K_{\mathbb{C}}$-orbits $\mathbb{O}_1, \dots, \mathbb{O}_m$, each of which has dimension equal to half the dimension of $\mathbb{O}_{\mathbb{C}}$.*
(3) *Some of $\mathbb{O}_i$ are contained in $\mathcal{V}_{\mathfrak{g}}(X)$ and they are precisely the $K_{\mathbb{C}}$-orbits of maximal dimension in $\mathcal{V}_{\mathfrak{g}}(X)$.*

The *Gelfand–Kirillov dimension* of X, to be denoted by $\operatorname{DIM}(X)$, is defined to be the dimension of $\mathcal{V}_{\mathfrak{g}}(X)$, or equivalently, half the dimension of $\mathcal{V}_{\mathfrak{g}}(U(\mathfrak{g}_{\mathbb{C}})/\operatorname{Ann} X)$. It follows from Proposition 2.2 that any infinite-dimensional $(\mathfrak{g}, K)$-module X satisfies $\operatorname{DIM}(X) \geq m(\mathfrak{g})$. The equality holds if and only if

$$\mathcal{V}_{\mathfrak{g}}(X) = \begin{cases} \overline{\mathbb{O}_{\min}} & (\mathfrak{g}\text{: not of Hermitian type}),\\ \overline{\mathbb{O}_{\min,+}}, \overline{\mathbb{O}_{\min,-}}, \text{ or } \overline{\mathbb{O}_{\min,+}} \cup \overline{\mathbb{O}_{\min,-}} & (\mathfrak{g}\text{: of Hermitian type}).\end{cases}$$

Since $\mathcal{V}_{\mathfrak{g}}(X)$ is $K_{\mathbb{C}}$-stable, the space of regular functions $\mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X))$ on $\mathcal{V}_{\mathfrak{g}}(X)$ can be viewed as a $K_{\mathbb{C}}$-module and hence as a K-module through the natural morphism $K \to K_{\mathbb{C}}$. The following proposition shows that the K-type of a $(\mathfrak{g}, K)$-module X can be approximated by that of $\mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X))$. We write $X \underset{K}{\leq} Y$ for K-modules X and Y if $\dim \operatorname{Hom}_K(\tau, X) \leq \dim \operatorname{Hom}_K(\tau, Y)$ for any irreducible K-module τ.

Proposition 3.3. *Let X be a finitely generated $(\mathfrak{g}, K)$-module. Then there exist finite-dimensional K-modules F and F' such that*

$$X|_K \underset{K}{\leq} \mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X)) \otimes F, \quad \text{and} \quad \mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X)) \underset{K}{\leq} X|_K \otimes F'.$$

Proof. Take a finite-dimensional K-submodule X_0 of X such that $U(\mathfrak{g}_{\mathbb{C}})X_0 = X$. We get a good filtration $\{X_i := U_i(\mathfrak{g}_{\mathbb{C}})X_0\}_{i\in\mathbb{N}}$ of X, where $U_i(\mathfrak{g}_{\mathbb{C}})$ is the standard increasing filtration of $U(\mathfrak{g}_{\mathbb{C}})$. The graded module $\operatorname{gr} X := \bigoplus_{i\in\mathbb{N}} X_i/X_{i-1}$ is a finitely generated $S(\mathfrak{p}_{\mathbb{C}})$-module and is isomorphic to X as a K-module. Let $I := \sqrt{\operatorname{Ann}_{S(\mathfrak{p}_{\mathbb{C}})}(\operatorname{gr} X)}$ be the radical of the annihilator of $\operatorname{gr} X$. Then I is $K_{\mathbb{C}}$-stable and there is an isomorphism

$$\mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X)) \simeq S(\mathfrak{p}_{\mathbb{C}})/I$$

of $S(\mathfrak{p}_{\mathbb{C}})$-modules, which respects the actions of $K_{\mathbb{C}}$.

Put $X'_j := I^j \cdot \operatorname{gr} X$ for $j \geq 0$. Then there exists n such that $X'_n = 0$. If n is the smallest such integer, we get a finite filtration:

$$0 = X'_n \subset X'_{n-1} \subset \cdots \subset X'_0 = \operatorname{gr} X$$

and each successive quotient X'_{j-1}/X'_j is an $(S(\mathfrak{p}_{\mathbb{C}})/I)$-module. Since X'_{j-1}/X'_j is finitely generated as an $(S(\mathfrak{p}_{\mathbb{C}})/I)$-module, we can take a finite-dimensional K-submodule F_j of X'_{j-1}/X'_j such that the map $(S(\mathfrak{p}_{\mathbb{C}})/I) \otimes F_j \to X'_{j-1}/X'_j$ is surjective. Then we have

$$X'_{j-1}/X'_j \underset{K}{\leq} (S(\mathfrak{p}_{\mathbb{C}})/I) \otimes F_j \simeq \mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X)) \otimes F_j$$

and hence

$$X|_K \simeq \bigoplus_{j=1}^{n} X'_{j-1}/X'_j \underset{K}{\leq} \bigoplus_{j=1}^{n} (S(\mathfrak{p}_{\mathbb{C}})/I) \otimes F_j \simeq \mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X)) \otimes \bigoplus_{j=1}^{n} F_j.$$

The first inequality in the lemma follows by setting $F = \bigoplus_{j=1}^{n} F_j$.

Let us prove the opposite estimate. We write $\mathcal{V}_{\mathfrak{g}}(X) = Z_1 \cup \cdots \cup Z_m$ for the irreducible decomposition, and P_i for the defining ideal of Z_i in $S(\mathfrak{p}_{\mathbb{C}})$. For each i, Z_i and P_i are $K_{\mathbb{C}}$-stable because $K_{\mathbb{C}}$ is connected. Since $P_1, \ldots, P_m$ are minimal prime ideals containing $\operatorname{Ann}_{S(\mathfrak{p}_{\mathbb{C}})}(\operatorname{gr} X)$, they are associated primes of the $S(\mathfrak{p}_{\mathbb{C}})$-module $\operatorname{gr} X$ (see [3, Theorem 3.1]). This means that there exists an element $v_i \in \operatorname{gr} X$ such that the kernel of the map $S(\mathfrak{p}_{\mathbb{C}}) \to \operatorname{gr} X$, $f \mapsto f v_i$ equals P_i. Let F_i be a finite-dimensional K-submodule of $\operatorname{gr} X$ that contains v_i. Then we get a map

$$\varphi_i : S(\mathfrak{p}_{\mathbb{C}}) \to \operatorname{Hom}_{\mathbb{C}}(F_i, \operatorname{gr} X), \quad f \mapsto (v \mapsto f v),$$

which respects the actions of K. Let e_{v_i} be the evaluation map

$$\mathrm{e}_{v_i} : \operatorname{Hom}_{\mathbb{C}}(F_i, \operatorname{gr} X) \to \operatorname{gr} X, \quad \alpha \mapsto \alpha(v_i).$$

Then $\operatorname{Ker}(\varphi_i) \subset \operatorname{Ker}(\mathrm{e}_{v_i} \circ \varphi_i) = P_i$. As a consequence,

$$\begin{aligned}\mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X)) &\underset{K}{\leq} \bigoplus_{i=1}^{m} \mathcal{O}(Z_i) \simeq \bigoplus_{i=1}^{m} S(\mathfrak{p}_{\mathbb{C}})/P_i \\ &\underset{K}{\leq} \bigoplus_{i=1}^{m} S(\mathfrak{p}_{\mathbb{C}})/\operatorname{Ker}(\varphi_i) \underset{K}{\leq} \bigoplus_{i=1}^{m} \operatorname{Hom}_{\mathbb{C}}(F_i, \operatorname{gr} X).\end{aligned}$$

By combining these inequalities with the natural isomorphisms of K-modules

$$\bigoplus_{i=1}^{m} \operatorname{Hom}_{\mathbb{C}}(F_i, \operatorname{gr} X) \simeq \bigoplus_{i=1}^{m} \operatorname{gr} X \otimes F_i^* \simeq X|_K \otimes \bigoplus_{i=1}^{m} F_i^*,$$

we obtain the second inequality in the lemma by setting $F' = \bigoplus_{i=1}^{m} F_i^*$. □

An irreducible $\mathfrak{g}$-module X is called a highest weight module if there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbb{C}}$ such that X has a one-dimensional $\mathfrak{b}$-stable subspace. If a simple Lie group G allows an infinite-dimensional irreducible $(\mathfrak{g}, K)$-module which is simultaneously a highest weight module, then the group G must be of Hermitian type and the Borel subalgebra $\mathfrak{b}$ is compatible with the decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$, namely, either $\mathfrak{b} \supset \mathfrak{p}_+$ or $\mathfrak{b} \supset \mathfrak{p}_-$ holds.

Definition 3.4. Suppose that G is a simple Lie group of Hermitian type. An irreducible $(\mathfrak{g}, K)$-module X is called a *highest weight* $(\mathfrak{g}, K)$*-module* (resp. *lowest weight* $(\mathfrak{g}, K)$*-module*) if X has a non-zero vector annihilated by $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$).

The highest weight $(\mathfrak{g}, K)$-modules and the lowest weight $(\mathfrak{g}, K)$-modules are characterized by their associated varieties:

Lemma 3.5. *Suppose that G is a connected simple Lie group of Hermitian type, and X is an irreducible $(\mathfrak{g}, K)$-module. Then X is a highest weight $(\mathfrak{g}, K)$-module if and only if $\mathcal{V}_{\mathfrak{g}}(X) \subset \mathfrak{p}_-^*$. Likewise, X is a lowest weight $(\mathfrak{g}, K)$-module if and only if $\mathcal{V}_{\mathfrak{g}}(X) \subset \mathfrak{p}_+^*$.*

Proof. If X is a highest weight $(\mathfrak{g}, K)$-module, Lemma 3.1 (3) gives $\mathcal{V}_{\mathfrak{g}}(X) \subset \mathfrak{p}_{\mathbb{C}}^* \cap (\mathfrak{p}_+)^{\perp} = \mathfrak{p}_-^*$.

Suppose that $\mathcal{V}_{\mathfrak{g}}(X) \subset \mathfrak{p}_-^*$. Then Proposition 3.3 yields an estimate of the K-type of X:

$$X|_K \underset{K}{\leq} \mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X)) \otimes F \underset{K}{\leq} \mathcal{O}(\mathfrak{p}_-^*) \otimes F, \tag{3.1}$$

where F is a finite-dimensional K-module. Let $\mathfrak{z}_K$ be the center of $\mathfrak{k}$ and choose $z \in \sqrt{-1}\mathfrak{z}_K$ such that $\operatorname{ad}(z) = 1$ on $\mathfrak{p}_+$ and $\operatorname{ad}(z) = -1$ on $\mathfrak{p}_-$. Since $\mathcal{O}(\mathfrak{p}_-^*)$ is isomorphic to the symmetric algebra $S(\mathfrak{p}_-)$, the eigenvalues of z on $\mathcal{O}(\mathfrak{p}_-^*)$ are all negative. By (3.1), the set of eigenvalues of z on X is bounded above. Hence there exists a maximal eigenvalue of z and then $\mathfrak{p}_+$ annihilates the corresponding eigenspace, which implies that X is a highest weight $(\mathfrak{g}, K)$-module.

The proof for lowest weight $(\mathfrak{g}, K)$-modules is similar. $\square$

4. Discrete Decomposability

Let G be a real reductive Lie group and σ an involutive automorphism of G. Then σ induces involutions of the Lie algebra $\mathfrak{g}$, its complexification $\mathfrak{g}_{\mathbb{C}}$, the inner automorphism group $\operatorname{Int}\mathfrak{g}_{\mathbb{C}}$, etc., for which we use the same letter σ. The subgroup $G^\sigma := \{g \in G : \sigma(g) = g\}$ is a reductive Lie group with Lie algebra $\mathfrak{g}^\sigma = \{x \in \mathfrak{g} : \sigma(x) = x\}$, and the pair (G, H) is called a reductive symmetric pair if H is an open subgroup of G^σ. Since the discrete decomposability of the restriction (see Definition 4.1 below) does not depend on (finitely many) connected components of the subgroup, we shall consider the case $H = G^\sigma$ without loss of generality. We can and do take a Cartan involution θ of G that commutes with σ. Then $\theta|_{G^\sigma}$ is a Cartan involution of G^σ. We set $K = G^\theta$ and $K^\sigma = G^\sigma \cap K$.

The notion of discrete decomposability of $\mathfrak{g}$-modules was introduced in [12]. We apply it to the restriction with respect to symmetric pairs, from $(\mathfrak{g}, K)$-modules to $(\mathfrak{g}^\sigma, K^\sigma)$-modules.

Definition 4.1. A $(\mathfrak{g}, K)$-module X is said to be *discretely decomposable as a* $(\mathfrak{g}^\sigma, K^\sigma)$*-module* if there exists an increasing filtration $\{X_i\}_{i\in\mathbb{N}}$ of $(\mathfrak{g}^\sigma, K^\sigma)$-modules such that

- $\bigcup_{i\in\mathbb{N}} X_i = X$ and
- X_i is of finite length as a $(\mathfrak{g}^\sigma, K^\sigma)$-module for any $i \in \mathbb{N}$.

Discrete decomposability is preserved by taking submodules, quotients, and the tensor product with finite-dimensional representations.

Remark 4.2 (see [12, Lemma 1.3])**.** Suppose that X is a unitarizable $(\mathfrak{g}, K)$-module. Then X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module if and only if X is isomorphic to an algebraic direct sum of irreducible $(\mathfrak{g}, K)$-modules.

We will state a necessary and a sufficient condition for the discrete decomposability, which were established in [11], [12].

We write

$$\mathrm{pr} : \mathfrak{g}_{\mathbb{C}}^* \to \mathfrak{g}_{\mathbb{C}}^{\sigma *}$$

for the restriction map.

Fact 4.3 (necessary condition [12, Corollary 3.5])**.** *Let X be a $(\mathfrak{g}, K)$-module of finite length and suppose that X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module. Then $\mathrm{pr}(\mathcal{V}_{\mathfrak{g}}(X)) \subset \mathcal{N}(\mathfrak{g}_{\mathbb{C}}^{\sigma *})$, where $\mathcal{N}(\mathfrak{g}_{\mathbb{C}}^{\sigma *})$ is the nilpotent variety of $\mathfrak{g}_{\mathbb{C}}^{\sigma *}$.*

We take a σ-stable Cartan subalgebra $\mathfrak{t} = \mathfrak{t}^\sigma + \mathfrak{t}^{-\sigma}$ of $\mathfrak{k}$ such that $\mathfrak{t}^{-\sigma}$ is a maximal abelian subalgebra of $\mathfrak{k}^{-\sigma}$. We say a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is $(-\sigma)$-*compatible* if $\{\alpha|_{\mathfrak{t}_{\mathbb{C}}^{-\sigma}} : \alpha \in \Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\} \setminus \{0\}$ is a positive system of the restricted root system $\Sigma(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}^{-\sigma})$. Write B_K for the Borel subgroup of $K_{\mathbb{C}}$ corresponding to $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. If $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is $(-\sigma)$-compatible, then $(K_{\mathbb{C}}^\sigma \cdot B_K)/B_K$ is an open dense subset of the flag variety $K_{\mathbb{C}}/B_K$. In Section 4 and Section 5, we always take a $(-\sigma)$-compatible positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

The asymptotic K-support $\mathrm{AS}_K(X)$ of a K-module X is a closed cone in $\sqrt{-1}\mathfrak{t}^* \setminus \{0\}$, which is defined as the limit cone of the highest weights of irreducible K-modules occurring in X. The asymptotic K-support is preserved by taking the tensor product of X with a finite-dimensional representation. An estimate of the singularity spectrum of a hyperfunction character of X yields a criterion of 'K'-admissibility' of X for a subgroup K' of K. See [11, Theorem 2.8]. When it is applied to the restriction with respect to reductive symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ we have:

Fact 4.4 (sufficient condition [11, Example 2.14])**.** *Let X be a $(\mathfrak{g}, K)$-module of finite length and suppose that $\mathrm{AS}_K(X) \cap \sqrt{-1}(\mathfrak{t}^\sigma)^\perp = \emptyset$. Then X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module.*

Remark 4.5. Let θ be a Cartan involution of G such that $\theta\sigma = \sigma\theta$. Then $\theta\sigma$ becomes another involution of G and the symmetric pair $(\mathfrak{g}, \mathfrak{g}^{\theta\sigma})$ is called the *associated pair* of $(\mathfrak{g}, \mathfrak{g}^\sigma)$. We note that $K^\sigma = K^{\theta\sigma}$. We can prove that a $(\mathfrak{g}, K)$-module X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module if and only if it is discretely decomposable as a $(\mathfrak{g}^{\theta\sigma}, K^{\theta\sigma})$-module, though we do not use this in the paper.

In the rest of this section, we suppose that G is a non-compact connected simple Lie group.

Lemma 4.6. *Let G be a non-compact connected simple Lie group and let β be the highest non-compact root given in Definition 2.1. Then $\mathrm{pr}(K_{\mathbb{C}} \cdot \mathfrak{p}_\beta^*) \subset \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$ if and only if $\sigma\beta \neq -\beta$.*

Proof. Suppose that $\sigma\beta = -\beta$. Take a non-zero vector $x \in \mathfrak{p}_\beta^*$. Then $\sigma(x) \in \mathfrak{p}_{-\beta}^*$, $\overline{x} \in \mathfrak{p}_{-\beta}^*$, and $\overline{\sigma(x)} \in \mathfrak{p}_\beta^*$. Here, $\overline{x}$ denotes the complex conjugate of x with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$. Replacing x by cx ($c \in \mathbb{C}$) if necessary, we may assume that $y := x + \overline{\sigma(x)}$ is non-zero. Since $y \in \mathfrak{p}_\beta^*$ and $\sigma(y) \in \mathfrak{p}_{\sigma\beta}^* = \mathfrak{p}_{-\beta}^*$, the projection $\mathrm{pr}(y) = \frac{1}{2}(y + \sigma(y))$ is non-zero. We have moreover

$$\mathrm{pr}(y) = \frac{1}{2}(y + \sigma(y)) = \frac{1}{2}(x + \overline{x} + \sigma(x) + \overline{\sigma(x)}) \in \mathfrak{p}^*,$$

which is a semisimple element. Therefore, $\mathrm{pr}(y) \notin \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$ and hence $\mathrm{pr}(K_{\mathbb{C}} \cdot \mathfrak{p}_\beta^*) \not\subset \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$.

Conversely, suppose that $\sigma\beta \neq -\beta$. We can choose a vector $a \in \sqrt{-1}\mathfrak{t}$ such that $\beta(a) > 0$ and $\sigma\beta(a) > 0$. This implies that the subspace $\mathfrak{p}_\beta^* + \mathfrak{p}_{\sigma\beta}^*$ of $\mathfrak{p}_{\mathbb{C}}^*$ is contained in the nilradical of some Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$. In particular, $\mathfrak{p}_\beta^* + \mathfrak{p}_{\sigma\beta}^* \subset \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^*)$

and hence $\mathrm{pr}(x) = \frac{1}{2}(x + \sigma(x)) \in \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$ for $x \in \mathfrak{p}_\beta^*$. We regard $\mathfrak{p}_\beta^*$ as a one-dimensional B_K-module and let $K_{\mathbb{C}} \times_{B_K} \mathfrak{p}_\beta^*$ be the $K_{\mathbb{C}}$-equivariant line bundle on the flag variety $K_{\mathbb{C}}/B_K$ with typical fiber $\mathfrak{p}_\beta^*$. Let $\mu : K_{\mathbb{C}} \times_{B_K} \mathfrak{p}_\beta^* \to \mathfrak{p}_{\mathbb{C}}^*$ be the map given by $[(k,x)] \mapsto k(x)$. Then, we have $\mathrm{Image}\,\mu = K_{\mathbb{C}} \cdot \mathfrak{p}_\beta^*$. Let us consider the composition of the maps

$$K_{\mathbb{C}} \times_{B_K} \mathfrak{p}_\beta^* \xrightarrow{\mu} \mathfrak{p}_{\mathbb{C}}^* \xrightarrow{\mathrm{pr}} \mathfrak{p}_{\mathbb{C}}^{\sigma *}.$$

Since $\mathrm{pr}(x) \in \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$ for $x \in \mathfrak{p}_\beta^*$ and the composition $\mathrm{pr} \circ \mu$ is $K_{\mathbb{C}}^\sigma$-equivariant, we have $\mathrm{pr} \circ \mu([(k,x)]) = k \cdot \mathrm{pr}(x) \in \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$ for $k \in K_{\mathbb{C}}^\sigma$ and $x \in \mathfrak{p}_\beta^*$. On the other hand, since we have chosen $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ to be $(-\sigma)$-compatible, $(K_{\mathbb{C}}^\sigma \cdot B_K)/B_K$ is dense in $K_{\mathbb{C}}/B_K$. Hence the subset $\{[(k,x)] : k \in K_{\mathbb{C}}^\sigma, x \in \mathfrak{p}_\beta^*\}$ is dense in $K_{\mathbb{C}} \times_{B_K} \mathfrak{p}_\beta^*$. We therefore have

$$\mathrm{pr}(K_{\mathbb{C}} \cdot \mathfrak{p}_\beta^*) = \mathrm{pr} \circ \mu(K_{\mathbb{C}} \times_{B_K} \mathfrak{p}_\beta^*) \subset \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$$

because $\mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$ is closed in $\mathfrak{p}_{\mathbb{C}}^{\sigma *}$. ☐

Proposition 4.7. *Let X be an infinite-dimensional irreducible $(\mathfrak{g}, K)$-module. If X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module, then $\sigma\beta \neq -\beta$. Here β is the highest non-compact root given in Definition 2.1.*

Proof. The associated variety $\mathcal{V}_{\mathfrak{g}}(X)$ is a non-zero $K_{\mathbb{C}}$-stable closed subset of $\mathfrak{p}_{\mathbb{C}}^*$. By Proposition 2.2, it follows that $\mathcal{V}_{\mathfrak{g}}(X) \supset \overline{\mathbb{O}_{\min}}$ if $\mathfrak{g}$ is not of Hermitian type, and that $\mathcal{V}_{\mathfrak{g}}(X) \supset \overline{\mathbb{O}_{\min,+}}$ or $\mathcal{V}_{\mathfrak{g}}(X) \supset \overline{\mathbb{O}_{\min,-}}$ if $\mathfrak{g}$ is of Hermitian type. In either case, we have $\mathcal{V}_{\mathfrak{g}}(X) \supset K_{\mathbb{C}} \cdot \mathfrak{p}_\beta^*$ or $\mathcal{V}_{\mathfrak{g}}(X) \supset K_{\mathbb{C}} \cdot \mathfrak{p}_{-\beta}^*$. Hence $\mathrm{pr}(K_{\mathbb{C}} \cdot \mathfrak{p}_\beta^*) \subset \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$ or $\mathrm{pr}(K_{\mathbb{C}} \cdot \mathfrak{p}_{-\beta}^*) \subset \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$ by Fact 4.3. For the former case, the claim $\sigma\beta \neq -\beta$ follows from Lemma 4.6. For the latter case, the claim can be proved by using an argument similar to the proof of Lemma 4.6. ☐

The following lemma relates the asymptotic K-support to the associated variety of a $(\mathfrak{g}, K)$-module.

Lemma 4.8. *Let X be a $(\mathfrak{g}, K)$-module of finite length. Let $\mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X))$ be the coordinate ring of the associated variety $\mathcal{V}_{\mathfrak{g}}(X)$, which is endowed with a natural $K_{\mathbb{C}}$-module structure and hence with a K-module structure through the morphism $K \to K_{\mathbb{C}}$. Then we have*

$$\mathrm{AS}_K(X) = \mathrm{AS}_K(\mathcal{O}(\mathcal{V}_{\mathfrak{g}}(X))).$$

Proof. This is immediate from Proposition 3.3. ☐

Lemma 4.9. *Let β be as in Definition 2.1. Suppose X is an irreducible $(\mathfrak{g}, K)$-module whose associated variety $\mathcal{V}_{\mathfrak{g}}(X)$ is equal to $K_{\mathbb{C}} \cdot \mathfrak{p}_\beta^*$. Then*

$$\mathrm{AS}_K(X) = \mathbb{R}_{>0}(-w_0\beta) \equiv \{-sw_0\beta : s > 0\},$$

where w_0 is the longest element of the Weyl group for $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

Proof. Put $Z := K_{\mathbb{C}} \cdot \mathfrak{p}_\beta^*$. Let $S(\mathfrak{p}_{\mathbb{C}})$ be the symmetric algebra of $\mathfrak{p}_{\mathbb{C}}$, which is identified with the space of regular functions on $\mathfrak{p}_{\mathbb{C}}^*$. Write $I \subset S(\mathfrak{p}_{\mathbb{C}})$ for the defining ideal of Z and write $\mathcal{O}(Z)$ for the coordinate ring of Z so that $\mathcal{O}(Z) \simeq S(\mathfrak{p}_{\mathbb{C}})/I$. By Lemma 4.8, it is enough to prove that

$$\mathrm{AS}_K(\mathcal{O}(Z)) = \mathbb{R}_{>0}(-w_0\beta). \tag{4.1}$$

Let $\mu : K_{\mathbb{C}} \times_{B_K} \mathfrak{p}_\beta^* \to \mathfrak{p}_{\mathbb{C}}^*$ be the map as in the proof of Lemma 4.6. Since μ maps onto Z, the pull-back map $\mu^* : \mathcal{O}(Z) \to \mathcal{O}(K_{\mathbb{C}} \times_{B_K} \mathfrak{p}_\beta^*)$ is injective. As a representation of B_K, the contragredient representation of $\mathfrak{p}_\beta^*$ is isomorphic to $\mathbb{C}_{-\beta}$, the character of B_K corresponding to $-\beta \in \mathfrak{t}_{\mathbb{C}}^*$. Therefore the regular functions $\mathcal{O}(K_{\mathbb{C}} \times_{B_K} \mathfrak{p}_\beta^*)$ are identified with the regular sections of the vector bundle $K_{\mathbb{C}} \times_{B_K} S(\mathbb{C}_{-\beta})$ on $K_{\mathbb{C}}/B_K$, where $S(\mathbb{C}_{-\beta})$ is the symmetric tensor of $\mathbb{C}_{-\beta}$. By the Borel–Weil theorem, the space of regular sections of $K_{\mathbb{C}} \times_{B_K} S^n(\mathbb{C}_{-\beta})$ is irreducible as a K-module and has highest weight $-nw_0\beta$. Hence (4.1) follows. □

Theorem 4.10. *Let G be a non-compact connected simple Lie group and suppose that X is an infinite-dimensional irreducible $(\mathfrak{g}, K)$-module having the smallest Gelfand–Kirillov dimension, namely $\mathrm{DIM}(X) = m(\mathfrak{g})$. Then X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module if and only if $\sigma\beta \neq -\beta$, where β is the highest non-compact root given in Definition 2.1.*

Proof. The 'only if' part follows from Proposition 4.7.

Conversely, suppose that $\sigma\beta \neq -\beta$. We then have $\sigma w_0\beta \neq -w_0\beta$, where w_0 is the longest element of the Weyl group for $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Indeed, since $K_{\mathbb{C}} \cdot \mathfrak{p}_\beta = K_{\mathbb{C}} \cdot \mathfrak{p}_{w_0\beta}$, Lemma 4.6 shows $\mathrm{pr}(K_{\mathbb{C}} \cdot \mathfrak{p}_{w_0\beta}) \subset \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$. Then by using an argument similar to Lemma 4.6 we can prove that $\sigma w_0\beta \neq -w_0\beta$. We prove the 'if' part of the theorem in the case where $\mathfrak{g}$ is of Hermitian type and $\mathcal{V}_{\mathfrak{g}}(X) = \overline{\mathbb{O}_{\min,+}} \cup \overline{\mathbb{O}_{\min,-}}$. Then as in the proof of Lemma 4.9, we see that $\mathrm{AS}_K(X) = \mathbb{R}_{>0}(-w_0\beta) \cup \mathbb{R}_{>0}\beta$. Therefore, $\sigma\beta \neq -\beta$ and $\sigma w_0\beta \neq -w_0\beta$ imply that $\mathrm{AS}_K(X) \cap \sqrt{-1}(\mathfrak{t}^\sigma)^\perp = \emptyset$. Hence the theorem in this case follows from Fact 4.4. The proof is similar for other cases. □

For most non-compact simple Lie groups G, there exist $(\mathfrak{g}, K)$-modules satisfying the assumption of Theorem 4.10 (by replacing G with a covering group of G if necessary). However, if G is $SO_0(p, q)$ ($p + q$: odd, $p, q \geq 4$) or its covering group, then no irreducible $(\mathfrak{g}, K)$-module X satisfies $\mathrm{DIM}(X) = m(\mathfrak{g})$ (see [26]).

A typical example of $(\mathfrak{g}, K)$-modules X that satisfy the assumption of Theorem 4.10 is a minimal representation.

Definition 4.11. Suppose that G is a simple Lie group without complex structure. This means that the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is still a simple Lie algebra. An irreducible $(\mathfrak{g}, K)$-module X is said to be a *minimal representation* of G if the annihilator of the $U(\mathfrak{g}_{\mathbb{C}})$-module X is the Joseph ideal of $U(\mathfrak{g}_{\mathbb{C}})$ ([7]).

By the definition of the Joseph ideal, we have:

Proposition 4.12. *Let G be a connected simple Lie group without complex structure. Suppose that X is a minimal representation of G. Then*

$$\mathcal{V}_{\mathfrak{g}}(X) = \begin{cases} \overline{\mathbb{O}_{\min}} & \textit{if } \mathfrak{g} \textit{ is not of Hermitian type,} \\ \overline{\mathbb{O}_{\min,+}}, \overline{\mathbb{O}_{\min,-}}, \textit{ or } \overline{\mathbb{O}_{\min,+}} \cup \overline{\mathbb{O}_{\min,-}} & \textit{if } \mathfrak{g} \textit{ is of Hermitian type.} \end{cases}$$

Proof. Let J be the Joseph ideal of $U(\mathfrak{g}_{\mathbb{C}})$, which implies that $\mathcal{V}_{\mathfrak{g}}(U(\mathfrak{g}_{\mathbb{C}})/J) = \overline{\mathbb{O}_{\min,\mathbb{C}}}$. Here $\mathbb{O}_{\min,\mathbb{C}}$ is the minimal nilpotent $(\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}})$-orbit in $\mathfrak{g}_{\mathbb{C}}^*$. Then the proposition follows from Lemma 2.3 and Fact 3.2. □

Remark 4.13. Actually, we can sharpen Proposition 4.12 slightly: if G is a connected simple Lie group of Hermitian type and X is a minimal representation of G, then $\mathcal{V}_{\mathfrak{g}}(X)$ is either $\overline{\mathbb{O}_{\min,+}}$ or $\overline{\mathbb{O}_{\min,-}}$. This is deduced from the following fact [27]: if $\mathbb{O}$ is a $K_{\mathbb{C}}$-orbit in $\mathcal{N}(\mathfrak{p}_{\mathbb{C}}^*)$ and if $\overline{\mathbb{O}}$ is an irreducible component of $\mathcal{V}_{\mathfrak{g}}(X)$, then at least one of the following two conditions holds:

- $\mathcal{V}_{\mathfrak{g}}(X) = \overline{\mathbb{O}}$,
- $\overline{\mathbb{O}} \setminus \mathbb{O}$ has codimension one in $\overline{\mathbb{O}}$.

As a special case of Theorem 4.10, we obtain a criterion for discrete decomposability of the restriction of minimal representations.

Corollary 4.14. *Let G be a connected simple Lie group without complex structure. Suppose that G has a minimal representation X. Then X is discretely decomposable* *as a* $(\mathfrak{g}^\sigma, K^\sigma)$*-module if and only if* $\sigma\beta \neq -\beta$*. Here* β *is the highest non-compact root given in Definition 2.1.*

Remark 4.15. The converse statement of Proposition 4.12 is not true in general.

(1) Let $G = SL(n,\mathbb{R})$. The Joseph ideal of $U(\mathfrak{g}_\mathbb{C})$ is not defined for $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(n,\mathbb{C})$, but there exists an irreducible $(\mathfrak{g}, K)$-module X isomorphic to the underlying $(\mathfrak{g}, K)$-module of some degenerate principal series representation such that $\mathcal{V}_\mathfrak{g}(X) = \overline{\mathbb{O}_{\min}}$.

(2) Let $G = Sp(m,n)$. Then $\mathbb{O}_{\min,\mathbb{C}}$ does not intersect with $\mathfrak{p}^*_\mathbb{C}$ (see Corollary 5.9). From Fact 3.2, there exists no minimal representation of G. However, there exists an irreducible $(\mathfrak{g}, K)$-module X isomorphic to some $A_\mathfrak{q}(\lambda)$ such that $\mathcal{V}_\mathfrak{g}(X) = \overline{\mathbb{O}_{\min}}$.

(3) If X is a minimal representation, then any infinite-dimensional $(\mathfrak{g}, K)$-module in its coherent family has the same associated variety as X. However, most of them are not a minimal representation because a minimal representation must have a fixed infinitesimal character.

Theorem 4.10 can be applied to these representations as well.

5. Classification

In this section we assume G to be a non-compact connected simple Lie group. Let K be the connected subgroup of G associated to a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The Cartan involution θ is chosen to satisfy $\sigma\theta = \theta\sigma$ and the positive system $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ is chosen to be $(-\sigma)$-compatible if an involutive automorphism σ of G is given.

Definition 5.1. Let $\mathfrak{g}$ be a non-compact real simple Lie algebra and $(\mathfrak{g}, \mathfrak{g}^\sigma)$ a symmetric pair. We say $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of *holomorphic type* if $\mathfrak{g}$ is of Hermitian type and the center $\mathfrak{z}_K$ of $\mathfrak{k}$ is contained in $\mathfrak{g}^\sigma$, or equivalently, σ induces a holomorphic involution on the Hermitian symmetric space G/K.

For example, the symmetric pairs $(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{u}(m, n-m))$ and $(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{sp}(m,\mathbb{R}) \oplus \mathfrak{sp}(n-m,\mathbb{R}))$ are of holomorphic type for any m and n, whereas the symmetric pair $(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{gl}(n,\mathbb{R}))$ is not of holomorphic type.

Here is the main result of this paper:

Theorem 5.2 (classification)**.** *Let* $\mathfrak{g}$ *be a non-compact real simple Lie algebra and* $(\mathfrak{g}, \mathfrak{g}^\sigma)$ *a symmetric pair. The following three conditions on the symmetric pair* $(\mathfrak{g}, \mathfrak{g}^\sigma)$ *are equivalent:*

(i) *There exists an infinite-dimensional irreducible* $(\mathfrak{g}, K)$*-module* X *(by replacing* G *with a covering group of* G *if necessary) such that* X *is discretely decomposable as a* $(\mathfrak{g}^\sigma, K^\sigma)$*-module.*

(ii) $\sigma\beta \neq -\beta$ *(*β *is the highest non-compact root given in Definition 2.1).*

(iii) *The pair* $(\mathfrak{g}, \mathfrak{g}^\sigma)$ *satisfies one of the following.*

  (a) σ *is a Cartan involution, i.e.* $\mathfrak{g}^\sigma = \mathfrak{k}$.

  (b) $(\mathfrak{g}, \mathfrak{g}^\sigma)$ *is of holomorphic type (see* [20, Table 2] *for a classification of symmetric pairs of holomorphic type).*

  (c) *The pair* $(\mathfrak{g}, \mathfrak{g}^\sigma)$ *appears in Table 1 (up to isomorphisms).*

Remark 5.3. In Table 1, a symmetric pair and its associated pair are listed in the same row. For example, we list two symmetric pairs $(\mathfrak{sl}(2n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{C}) \oplus \mathfrak{u}(1))$, $(\mathfrak{sl}(2n,\mathbb{R}), \mathfrak{sp}(n,\mathbb{R}))$ in the first row and one is the associated pair of the other. In the second row, only one symmetric pair $(\mathfrak{su}(2m, 2n), \mathfrak{sp}(m,n))$ is listed. This means that the pair $(\mathfrak{su}(2m, 2n), \mathfrak{sp}(m,n))$ is self-associated.

Remark 5.4. Here is a guidance to the notation used in Table 1.

$\mathfrak{g}$	$\mathfrak{g}^\sigma$	minimal	$A_\mathfrak{q}(\lambda)$
$\mathfrak{sl}(2n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{C})\oplus\mathfrak{u}(1)$, $\mathfrak{sp}(n,\mathbb{R})$		$n=2$
$\mathfrak{su}(2m,2n)$	$\mathfrak{sp}(m,n)$		$\bigcirc$
$\mathfrak{so}(m,n)$	$\mathfrak{u}(\frac{m}{2},\frac{n}{2})$	$(*)$	$\bigcirc$
	$\mathfrak{so}(m,k)\oplus\mathfrak{so}(n-k)$ $(m>1)$	$(*)$	$\bigcirc$
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{C})$	$\bigcirc$	$n=1$
$\mathfrak{sp}(m,n)$	$\mathfrak{sp}(k,l)\oplus\mathfrak{sp}(m-k,n-l)$		$\bigcirc$
$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{C})$, $\mathfrak{su}^*(2n)$		$\bigcirc$
$\mathfrak{so}(n,\mathbb{C})$	$\mathfrak{so}(n-1,\mathbb{C})$, $\mathfrak{so}(n-1,1)$ $(n\geq 5)$		n : even
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(k,\mathbb{C})\oplus\mathfrak{sp}(n-k,\mathbb{C})$, $\mathfrak{sp}(k,n-k)$		
$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(2,1)\oplus\mathfrak{su}(2)$, $\mathfrak{so}(5,4)$	$\bigcirc$	$\bigcirc$
$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(8,1)$		$\bigcirc$
$\mathfrak{e}_{6(6)}$	$\mathfrak{su}^*(6)\oplus\mathfrak{su}(2)$, $\mathfrak{f}_{4(4)}$	$\bigcirc$	
$\mathfrak{e}_{6(2)}$	$\mathfrak{so}(6,4)\oplus\mathfrak{so}(2)$, $\mathfrak{su}(4,2)\oplus\mathfrak{su}(2)$	$\bigcirc$	$\bigcirc$
	$\mathfrak{sp}(3,1)$, $\mathfrak{f}_{4(4)}$	$\bigcirc$	$\bigcirc$
	$\mathfrak{so}^*(10)\oplus\mathfrak{so}(2)$	$\bigcirc$	$\bigcirc$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{f}_{4(-20)}$	$\bigcirc$	$\bigcirc$
$\mathfrak{e}_{7(7)}$	$\mathfrak{so}^*(12)\oplus\mathfrak{su}(2)$, $\mathfrak{e}_{6(2)}\oplus\mathfrak{so}(2)$	$\bigcirc$	
$\mathfrak{e}_{7(-5)}$	$\mathfrak{su}(6,2)$, $\mathfrak{e}_{6(2)}\oplus\mathfrak{so}(2)$	$\bigcirc$	$\bigcirc$
	$\mathfrak{so}(8,4)\oplus\mathfrak{su}(2)$	$\bigcirc$	$\bigcirc$
	$\mathfrak{e}_{6(-14)}\oplus\mathfrak{so}(2)$	$\bigcirc$	$\bigcirc$
$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{7(-5)}\oplus\mathfrak{su}(2)$	$\bigcirc$	
$\mathfrak{e}_{8(-24)}$	$\mathfrak{so}(12,4)$, $\mathfrak{e}_{7(-5)}\oplus\mathfrak{su}(2)$	$\bigcirc$	$\bigcirc$
$\mathfrak{f}_4^\mathbb{C}$	$\mathfrak{so}(9,\mathbb{C})$, $\mathfrak{f}_{4(-20)}$		
$\mathfrak{e}_6^\mathbb{C}$	$\mathfrak{f}_4^\mathbb{C}$, $\mathfrak{e}_{6(-26)}$		

TABLE 1.

(1) The circle $\bigcirc$ below "minimal" means that there exists a minimal representation for some Lie group G with Lie algebra $\mathfrak{g}$. For these pairs in Table 1, a minimal representation X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module by Corollary 4.14, and thus the condition (i) is fulfilled.

The asterisk $(*)$ for $\mathfrak{g}=\mathfrak{so}(m,n)$ reflects the fact that the existence of minimal representations depends on the parameters m and n: there exists a minimal representation for some Lie group G with Lie algebra $\mathfrak{so}(m,n)$ if and only if (m,n) satisfies one of the following.
- $m+n$ is even, $m,n\geq 2$, and $m+n\geq 8$.
- $(m,n)=(3,2l),(2l,3)$ for $l\geq 2$.
- $(m,n)=(2,2l+1),(2l+1,2)$ for $l\geq 1$.

(2) The circle $\bigcirc$ below "$A_\mathfrak{q}(\lambda)$" means that there exists a θ-stable parabolic subalgebra $\mathfrak{q}$ $(\neq \mathfrak{g}_\mathbb{C})$ such that the Zuckerman derived functor modules $A_\mathfrak{q}(\lambda)$ are discretely decomposable as $(\mathfrak{g}^\sigma, K^\sigma)$-modules.

(3) For real exceptional Lie algebras, we follow the notation of [5, Chapter X].

Remark 5.5. We did not intend to make the conditions (a), (b), and (c) in Theorem 5.2 to be exclusive with one another. For example, the pair $(\mathfrak{so}(m,n), \mathfrak{u}(\frac{m}{2}, \frac{n}{2}))$ is of holomorphic type if $m = 2$.

Before giving a proof of Theorem 5.2, we prepare the following:

Lemma 5.6. *Let $\mathfrak{g}$ be a non-compact real simple Lie algebra. Assume that the symmetric pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is not of holomorphic type. Then the following three conditions on σ are equivalent:*

(i) *$\sigma\beta \neq -\beta$.*
(ii) *$-\sigma\beta$ is not dominant with respect to $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$.*
(iii) *The pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ satisfies (a) or (b).*
    (a) *σ is a Cartan involution, i.e. $\mathfrak{g}^\sigma = \mathfrak{k}$.*
    (b) *The pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ appears in Table 1 (up to isomorphisms).*

Proof. (i) $\Leftrightarrow$ (ii) We set

$$V := \begin{cases} \mathfrak{p}_\mathbb{C}^* & \text{if } \mathfrak{g} \text{ is not of Hermitian type,} \\ \mathfrak{p}_+^* & \text{if } \mathfrak{g} \text{ is of Hermitian type.} \end{cases}$$

Then $K_\mathbb{C}$ acts irreducibly on V and β is the highest weight of V. We claim that the set $\Delta(V, \mathfrak{t}_\mathbb{C})$ of weights is preserved by $-\sigma$. In fact, if $\mathfrak{g}$ is not of Hermitian type, $\sigma\mathfrak{p}_\mathbb{C}^* = \mathfrak{p}_\mathbb{C}^*$ and hence $\sigma(\Delta(\mathfrak{p}_\mathbb{C}^*, \mathfrak{t}_\mathbb{C})) = \Delta(\mathfrak{p}_\mathbb{C}^*, \mathfrak{t}_\mathbb{C}) = -\Delta(\mathfrak{p}_\mathbb{C}^*, \mathfrak{t}_\mathbb{C})$. If $\mathfrak{g}$ is of Hermitian type, let $\mathfrak{z}_K$ be the center of $\mathfrak{k}$. Then $\sigma(z) = -z$ for $z \in \sqrt{-1}\mathfrak{z}_K$ because $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is not of holomorphic type. Hence $\sigma\mathfrak{p}_+^* = \mathfrak{p}_-^*$ and $\sigma(\Delta(\mathfrak{p}_+^*, \mathfrak{t}_\mathbb{C})) = \Delta(\mathfrak{p}_-^*, \mathfrak{t}_\mathbb{C}) = -\Delta(\mathfrak{p}_+^*, \mathfrak{t}_\mathbb{C})$. Thus $-\sigma(\Delta(V, \mathfrak{t}_\mathbb{C})) = \Delta(V, \mathfrak{t}_\mathbb{C})$ in either case.

Since β, $-\sigma\beta \in \Delta(V, \mathfrak{t}_\mathbb{C})$ are of the same length, $-\sigma\beta$ is dominant if and only if $-\sigma\beta$ coincides with the highest weight β of the irreducible representation V. Hence the equivalence (i) $\Leftrightarrow$ (ii) is proved.

(ii) $\Leftrightarrow$ (iii) We recall that a classification of symmetric pairs with $-\sigma\beta$ dominant was carried out in [20]. In the case that $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is a symmetric pair not of holomorphic type, the weight $-\sigma\beta$ is dominant if and only if the real form $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$ is split or $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is one of those listed in [20, Appendix B.1]. Consequently, $(\mathfrak{g}, \mathfrak{g}^\sigma)$ satisfies $-\sigma\beta \neq \beta$ if and only if the following two conditions hold:

- $\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}$ is not a split real form of $\mathfrak{k}_\mathbb{C}$;
- $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is not listed in [20, Appendix B.1].

Table 1 is obtained as the complementary subset of these pairs in all the symmetric pairs with $\mathfrak{g}$ simple (not of holomorphic type), for which the classification was established earlier by M. Berger [1]. Hence the equivalence (ii) $\Leftrightarrow$ (iii) is proved. $\square$

We are ready to prove Theorem 5.2.

Proof of Theorem 5.2. (i) $\Rightarrow$ (ii) This is Proposition 4.7.

(ii) $\Leftrightarrow$ (iii) If the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of holomorphic type, then we can take a non-zero element z in the center $\mathfrak{z}_K$ of $\mathfrak{k}$ and we have $\beta(z) \neq 0$. Since σ acts as the identity on $\mathfrak{z}_K$, it follows that $(\sigma\beta)(z) = \beta(\sigma(z)) = \beta(z)$ and hence $-\sigma\beta \neq \beta$.

If the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is not of holomorphic type, our assertion follows from Lemma 5.6.

(iii) $\Rightarrow$ (i) To prove this implication, we have to find a discretely decomposable $(\mathfrak{g}, K)$-module X. If $\mathfrak{g}^\sigma = \mathfrak{k}$, then any irreducible $(\mathfrak{g}, K)$-module is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module. If $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of holomorphic type, then $\mathfrak{g}$ is of Hermitian type and there exist infinite-dimensional highest weight $(\mathfrak{g}, K)$-modules. It is known that any highest weight $(\mathfrak{g}, K)$-module is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module if $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of holomorphic type (see [13, Theorem 7.4]).

Suppose that the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is isomorphic to one of those listed in Table 1. We give three sufficient conditions for (i):

(1) There exists a minimal representation X for some connected covering group of G.
(2) $\mathfrak{g}$ is a complex Lie algebra and $\mathfrak{g} \not\simeq \mathfrak{sl}(n, \mathbb{C})$.
(3) There exists a θ-stable parabolic subalgbra $\mathfrak{q}(\neq \mathfrak{g}_{\mathbb{C}})$ such that the Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$ are discretely decomposable as $(\mathfrak{g}^\sigma, K^\sigma)$-modules.

(1) is satisfied for $\mathfrak{g} = \mathfrak{so}(m, n)$ with a certain condition on m, n (see Remark 5.4), $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$, $\mathfrak{f}_{4(4)}$, $\mathfrak{e}_{6(6)}$, $\mathfrak{e}_{6(2)}$, $\mathfrak{e}_{6(-14)}$, $\mathfrak{e}_{7(7)}$, $\mathfrak{e}_{7(-5)}$, $\mathfrak{e}_{8(8)}$, $\mathfrak{e}_{8(-24)}$ (see [25]). Then by Theorem 4.14, a minimal representation X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module.

If (2) holds, then put $X = U(\mathfrak{g})/J$, where J is the Joseph ideal of $U(\mathfrak{g})$. We can regard X as a $(\mathfrak{g}, K)$-module (sometimes referred to as a Harish-Chandra bimodule) and we have that $\mathcal{V}_{\mathfrak{g}}(X)$ is the closure of the minimal nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}^*$. Hence Theorem 4.10 shows that X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module.

By the classification [20, Table 3 and Table 4], (3) is satisfied for the pairs in Table 1 with $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R})$, $\mathfrak{su}(2m, 2n)$, $\mathfrak{so}(m, n)$, $\mathfrak{sp}(2, \mathbb{R})$, $\mathfrak{sp}(m, n)$, $\mathfrak{sl}(2n, \mathbb{C})$, $\mathfrak{so}(2n, \mathbb{C})$, $\mathfrak{f}_{4(4)}$, $\mathfrak{f}_{4(-20)}$, $\mathfrak{e}_{6(2)}$, $\mathfrak{e}_{6(-14)}$, $\mathfrak{e}_{7(-5)}$, $\mathfrak{e}_{8(-24)}$.

The only remaining pairs that are not covered by (1), (2) and (3) are $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{u}(1))$ and $(\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sp}(n, \mathbb{R}))$. In this case, let $G = SL(2n, \mathbb{R})$ and P a maximal parabolic subgroup of G with Levi part $L = S(GL(2n-1, \mathbb{R}) \times GL(1, \mathbb{R}))$. Let X be the underlying $(\mathfrak{g}, K)$-module of a degenerate principal series representation of G induced from a character of P. Then it turns out that $\mathrm{AS}_K(X) = \mathbb{R}_{>0}(-w_0\beta)$ and hence X is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module by the criterion given in Fact 4.4.

Thus we have found at least one discretely decomposable $(\mathfrak{g}, K)$-module for all the pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ in Table 1. This completes the proof of the theorem. □

Remark 5.7. Concrete branching laws are given in [19] for the last two cases in the proof above, that is, $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{u}(1))$ and $(\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sp}(n, \mathbb{R}))$.

From the proof of Theorem 5.2, we can take X in the condition (i) of Theorem 5.2 to be unitarizable.

Corollary 5.8 (unitarizable X)**.** *In the setting of Theorem 5.2, the conditions* (i)*,* (ii)*, and* (iii) *are also equivalent to* (i′)*.*

(i′) *there exists an infinite-dimensional irreducible unitarizable* $(\mathfrak{g}, K)$*-module* X *(by replacing* G *with a covering group of* G *if necessary) such that* X *is discretely decomposable as a* $(\mathfrak{g}^\sigma, K^\sigma)$*-module.*

Proof. It is enough to see that the $(\mathfrak{g}, K)$-modules X in the proof of Theorem 5.2 can be taken to be unitarizable in all cases. For highest weight $(\mathfrak{g}, K)$-modules, we can take (for example) holomorphic discrete series representations. For minimal representations, see [25]. For $X = U(\mathfrak{g})/J$ with $\mathfrak{g}$ complex and J the Joseph ideal, see [6, §12.4] and [26]. For $A_{\mathfrak{q}}(\lambda)$ and degenerate principal series representations, we use the fact that the Zuckerman derived functor preserves unitarity under a certain positivity condition and that the classical parabolic induction always preserves unitarity. □

We pin down a special case that $\mathfrak{g}$ is a complex simple Lie algebra:

Corollary 5.9. *Suppose that* $\mathfrak{g}$ *is a complex simple Lie algebra and* $\mathfrak{g}^\sigma$ *is a real form of* $\mathfrak{g}$*. We regard the pair* $(\mathfrak{g}, \mathfrak{g}^\sigma)$ *as a symmetric pair of real Lie algebras. Then the following six conditions on* $(\mathfrak{g}, \mathfrak{g}^\sigma)$ *are equivalent.*

(i) *There exists an infinite-dimensional irreducible* $(\mathfrak{g}, K)$*-module* X *such that* X *is discretely decomposable as a* $(\mathfrak{g}^\sigma, K^\sigma)$*-module.*

(ii) $\mathrm{pr}(K_{\mathbb{C}} \cdot \mathfrak{p}_\beta^*) \subset \mathcal{N}(\mathfrak{p}_{\mathbb{C}}^{\sigma *})$.
(iii) $\sigma\beta \neq -\beta$ *(β is the highest non-compact root given in Definition 2.1).*
(iv) *The minimal nilpotent orbit of $\mathfrak{g}$ does not intersect with the real form $\mathfrak{g}^\sigma$.*
(v) *The minimal nilpotent orbit of $\mathfrak{g}_{\mathbb{C}}^\sigma (\simeq \mathfrak{g})$ does not intersect with $\mathfrak{p}_{\mathbb{C}}^\sigma$.*
(vi) *The real form $\mathfrak{g}^\sigma$ of $\mathfrak{g}$ is compact, or is isomorphic to $\mathfrak{su}^*(2n)$, $\mathfrak{so}(n-1,1)(n \geq 5)$, $\mathfrak{sp}(m,n)$, $\mathfrak{f}_{4(-20)}$, or $\mathfrak{e}_{6(-26)}$.*

Remark 5.10. (1) Corollary 5.9 generalizes [12, Theorem 8.1], which dealt with split real forms $\mathfrak{g}^\sigma$ of $\mathfrak{g}$.
(2) In the condition (vi) of Corollary 5.9, the associated symmetric pair $(\mathfrak{g}, \mathfrak{g}^{\theta\sigma})$ is $(\mathfrak{g}, \mathfrak{g})$, or is a complex symmetric pair $(\mathfrak{sl}(2n,\mathbb{C}), \mathfrak{sp}(n,\mathbb{C}))$, $(\mathfrak{so}(n,\mathbb{C}), \mathfrak{so}(n-1,\mathbb{C}))(n \geq 5)$, $(\mathfrak{sp}(m+n,\mathbb{C}), \mathfrak{sp}(m,\mathbb{C}) \oplus \mathfrak{sp}(n,\mathbb{C}))$, $(\mathfrak{f}_4^{\mathbb{C}}, \mathfrak{so}(9,\mathbb{C}))$, or $(\mathfrak{e}_6^{\mathbb{C}}, \mathfrak{f}_4^{\mathbb{C}})$, respectively.

Proof. The equivalence of (i), (ii), (iii) and (vi) follows from Theorem 5.2 and Lemma 4.6. If $\mathfrak{g}$ is a complex simple Lie algebra, then $\mathfrak{k}$ is a real form of $\mathfrak{g}$ and there is a natural isomorphism of complex Lie algebras $\iota : \mathfrak{k}_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{g}$ that is identity on $\mathfrak{k}$. Then $\iota(\mathfrak{k}^\sigma + \sqrt{-1}\mathfrak{k}^{-\sigma}) = \mathfrak{g}^\sigma$. Put $\mathfrak{a} := \iota(\sqrt{-1}\mathfrak{t}^{-\sigma})$, $\mathfrak{h} := \iota(\mathfrak{t}_{\mathbb{C}})$, and let $\Delta^+(\mathfrak{g}, \mathfrak{h})$ be the positive system corresponding to $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}^\sigma$, and $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is compatible with some positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ of the restricted root system. Under the isomorphism ι, the $\mathfrak{k}_{\mathbb{C}}$-module $\mathfrak{p}_{\mathbb{C}}^*$ can be identified with the adjoint representation of $\mathfrak{g}$. Then the weight β corresponds to the highest root in $\Delta^+(\mathfrak{g}, \mathfrak{h})$ and the condition (iii) amounts to that the highest root is zero on $\mathfrak{t}^\sigma$ (i.e. a real root). By a result of T. Okuda [22], this is equivalent to (iv). The equivalence of (iv) and (v) follows from the Kostant–Sekiguchi correspondence [23, Proposition 1.11]. $\square$

6. Discretely Decomposable Tensor Product

The tensor product of two irreducible representations is regarded as a special case of our setting.

Let G be a connected simple Lie group. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the Lie algebra and K the connected subgroup with Lie algebra $\mathfrak{k}$. Put $\widetilde{G} = G \times G$, $\widetilde{K} = K \times K$ and let σ act on $\widetilde{G}$ by switching factors. Then any irreducible $(\widetilde{\mathfrak{g}}, \widetilde{K})$-module X is of the form of the exterior product $X_1 \boxtimes X_2$ with two irreducible $(\mathfrak{g}, K)$-modules X_1 and X_2. Then X, regarded as a $(\widetilde{\mathfrak{g}}^\sigma, \widetilde{K}^\sigma)$-module by restriction, is nothing but the tensor product representation $X_1 \otimes X_2$. The following theorem determines when $X_1 \otimes X_2$ is discretely decomposable.

Theorem 6.1. *Let G be a non-compact connected simple Lie group. Let X_1 and X_2 be infinite-dimensional irreducible $(\mathfrak{g}, K)$-modules. Then the tensor product representation $X_1 \otimes X_2$ is discretely decomposable as a $(\mathfrak{g}, K)$-module if and only if G is of Hermitian type and both X_1 and X_2 are simultaneously highest weight $(\mathfrak{g}, K)$-modules or simultaneously lowest weight $(\mathfrak{g}, K)$-modules.*

Proof. If X_1 and X_2 are both highest weight $(\mathfrak{g}, K)$-modules or they are both lowest weight $(\mathfrak{g}, K)$-modules, it is known that the tensor product $X_1 \otimes X_2$ is discretely decomposable (see [13, Theorem 7.4]).

Conversely, let us prove that $X_1 \otimes X_2$ is not discretely decomposable as a $(\mathfrak{g}, K)$-module unless X_1 and X_2 are highest weight modules or they are lowest weight modules. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. Fix a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. We set $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$, $\widetilde{\mathfrak{k}} = \mathfrak{k} \oplus \mathfrak{k}$, and $\widetilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathfrak{t}$. Then $\widetilde{\mathfrak{t}}$ is a Cartan subalgebra of $\widetilde{\mathfrak{k}}$. We have an isomorphism $\widetilde{\mathfrak{g}}^\sigma \simeq \mathfrak{g}$ and the restriction map $\mathrm{pr} : \widetilde{\mathfrak{g}}_{\mathbb{C}}^* \to \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\sigma *}$ is identified with the map $\mathfrak{g}_{\mathbb{C}}^* \oplus \mathfrak{g}_{\mathbb{C}}^* \to \mathfrak{g}_{\mathbb{C}}^*$ given by $(x, y) \mapsto x + y$. We take a positive system $\Delta^+(\widetilde{\mathfrak{k}}_{\mathbb{C}}, \widetilde{\mathfrak{t}}_{\mathbb{C}})$ to be the union of $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ in the first factor and $-\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ in the second factor so that $\Delta^+(\widetilde{\mathfrak{k}}_{\mathbb{C}}, \widetilde{\mathfrak{t}}_{\mathbb{C}})$ is $(-\sigma)$-compatible. Let $\beta \in \sqrt{-1}\mathfrak{t}$ be the highest non-compact root given in Definition 2.1.

Suppose that $\mathfrak{g}$ is not of Hermitian type. Since X_1 and X_2 are infinite-dimensional, we have $\mathcal{V}_{\mathfrak{g}}(X_1), \mathcal{V}_{\mathfrak{g}}(X_2) \neq \{0\}$ by Lemma 3.1(2). Hence they contain $K_{\mathbb{C}} \cdot \mathfrak{p}^*_\beta$ and $K_{\mathbb{C}} \cdot \mathfrak{p}^*_{-\beta}$, in particular $\mathcal{V}_{\mathfrak{g}}(X_1) \supset \mathfrak{p}^*_\beta$ and $\mathcal{V}_{\mathfrak{g}}(X_2) \supset \mathfrak{p}^*_{-\beta}$. We therefore have

$$\mathcal{V}_{\widetilde{\mathfrak{g}}}(X_1 \boxtimes X_2) = \mathcal{V}_{\mathfrak{g}}(X_1) \oplus \mathcal{V}_{\mathfrak{g}}(X_2) \supset \mathfrak{p}^*_\beta \oplus \mathfrak{p}^*_{-\beta}.$$

As in the proof of Lemma 4.6, we can see $\mathrm{pr}(\mathfrak{p}^*_\beta \oplus \mathfrak{p}^*_{-\beta}) \not\subset \mathcal{N}(\mathfrak{p}^*_{\mathbb{C}})$ and hence $\mathrm{pr}(\mathcal{V}_{\widetilde{\mathfrak{g}}}(X_1 \boxtimes X_2)) \not\subset \mathcal{N}(\mathfrak{p}^*_{\mathbb{C}})$. Therefore Fact 4.3 shows that $X_1 \otimes X_2$ is not discretely decomposable.

Suppose that $\mathfrak{g}$ is of Hermitian type. By Lemma 3.1 (2) and Lemma 3.5, if a highest weight $(\mathfrak{g}, K)$-module X is also a lowest weight $(\mathfrak{g}, K)$-module, then X is finite-dimensional. Since X_1 and X_2 are infinite-dimensional, at least one of the following holds.

(1) X_1 and X_2 are highest weight modules.
(2) X_1 and X_2 are lowest weight modules.
(3) X_1 is not a lowest weight module and X_2 is not a highest weight module.
(4) X_1 is not a highest weight module and X_2 is not a lowest weight module.

By switching X_1 and X_2 if necessary, it is enough to prove that $X_1 \otimes X_2$ is not discretely decomposable under the assumption (3). We thus assume that X_1 is not a lowest weight $(\mathfrak{g}, K)$-module and X_2 is not a highest weight $(\mathfrak{g}, K)$-module. By Lemma 3.5, this assumption is equivalent to $\mathcal{V}_{\mathfrak{g}}(X_1) \not\subset \mathfrak{p}^*_-$ and $\mathcal{V}_{\mathfrak{g}}(X_2) \not\subset \mathfrak{p}^*_+$. Hence it follows from the proof of Proposition 2.2 that $\mathcal{V}_{\mathfrak{g}}(X_1) \supset \mathfrak{p}^*_\beta$ and $\mathcal{V}_{\mathfrak{g}}(X_2) \supset \mathfrak{p}^*_{-\beta}$. Then by using the previous argument we see that $X_1 \otimes X_2$ is not discretely decomposable. $\square$

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