Symmetry Breaking under Translations

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This work

Some of the results from this talk will be used
as a basis in the forthcoming joint papers:

Harris–TK–Speh (in preparation),
Translation functors and restriction of coherent cohomology of Shimura varieties.

TK–Speh (in preparation),
How does the restriction of representations change under translations?

Classical branching law: finite-dim'l reps of $U(n)$

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ satisfying $\mu_1 \geq \cdots \geq \mu_n$,
$F^{U(n)}(\mu)$: the irreducible rep of $U(n)$ with highest weight μ.

Weyl's branching law $(U(n) \downarrow U(n-1))$

$$F^{U(n)}(\mu)|_{U(n-1)} \simeq \bigoplus_{\tau} F^{U(n-1)}(\tau)$$

where the highest weight τ runs over $\mathbb{Z}^{n-1}$ satisfying

$$\mu_1 \geq \tau_1 \geq \mu_2 \geq \tau_2 \geq \cdots \geq \tau_{n-1} \geq \mu_n.$$

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Example $(U(4) \downarrow U(3))$

$$\begin{array}{ccccccc} \mu_1 & & \mu_2 & & \mu_3 & & \mu_4 \\ & \tau_1 & & \tau_2 & & \tau_3 & \end{array}$$

Reformulation $\mathfrak{gl}(n) \downarrow \mathfrak{gl}(n-1)$

$\mathfrak{Z}(\mathfrak{gl}_n)$: the center of the enveloping algebra $U(\mathfrak{gl}_n)$

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{gl}_n), \mathbb{C}) \simeq \mathbb{C}^n / \mathfrak{S}_n \quad \text{(Harish-Chandra isomorphism).}$$

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highest wt		infinitesimal character
$F^{U(n)}(\mu)$	$\rightsquigarrow$	$x := \mu + (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2})$,
$F^{U(n-1)}(\tau)$	$\rightsquigarrow$	$y := \tau + (\frac{n-2}{2}, \ldots, \frac{2-n}{2})$.

Remark $x_i - y_j \in \mathbb{Z} + \frac{1}{2}$.

$$\mu_1 \geq \tau_1 \geq \mu_2 \geq \tau_2 \geq \cdots \geq \tau_{n-1} \geq \mu_n$$

$$\iff \quad x_1 > y_1 > x_2 > y_2 > \cdots > y_{n-1} > x_n$$

Wall for G *vs* Fence for $G \supset G'$

$$\mathbb{R}^n_> := \{x \in \mathbb{R}^n : x_1 > \cdots > x_n\},$$
$$\mathbb{R}^n_\geq := \{x \in \mathbb{R}^n : x_1 \geq \cdots \geq x_n\}.$$

This is viewed as the dominant Weyl chamber for $\mathfrak{gl}_n$.

Wall of $\mathbb{R}^n_\geq$: the hyperplane defined by $x_i = x_{i+1}$, corresponding to an adjacent inequality $x_i > x_{i+1}$.

Fence in $\mathbb{R}^n_\geq \times \mathbb{R}^m_\geq$: shall be defined in the next slide.

Fence: definition

Definition An interlacing pattern D in $\mathbb{R}^n_> \times \mathbb{R}^m_>$ is a total order among $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_>$ and $y = (y_1, \ldots, y_m) \in \mathbb{R}^m_>$.

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Example

$$D_1 = \{(x, y) \in \mathbb{R}^{3+2} : x_1 > y_1 > x_2 > x_3 > y_2\},$$
$$D_2 = \{(x, y) \in \mathbb{R}^{3+2} : y_1 > y_2 > x_1 > x_2 > x_3\},$$

are examples of interlacing patterns in $\mathbb{R}^3_> \times \mathbb{R}^2_>$.

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are examples of interlacing patterns in $\mathbb{R}^3_> \times \mathbb{R}^2_>$.

Definition A fence of an interlacing pattern D is the hyperplane corresponding to the adjacent inequality between x_i and y_j.

Example D_1 has 3 fences: $x_1 = y_1$, $y_1 = x_2$, and $x_3 = y_2$, while D_2 has only one fence $y_2 = x_1$.

Harish-Chandra modules, Casselman–Wallach globalization

Let G be a real reductive linear Lie group.

$\mathcal{M}(G)$: the category of admissible smooth representations of G of finite length having moderate growth.

$\mathrm{Irr}(G)$: Irreducible objects in $\mathcal{M}(G)$.

Harish-Chandra modules, Casselman–Wallach globalization

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$\mathrm{Irr}(G)$: Irreducible objects in $\mathcal{M}(G)$.

K: maximal compact subgroup of G.

$\mathcal{M}(\mathfrak{g}, K)$: the category of $(\mathfrak{g}, K)$-modules of finite length.

There is a natural category equivalence (Casselman–Wallach):

$$\mathcal{M}(G) \simeq \mathcal{M}(\mathfrak{g}, K).$$

Virtual $(\mathfrak{g}, K)$-modules: reminder

G: real reductive Lie group $\supset K$: max compact subgroup

$\mathcal{M}(\mathfrak{g}, K)$: the category of $(\mathfrak{g}, K)$-modules of finite length.

$\mathcal{V}(\mathfrak{g}, K)$: <u>the Grothendieck group of $\mathcal{M}(\mathfrak{g}, K)$</u>.

$\cdots$ the abelian group generated by $X \in \mathcal{M}(\mathfrak{g}, K)$
modulo the equivalence relation

$$X \sim Y + Z$$

whenever there is a short exact sequence $0 \to Y \to X \to Z \to 0$.

Similarly,

$\mathcal{M}(G)$: the category of admissible smooth representation of G of finite length having moderate growth.

$\mathcal{V}(G)$: <u>the Grothendieck group of $\mathcal{M}(G)$</u>.

Reminder: Coherent family

$G \supset J$ (maximally split) Cartan subgroup,
W: Weyl group for $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$,
$\Lambda \subset \mathfrak{j}^*$: weight lattice of finite-dimensional reps of G.
(To be precise , $\Lambda \subset \widehat{J}$)

Definition (Coherent family) Let $\xi \in \mathfrak{j}_{\mathbb{C}}^*$. A map

$$\Pi : \xi + \Lambda \to \mathcal{V}(\mathfrak{g}, K)$$

is a coherent family if, for any $x \in \xi + \Lambda$,

(1) Π_x has infinitesimal character $x \in \mathfrak{j}_{\mathbb{C}}^* \mod W$;
(2) $\Pi_x \otimes F \simeq \sum_{u \in \Delta(F)} \Pi_{x+u}$ in $\mathcal{V}(\mathfrak{g}, K)$

for any finite-dimensional rep F of G.

Similarly, one can define coherent families in $\mathcal{V}(G)$.

Coherent continuation: Basic property and examples

Fact (1) (Existence and Uniqueness) Let Π be an irreducible rep of G with non-singular infinitesimal character $\xi \in \mathfrak{j}_{\mathbb{C}}^*$. Then there exists a unique coherent family

$$\Pi \colon \xi + \Lambda \to \mathcal{V}(G)$$

starting from $\Pi_{\xi} + 0 := \Pi$.

(2) (Irreducibility) Π_x is irreducible as long as $x \in \xi + \Lambda$ is non-singular and stays inside the same Weyl chamber containing ξ.

Coherent continuation: Basic property and examples

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Example (1) (Finite-dim'l rep) Let $F^G(\mu)$ denote an irreducible finite-dim'l rep of G with highest weight μ:

$$\Pi_{\rho_G+\mu} := F^G(\mu) \qquad (\mu\text{: dominant integral}).$$

(2) (Principal series representations).

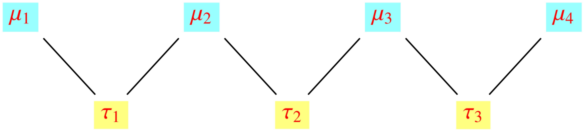
(3) (Discrete series representations).

“Fence Crossing” and “Symmetry Breaking”

Let $\Pi\colon \xi + \Lambda \to \mathcal{V}(G)$ be a coherent family through the initial point $\Pi_\xi := \Pi \in \mathrm{Irr}(G)$ at non-singular ξ.

- Representations (a single group G)
  Many of representation-theoretic properties of Π_x are stable, if $x \in \xi + \Lambda$ is (strictly) inside the same Weyl chamber with ξ.

- Symmetry Breaking (a pair of reductive groups $G \supset G'$)
  The multiplicity $[\Pi_x|_{G'} : \pi]$ can alter significantly even if $x \in \xi + \Lambda$ is inside the same Weyl chamber with ξ.



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Wall Crossing	$\rightsquigarrow$	“Fence Crossing”
G		$G \supset G'$

Multiplicity for Symmetry Breaking $G \downarrow G'$

Let $G \supset G'$ be a pair of real reductive linear Lie groups.

For $\Pi \in \mathrm{Irr}(G)$ and $\pi \in \mathrm{Irr}(G')$, the multiplicity is defined by

$$[\Pi|_{G'} : \pi] := \dim \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \{0\} \cup \mathbb{Z}_+ \cup \{\infty\},$$

where

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) := \{\text{symmetry breaking operators}\}.$$

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TK–T. Oshima*

$[\Pi|_{G'} : \pi] < \infty \quad ({}^\forall\Pi, {}^\forall\pi) \iff (G \times G')/\operatorname{diag} G'$ is real spherical,

$[\Pi|_{G'} : \pi] \leq C \quad ({}^\forall\Pi, {}^\forall\pi) \iff (G'_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag} G'_{\mathbb{C}}$ is spherical,

Sun–Zhu**

$[\Pi|_{G'} : \pi] \in \{0, 1\} \quad ({}^\forall\Pi, {}^\forall\pi)$ if (G, G') is

$$(GL(n, \mathbb{R}), GL(n-1, \mathbb{R})),\ (U(p, q), U(p-1, q)), \text{ etc.}$$

* TK, Proc. Summer School, 1995, TK–T. Oshima, Adv. Math. 2013; ** Sun–Zhu, Ann. Math. 2012.

Symmetry Breaking inside "Fence"

Let (G, G') be real forms of $(GL(n, \mathbb{C}), GL(n-1, \mathbb{C}))$.
E.g. $(G, G') = (GL(n, \mathbb{R}), GL(n-1, \mathbb{R}))$ or $(U(p, q), U(p-1, q))$.

π: irreducible rep of the subgroup G' with infinitesimal character y.
Π: irreducible rep of G with non-singular infinitesimal character ξ.

Let $\Pi : \xi + \Lambda \to \mathcal{V}(G)$ be a coherent family starting at $\Pi_{\xi+0} = \Pi$.

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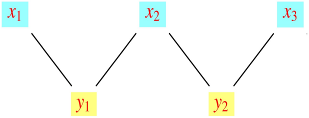
Theorem A For any $x \in \xi + \Lambda$ such that (x, y) satisfies the same interlacing pattern with (ξ, y), one has

$$[\Pi_x|_{G'} : \pi] = [\Pi|_{G'} : \pi].$$

Example: $U(3) \downarrow U(2)$ revisited

Weyl's branching law: If $\xi_1 \geq \tau_1 \geq \xi_2 \geq \tau_2 \geq \xi_3$, then

$$[F^{U(3)}(\xi_1, \xi_2, \xi_3)|_{U(2)} : F^{U(2)}(\tau_1, \tau_2)] = 1.$$



$\Uparrow$ Theorem A

A simple and easy case

Example: $U(3) \downarrow U(2)$ revisited

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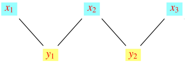


$\Uparrow$ Theorem A

A simple and easy case:

$$[F^{U(3)}(\tau_1, \tau_2, \xi_3)|_{U(2)} : F^{U(2)}(\tau_1, \tau_2)] = 1.$$

The infinitesimal characters y and x of π and Π, respectively:

$$(\tau_1 + \tfrac{1}{2}, \tau_2 - \tfrac{1}{2}) \equiv (y_1, y_2)$$

$$(\tau_1 + 1, \tau_2, \xi_3 - 1) \equiv (x_1, x_2, x_3)$$

satisfies the interlacing pattern $x_1 > y_1 > x_2 > y_2 > x_3$.

Example: Gan–Gross–Prasad conjecture

- $(G, G') = (U(p,q), U(p-1,q))$
  We review Harish-Chandra's discrete series representations for G and G' in a simple setting where $(p,q) = (2,1)$.

- $G = U(2,1)$
  Three families: for $x = (x_1, x_2, x_3) \in \mathbb{Z}^3_{>}$.

$$\Pi^{\mathrm{holo}}(x), \quad \Pi^{\mathrm{non}}(x), \quad \Pi^{\mathrm{anti}}(x).$$

- $G = U(1,1)$
  Two families: for $y = (y_1, y_2) \in (\mathbb{Z} + \frac{1}{2})^2_{>}$,

$$\pi^{\mathrm{holo}}(y), \quad \pi^{\mathrm{anti}}(y).$$

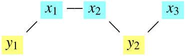
Here, x, y stand for infinitesimal characters.

Inside Fence: Example: $(G, G') = (U(2,1), U(1,1))$

$$x = (x_1, x_2, x_3) \in \mathbb{Z}^3_>, \qquad y = (y_1, y_2) \in (\mathbb{Z} + \tfrac{1}{2})^2_>.$$

- $[\Pi^{\mathrm{holo}}(x)|_{G'} : \pi^{\mathrm{holo}}(y)] = 1$

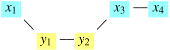
  $\iff \quad y_1 > x_1 > x_2 > y_2 > x_3.$

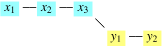


- $[\Pi^{\mathrm{non}}(x)|_{G'} : \pi^{\mathrm{holo}}(y)] = 1$

  $\underset{\text{H. He}^*}{\iff} \quad x_1 > y_1 > y_2 > x_2 > x_3$ or

  $x_1 > x_2 > x_3 > y_1 > y_2.$





These conditions on parameters fit well with our general theorem:

Theorem A For any $x \in \xi + \Lambda$ such that (x, y) satisfies the same interlacing pattern with (ξ, y), one has

$$[\Pi_x|_{G'} : \pi] = [\Pi|_{G'} : \pi].$$

* H. He, *On the Gan–Gross–Prasad conjecture for $U(p,q)$*, Invent. Math., **209** (2017), 837–884.

Jumping Fence of Interlacing Pattern

Let (G, G') be real forms of $(GL(n,\mathbb{C}), GL(n-1,\mathbb{C}))$.

π: irreducible rep of G' with infinitesimal character y.
Π: irreducible rep of G with non-singular infinitesimal character x.

Theorem A is derived from: let $\phi_x^{x'}$ be a translation functor.

Theorem B Let x be non-singular. Then

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \simeq \mathrm{Hom}_{G'}(\phi_x^{x+f_i}(\Pi)|_{G'}, \pi)$$

if $x_i \notin \{y_1 - \frac{1}{2}, \ldots, y_{n-1} - \frac{1}{2}\}$.

- Previous examples $\cdots$ $x_i - y_j \in \mathbb{Z} + \frac{1}{2}$.
- If $x_i - y_j \in \mathbb{Z}$, then one can jump the fences!

Discrete series representations for G/H

$\widehat{G}$: the unitary dual of G.

Suppose that a homogeneous space G/H has a G-invariant measure.

$\rightsquigarrow$ $G \curvearrowright L^2(G/H)$ unitary representation.

The set of discrete series representations for G/H is defined by

$$\mathrm{Disc}(G/H) := \{\Pi \in \widehat{G} : \mathrm{Hom}_G(\Pi, L^2(G/H)) \neq \{0\}\}.$$

Examples of discrete series reps for G/H

$$\mathrm{Disc}(G/H) := \{\Pi \in \widehat{G} : \mathrm{Hom}_G(\Pi, L^2(G/H)) \neq \{0\}\}.$$

Example $X = G/H = GL(n,\mathbb{R})/(GL(p,\mathbb{R}) \times GL(q,\mathbb{R}))$ $(p+q=n)$.

$$\mathrm{Disc}(G/H) \simeq \{\Pi_\lambda : \lambda \in (2\mathbb{Z}+1)^\ell, \lambda_1 > \cdots > \lambda_\ell > 0\}.$$

Π_λ is a cohomological parabolic induction from the Levi subgroup

$$L \simeq (\mathbb{C}^\times)^\ell \times GL(2n-\ell,\mathbb{R}), \quad \ell := \min(p,q).$$

Our normalization: $\mathfrak{Z}(\mathfrak{g}_\mathbb{C})$-inf character of Π_λ is

$$x := \frac{1}{2}(\lambda_1, \ldots, \lambda_\ell, n-2\ell-1, \ldots, 1+2\ell-n, -\lambda_\ell, \cdots, -\lambda_1).$$

Examples of discrete series reps for G/H

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<u>Example</u> $X = G/H = GL(n,\mathbb{R})/(GL(p,\mathbb{R}) \times GL(q,\mathbb{R}))$ $(p+q=n)$.

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$$x := \tfrac{1}{2}(\lambda_1, \ldots, \lambda_\ell, n-2\ell-1, \ldots, 1+2\ell-n, -\lambda_\ell, \cdots, -\lambda_1).$$

Similarly for $Y = G'/H' = GL(n-1,\mathbb{R})/(GL(p,\mathbb{R}) \times GL(q-1,\mathbb{R}))$.

$$\mathrm{Disc}(G'/H') = \{\pi_\nu : \nu \in (2\mathbb{Z}+1)^{\ell'}, \nu_1 > \cdots > \nu_{\ell'} > 0\},$$

Here, $\ell' = \min(p, q-1)$. The $\mathfrak{Z}(\mathfrak{g}'_\mathbb{C})$-inf character of π_ν is

$$y := \tfrac{1}{2}(\nu_1, \ldots, \nu_{\ell'}, n-2\ell'-2, \ldots, 2+2\ell'-n, -\nu_{\ell'}, \cdots, -\nu_1).$$

Period integral

$\Pi \subset L^2(X)$: discrete series representation for $X = G/H$

$$\cup$$

$\pi \subset L^2(Y)$: discrete series representation for $Y = G'/H'$

Consider a period integral:

$$B\colon \Pi^\infty \times \pi^\infty \to \mathbb{C}, \quad (F, f) \mapsto \int_Y F\overline{f}$$

$$\rightsquigarrow \quad T_B \in \mathrm{Hom}_{G'}(\Pi^\infty|_{G'}, \pi^\infty) \quad \text{via} \quad \overline{\pi}^\vee \simeq \pi.$$

<u>Example</u> $(G, G') = (GL(n,\mathbb{R}), GL(n-1,\mathbb{R}))$ $\;p + q = n,\;\; 2p \leq n-1.$

$$H = GL(p,\mathbb{R}) \times GL(q,\mathbb{R}),$$
$$H' = GL(p,\mathbb{R}) \times GL(q-1,\mathbb{R}).$$

Jumping Fence of Interlacing Pattern

Theorem For any $\Pi \in \mathrm{Disc}(G/H)$ and any $\pi \in \mathrm{Disc}(G'/H')$ with non-singular inf characters, we have

$$[\Pi|_{G'} : \pi] = 1.$$

$$x_1 \text{ — } x_2 \text{ — } \cdots \text{ — } x_\ell$$

$$y_1 \text{ — } y_2 \text{ — } \cdots \text{ — } y_\ell$$

(no interlacing!)

Jumping Fence of Interlacing Pattern

Theorem For any $\Pi \in \mathrm{Disc}(G/H)$ and any $\pi \in \mathrm{Disc}(G'/H')$ with non-singular inf characters, we have

$$[\Pi|_{G'} : \pi] = 1.$$

Proof. For simplicity, $(G, G') = (GL(5,\mathbb{R}), GL(4,\mathbb{R}))$ and $p = 2$. The period integral does not vanish in the special setting where $(\lambda_1, \lambda_2) = (\nu_1, \nu_2)$, thus, $[\Pi_\lambda|_{G'} : \pi_\nu] = 1$.

The interlacing property in this specific case is given as

$$x_1 = y_1 > x_2 = y_2 .$$

$\rightsquigarrow$ One can jump fences for x_i and y_j, because

$$x_i - y_j \in \mathbb{Z}, \text{ whereas}$$

Theorem B involves an "obstruction" only when $x_i - y_j \in \mathbb{Z} + \frac{1}{2}$.

Another example: Branching of the Speh representation

$$(G, G') = (GL(2m, \mathbb{R}), GL(2m-1, \mathbb{R}))$$

For simplicity of the slide, consider the case $m = 2$.

- $G = GL(4, \mathbb{R})$
  Let $\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \in (\mathbb{Z}^4)_{\geq}$.
  $\Pi(\mu)$: coherent continuation of the Speh rep of $G = GL(4, \mathbb{R})$.

  Our normalization:
  $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character $x = \mu + (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$.
  $\mu = (\ell, \ell, -\ell, -\ell) \rightsquigarrow \Pi(\mu)$ is the 2ℓ-th Speh rep.

- $G' = GL(3, \mathbb{R})$
  For $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{Z}^3_{>}$, we consider a certain coherent family $\pi(\tau) \in \mathrm{Irr}(G')$ having an infinitesimal character $y = (\tau_1, \tau_2, \tau_3)$.

Another example: Branching of the Speh representation

$\Pi(\mu) \cdots$ (Speh representation)

cohomological parabolic induction

$L = GL(n, \mathbb{C}) \uparrow G = GL(2n, \mathbb{R})$

$\pi(\tau) \cdots$ cohomological parabolic induction

$L' = GL(n-1, \mathbb{C}) \times \mathbb{R}^\times \uparrow G' = GL(2n-1, \mathbb{R})$

Branching to Speh representation $GL(2m) \downarrow GL(2m-1)$

For simplicity, we consider the case $m = 2$. Let $\tau \in \mathbb{Z}^3_>$.

<u>Theorem</u> For any $\mu_1 \geq \tau_1 \geq \mu_2 > \mu_3 \geq \tau_3 \geq \mu_4$,

$$[\Pi(\mu_1, \mu_2, \mu_3, \mu_4)|_{G'} : \pi(\tau_1, \tau_2, \tau_3)] = 1.$$

This corresponds to the interlacing pattern of $\mathfrak{Z}(\mathfrak{g}_\mathbb{C})$-inf characters:

$$x_1 > y_1 > x_2 > x_3 > y_3 > x_4$$

$$\begin{array}{ccccccccc}
x_1 & & & & x_2 & — & x_3 & & & & x_4 \\
 & \diagdown & & \diagup & & & & \diagdown & & \diagup & \\
 & & y_1 & & & & & & y_3 & &
\end{array}$$

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x_1 & & & & x_2 & - & x_3 & & & & x_4 \\
& \diagdown & & \diagup & & & & \diagdown & & \diagup & \\
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Theorem A $\Uparrow$

<u>Easier case</u>: if $\mu_1 \geq \tau_1$, $\tau_3 \geq \mu_4$, and $\mu_1 + \mu_4 = \tau_2$, then

$$[\Pi(\mu_1, \tau_1, \tau_3, \mu_4)|_{G'} : \pi(\tau_1, \tau_2, \tau_3)] = 1.$$

Vanishing: Branching of Speh Representation

Changing the interlacing pattern

$$x_1 > y_1 > x_2 > x_3 > y_3 > x_4,$$

for the non-vanishing, one has a vanishing result.

Corollary If the following interlacing

$$x_1 > x_2 > y_1 > y_3 > x_3 > x_4$$

holds, then

$$[\Pi(\mu)|_{G'} : \pi(\tau)] = 0.$$

Here we recall $x = \mu + \frac{1}{2}(1, -1, 1, -1)$, $y = \tau$.

Proof for the vanishing uses again Theorem B to extend an easier and special case to the general case.

Scheme

$$\begin{tikzcd}[column sep=huge]
\text{Theorem A} \arrow[d, rightsquigarrow] & \text{Theorem B} \arrow[l, rightsquigarrow] \arrow[d, rightsquigarrow, "\text{(depending on parity)}"] \\
\text{Interlacing Pattern (inside fences)} & \text{Jumping Fences}
\end{tikzcd}$$

Thank you very much!