Structure of Tempered Homogeneous Spaces

I. Dynamical Approach

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Minicourses

Institut Henri Poincaré, France, 17–21 February 2025

References

The theme of the mini-course is joint with Yves Benoist.

Tempered Homogeneous Spaces:

- ——— I. (J. Euro Math., 2015)
  Method (Dynamical System)
- ——— II. (Margulis Festschrift, 2022, Chicago Univ. Press)
  Representation Theory
- ——— III. (J. Lie Theory, 2021)
  Classification Theory (Combinatorics)
- ——— IV. (J. Inst. Math. Jussieu, 2023)
  Limit algebra, geometric quantization

Tensor product of GL_n (J. Algebra, 2023)

Plan of Lectures

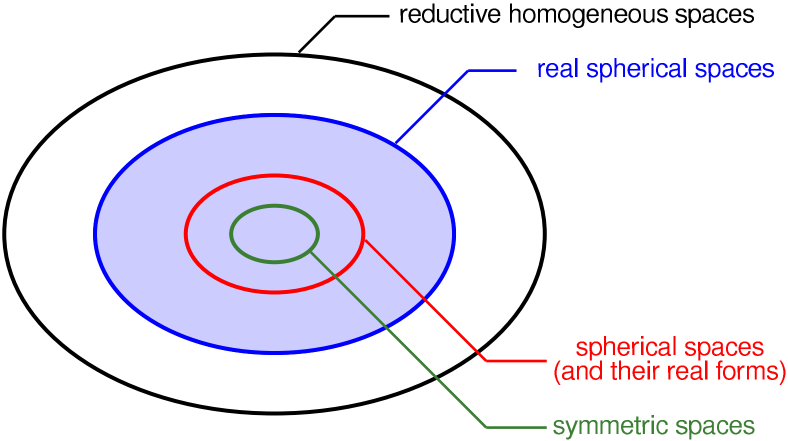
- Talk 1: (February 17, 2025)
  Tempered homogeneous spaces
  —Dynamical approach

- Talk 2: (February 19, 2025)
  Classification theory of tempered G/H
  —Combinatorics of convex polyhedra

- Talk 3: (February 21, 2025)
  Tempered homogeneous spaces
  —Interaction with topology and geometry

Reductive homogeneous space G/H

G: real reductive groups
H: reductive subgroup



We shall also discuss when G and H are not nesssarily reductive.

Plan for Today

Beyond spherical cases, we try to capture "coarse information".

<u>Basic Problem</u> Find a geometric criterion for $G \curvearrowright X$ that assures $L^2(X)$ to be almost L^p.

Change of approach

PDE $\rightsquigarrow$ Dynamical approach

Plan of Today (Lecture 1)

- Methods:
  - Optimal constant $q(G;X)$ for L^q-estimate of $\mathrm{vol}(gS \cap S)$.
  - Almost L^p-representation.
- Concept:
  - Tempered homogeneous spaces
  - Tempered subgroups

Learn from Dynamical System

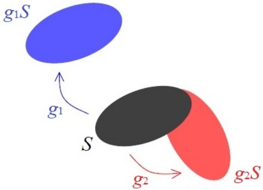
G : locally compact group

X : locally compact space

Definition A continuous action $G \curvearrowright X$ is called proper if the subset

$$G_S := \{g \in G : S \cap gS \neq \emptyset\}$$

is compact for any compact subset $S \subset X$.



$$\Longrightarrow \quad g_1 \not\in G_S, \quad g_2 \in G_S$$

Definition The action is free $\Longleftrightarrow$ $G_{\{x\}} = \{e\} \ ^\forall x \in X$.

Criterion for proper actions — topology

Basic problem (topology) Given a geometry X.
Find a criterion for a group L $(\subset \mathrm{Aut}(X))$ to act properly on X.

Group theoretic approach:

- Properness criterion was established for a homogeneous space X of a reductive group G (1989*–1996).
  $\cdots$ Applications include a solution (1989*) to the Calabi-Markus phenomenon (Ann. Math., 1962).
- Properness criterion for nilpotent Lie groups G up to 3-step (1995–**).
- Open problems in general.***

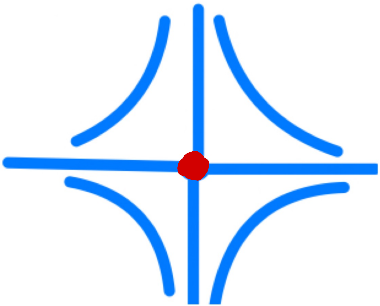
* T. Kobayashi (Math. Ann., '89 and JLT '96), Benoist (Ann. Math., '96);
** R. Lipsman (JLT '95), S. Nasrin ('01), T. Yoshino (IJM, '07), Baklouti–Khlif (IMM, '05) et al;
*** T. Kobayashi, Conjectures on reductive homogeneous spaces, Lect. Notes in Math., (2023).

Non-proper action — delicate example $\mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

Example Let $\mathbb{R} \ni t$ act on $\mathbb{R}^2$ by

$$(x, y) \mapsto (e^t x, e^{-t} y).$$

1) This action is neither free nor proper
because the origin $(0,0)$ is a fixed point.



Non-proper action — delicate example $\mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

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$$(x, y) \mapsto (e^t x, e^{-t} y).$$

1) This action is neither free nor proper
because the origin $(0,0)$ is a fixed point.
The removal of the origin makes the situation slightly better.

2) The action on $X := \mathbb{R}^2 \setminus \{(0,0)\}$ is free, but is not proper.

Non-proper action — delicate example $\mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

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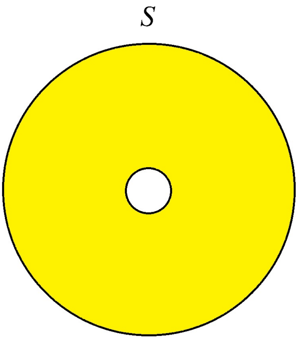

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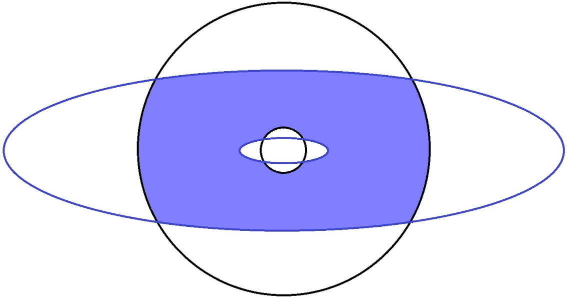
$S \cap gS$



Idea: Quantify proper actions

Locally compact group $G \curvearrowright X$ locally compact space

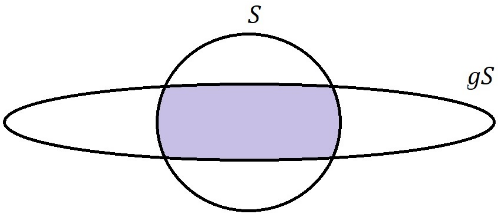
$G \curvearrowright X$ proper $\overset{\text{def}}{\Leftrightarrow} \{g \in G : S \cap gS \neq \emptyset\}$ is compact ${}^{\forall}S \subset X$ compact,

$\Leftrightarrow \mathrm{vol}(S \cap gS) \in C_c(G)$ ${}^{\forall}S \subset X$ compact,

where we fix an appropriate Radon measure on X.

Idea: Quantitative estimate for non-proper actions.

Look at asymptotic behavior of $\mathrm{vol}(S \cap gS)$ as g goes to infinity.



Volume estimate $\mathrm{vol}(t\cdot S\cap S)$: example $\quad \mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

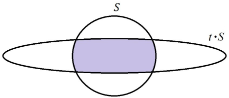
Example Let $\mathbb{R} \ni t$ act on $X = \mathbb{R}^2 \setminus \{(0,0)\}$ by

$$(x,y) \mapsto (e^t x, e^{-t} y)$$

- This action is free, but is not proper.
- Asymptotic behavior of $\mathrm{vol}(S \cap t\cdot S)$.

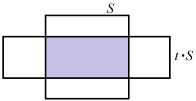
For any compact neighbourhood S of the origin in $\mathbb{R}^2$, one has

$$C_1 e^{-|t|} \le \mathrm{vol}(t\cdot S \cap S) \le C_2 e^{-|t|}.$$



For instance, if $S = \{(x,y)\in\mathbb{R}^2 : |x| \le 1, |y| \le 1\}$,

$$\mathrm{vol}(t\cdot S \cap S) = 4e^{-|t|}.$$



Almost L^p function

Z: locally compact space equipped with a Radon measure.
Eg. a locally compact group G with (left) Haar measure.

Definition A measurable function f on Z is almost L^p if

$$f \in \bigcap_{\varepsilon > 0} L^{p+\varepsilon}(Z).$$

Remark For $p \leq p'$, one has
f is almost $L^p \Longrightarrow f$ is almost $L^{p'}$.

We are interested in the best possible p for which f is almost L^p, in particular, when Z is a semisimple Lie group G.

(*e.g.*, $G = SL(n, \mathbb{R})$, $SU(p, q)$, $SO(p, q)$, $Sp(n, \mathbb{R})$. $\cdots$).

Example 1. L^p-estimate of K-finite eigenfunctions

$$D = \{z \in \mathbb{C} : |z| < 1\} \quad ds^2 = \frac{4(dx^2+dy^2)}{(1-|z|^2)^2} \quad \text{(Poincaré disc)}$$

Any K-finite function f satisfying $\Delta f = \lambda f$ is almost $L^{p(\lambda)}$ $(\lambda > 0)$,

where $p(\lambda) := \frac{2}{1-\sqrt{1-4\lambda}}$ $(0 \le \lambda \le \frac{1}{4})$; $= 2$ $(\frac{1}{4} \le \lambda)$.

Example 1. L^p-estimate of K-finite eigenfunctions

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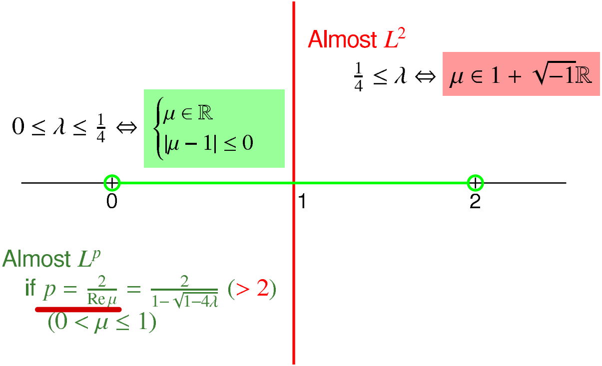
Figure in the μ-plane with $\lambda = -\frac{1}{4}(\mu^2 - 2\mu)$.



Example 2. L^p-estimate of $\mathrm{vol}(gS \cap S)$ for $G \curvearrowright G/N$

The example $\mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$, $(x, y) \mapsto (e^t x, e^{-t} y)$

$$A = \{a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R}\} \subset G = SL(2, \mathbb{R}) \supset N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}.$$

$$gN \mapsto g\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{array}{ccccc} G/N & \xrightarrow{\sim} & \mathbb{R}^2 \setminus \{(0,0)\} & & \begin{pmatrix} x \\ y \end{pmatrix} \\ a_t \cdot \downarrow & & \downarrow & & \downarrow \\ G/N & \xrightarrow{\sim} & \mathbb{R}^2 \setminus \{(0,0)\} & & \begin{pmatrix} e^t x \\ e^{-t} y \end{pmatrix} \end{array}$$

Example 2. L^p-estimate of $\mathrm{vol}(gS \cap S)$ for $G \curvearrowright G/N$

The example $\mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$, $(x,y) \mapsto (e^t x, e^{-t} y)$ is interpreted as

$$A \hookrightarrow G \curvearrowright G/N \Longleftrightarrow \mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}.$$

$$A = \{a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R}\} \subset G = SL(2,\mathbb{R}) \supset N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}.$$

- For any compact $S \subset G/N$ and $g = k_1 a_t k_2$ with $k_1, k_2 \in SO(2)$,

$$\mathrm{vol}(gS \cap S) \sim e^{-|t|} \quad \text{(previous example).}$$

Example 2. L^p-estimate of $\mathrm{vol}(gS \cap S)$ for $G \curvearrowright G/N$

The example $\mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$, $(x,y) \mapsto (e^t x, e^{-t} y)$ is interpreted as

$$A \hookrightarrow G \curvearrowright G/N \iff \mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}.$$

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- For any compact $S \subset G/N$ and $g = k_1 a_t k_2$ with $k_1, k_2 \in SO(2)$,

  $\mathrm{vol}(gS \cap S) \sim e^{-|t|}$ (previous example).

$\Downarrow$

$\mathrm{vol}(gS \cap S)$ is almost $L^2(G)$ for any compact subset $S \subset G/N$.

Example 2. L^p-estimate of $\mathrm{vol}(gS \cap S)$ for $G \curvearrowright G/N$

$$A = \{a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R}\} \subset G = SL(2,\mathbb{R}) \supset N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}.$$

Recall

Definition A measurable function f on Z is almost L^p if

$$f \in \bigcap_{\varepsilon>0} L^{p+\varepsilon}(Z).$$

Want to check:

$\mathrm{vol}(gS \cap S)$ is almost $L^2(G)$ for any compact subset $S \subset G/N$.

- For any compact $S \subset G/N$ and $g = k_1 a_t k_2$ with $k_1, k_2 \in SO(2)$,

$$\mathrm{vol}(gS \cap S) \sim e^{-|t|} \quad \text{(previous example)}.$$

- Haar measure on $g = k_1 a_t k_2 \in G = SL(2,\mathbb{R})$: One has

$$dg = \sinh(2t) dk_1 dt dk_2 \sim e^{2|t|} dk_1 dt dk_2.$$

Hence

$$\mathrm{vol}(gS \cap S) \in L^{p+\varepsilon}(G) \iff 2 - p - \varepsilon < 0.$$

Optimal constant $q(G;X)$ of volume estimate

$$G \curvearrowright X$$

Suppose X admits a G-invariant Radon measure.

Definition We write $q(G;X)$ for the optimal constant $q > 0$ such that

$$\mathrm{vol}(S \cap gS) \text{ is an almost } L^q\text{-function on } G$$

for every compact subset $S \subset X$.

Example $q(G;X) = 2$ if $(G, X) = (SL(2,\mathbb{R}), \mathbb{R}^2)$.

General Problem Find an explicit formula of $q(G;X)$.

Finding the optimal L^p-estimate of $\mathrm{vol}(gS \cap S)$

Let G be a semisimple Lie group acting on X.
$q(G;X)$: the optimal constant for L^q-estimate of $\mathrm{vol}(gS \cap S)$.

We shall give an explicit formula of $q(G;X)$
when $X = V$ (linear action) or $X = G/H$ (H: reductive).

			Method
(Theorem A)	Case 1	$G \curvearrowright V$ (semisimple, linear)	Dynamical approach
(Theorem B)	Case 2	$G \supset H$ (semisimple, reductive)	Global geometry + Case 1

L^p-estimate of $\mathrm{vol}(gS \cap S) \cdots$ Case 1. $H \curvearrowright V$ linear

Notation : $G \curvearrowright X \rightsquigarrow H \curvearrowright V$ (linear)

Let H be a semisimple Lie group, and $\tau \colon H \to SL_{\mathbb{R}}(V)$ a representation. Assume τ has a compact kernel.

The optimal constant $q(H;V)$ for $\mathrm{vol}(gS \cap S)$ to be almost L^q is given as follows.

Theorem A For a linear action $H \curvearrowright V$, one has

$$q(H;V) = p_V .$$

analysis combinatorics

$$p_V := \max_{Y \in \mathfrak{h} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$$

$\rho_{\mathfrak{h}}$, ρ_V $\cdots$ next page.

Piecewise linear function ρ_V associated to $\tau\colon \mathfrak{h} \to \mathrm{End}(V)$

For a finite-dimensional rep $\tau\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$, we introduce:

Definition (non-negative function ρ_V on the Lie algebra $\mathfrak{h}$)

$$\rho_V : \mathfrak{h} \to \mathbb{R}_{\geq 0}, \quad Y \mapsto \frac{1}{2} \sum |\mathrm{Re}\, \lambda(Y)|.$$

gen. eigenvalues of $\tau(Y) \in \mathrm{End}(V_{\mathbb{C}})$

Let $\mathfrak{a}$ be a maximal split abelian subspace of the Lie algebra $\mathfrak{h}$.

$\rightsquigarrow$
- ρ_V is determined by its restriction to $\mathfrak{a}$,
- The restriction $\rho_V|_{\mathfrak{a}}$ is piecewise linear.

Remark For a reductive $\mathfrak{h}$ and for $(\tau, V) = (\mathrm{ad}, \mathfrak{h})$,

$\rho_{\mathfrak{h}}|_{\mathfrak{a}}$ = twice the usual ρ on the dominant Weyl chamber,

however, our $\rho_{\mathfrak{h}}|_{\mathfrak{a}}$ is not linear whereas the usual ρ is linear.

A constant p_V associated to $\tau\colon \mathfrak{h} \to \mathrm{End}(V)$

Let $\mathfrak{a}$ be a maximally split abelian subspace of a Lie algebra $\mathfrak{h}$. For a finite-dimensional rep $\tau\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$, we introduce:

<u>Definition</u> $p_V := \max_{Y \in \mathfrak{h}\setminus\{0\}} \dfrac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)} = \max_{Y \in \mathfrak{a}\setminus\{0\}} \dfrac{\sum |\text{eigenvalue of } \mathrm{ad}(Y) \in \mathrm{End}(\mathfrak{h})|}{\sum |\text{eigenvalue of } \tau(Y) \in \mathrm{End}(V)|}$.

<u>Short Summary</u> $\tau\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$

$\rightsquigarrow$ ρ_V $\cdots$ piecewise linear function

p_V $\cdots$ positive number

<u>Example</u> $H_0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $\mathfrak{a} = \mathbb{R}H_0 \subset \mathfrak{h} = \mathfrak{sl}(2,\mathbb{R}) \curvearrowright V = \mathbb{R}^2$

$\rho_{\mathfrak{h}}(tH_0) = \frac{1}{2}(|2t| + 0 + |-2t|) = 2|t|.$

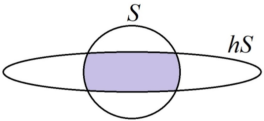
$\rho_V(tH_0) = \frac{1}{2}(|t| + |-t|) = |t|.$

$p_V = 2.$

Sketch of Proof for Theorem A: $H \curvearrowright V$ (linear)

Let H be a semisimple Lie group. Suppose $\tau \colon H \to GL_{\mathbb{R}}(V)$ has a compact kernel. As in the case $(H, V) = (SL(2, \mathbb{R}), \mathbb{R}^2)$, one has

> Theorem A For a linear action $H \curvearrowright V$, one has
> $$q(H; V) = p_V .$$
> analysis combinatorics



Proof. • For $H \ni h = k_1 e^Y k_2$, one has

$$\mathrm{vol}(hS \cap S) \sim e^{-\rho_V(Y)}.$$

• For the Haar measure dh on H, one has

$$dh \sim e^{\rho_{\mathfrak{h}}(Y)} dk_1 dY dk_2 \quad \text{(away from wall).}$$

Therefore the $L^{q+\varepsilon}$-estimate of $\mathrm{vol}(hS \cap S)$ amounts to

$$\mathrm{vol}(hS \cap S)^{q+\varepsilon} dh \sim e^{\rho_{\mathfrak{h}}(Y) - (q+\varepsilon)\rho_V(Y)} dk_1 dY dk_2. \qquad \square$$

Strategy: finding the optimal L^p-estimate of $\mathrm{vol}(gS \cap S)$

Let $G \curvearrowright X$.

$q(G;X)$: the optimal constant for L^q-estimate of $\mathrm{vol}(gS \cap S)$.

We discussed when $X = V$ (linear). Now consider $X = G/H$.

			Method
(Theorem A)	Case 1	G semisimple $\curvearrowright$ V linear	Dynamical approach
(Theorem B)	Case 2	G semisimple $\supset$ H reductive	Global geometry + Case 1

Case 2 G (semisimple) $\supset$ H (reductive)

Recall $q(G;X)$ is the optimal constant q for which $\mathrm{vol}(gS \cap S)$ is almost L^q for all compact subset $S \subset X$.

Theorem B* Let G be a semisimple Lie group, H a reductive subgroup, and $X = G/H$. Then one has

$$q(G;X) = p_{\mathfrak{g}/\mathfrak{h}} + 1.$$

analysis (for $q(G;X)$), combinatorics (for $p_{\mathfrak{g}/\mathfrak{h}}$)

Recall $p_V = \max_{\mathfrak{h} \ni Y \neq 0} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_{\mathfrak{g}/\mathfrak{h}}(Y)}$ is defined for a linear action $H \curvearrowright V$.

Point It turns out that one can control $\mathrm{vol}(gS \cap S)$ for $g \in G$ only by "ρ-function" for the subgroup H acting on $\mathfrak{g}/\mathfrak{h}$.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

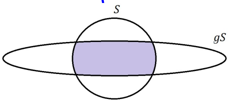
Case 2 $\underset{\text{semisimple}}{G} \supset \underset{\text{reductive}}{H}$ Global geometry + Case 1

Asymptotic estimate of volume

For any compact $S \subset G/H$, we want to find $m(g)$ and $M(g)$:

$$m(g) \le \mathrm{vol}(gS \cap S) \le M(g) \quad \text{for all } g \in G.$$

for $g \in H$



$$H \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g}/\mathfrak{h} \underset{\text{locally}}{\fallingdotseq} G/H.$$

Some difficulties to overcome:

- Need a lower bound $m(g)$ for $g \in G$, not only for $g \in H$.
-

Case 2 $\underset{\text{semisimple}}{G} \supset \underset{\text{reductive}}{H}$ Global geometry + Case 1

Asymptotic estimate of volume

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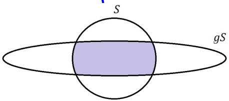
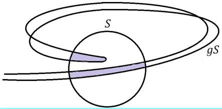




$$H \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g}/\mathfrak{h} \underset{\text{locally}}{\fallingdotseq} G/H.$$

Some difficulties to overcome:

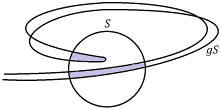
- Need a lower bound $m(g)$ for $g \in G$, not only for $g \in H$.
- An upper bound $M(g)$ is more involved.

Case 2 G semisimple $\supset$ H reductive Global geometry + Case 1

Theorem B* Let G be a semisimple Lie group, H a reductive subgroup, and $X := G/H$. Then one has

$$q(G;X) = p_{\mathfrak{g}/\mathfrak{h}} + 1.$$

analysis combinatorics



Key idea: Quantify the proof of the properness criterion** for subgroups L of G acting on G/H.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

** T. Kobayashi, Proper action on a homogeneous space of reductive type, Math. Ann., **285** (1989), 249–263.

Plan

- Methods and elementary examples
  - Optimal constant $q(G;X)$ for L^q-estimate $\mathrm{vol}(gS \cap S)$.
  - Almost L^p-representation, tempered representations.
- Tempered homogeneous spaces.
- Tempered subgroups.

Almost L^p representations

Almost L^p functions

⇝

Almost L^p representations

Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$.

Definition For $p \geq 1$, $(\pi, \mathcal{H})$ is called almost L^p if there is a dense subspace $D \subset \mathcal{H}$ such that matrix coefficients for $x, y \in D$ are almost L^p, namely,

$$(\pi(g)x, y)_{\mathcal{H}} \in \bigcap_{\varepsilon>0} L^{p+\varepsilon}(G) \qquad {}^\forall x, {}^\forall y \in D$$

Almost L^p representations

Almost L^p functions

$\Downarrow$

Almost L^p representations

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for $x, y \in D$ are almost L^p, namely,

$$(\pi(g)x, y)_{\mathcal{H}} \in \bigcap_{\varepsilon>0} L^{p+\varepsilon}(G) \quad \forall x, \forall y \in D$$

Example Suppose that G acts on (X, μ), preserving the measure μ.
Then the unitary rep $G \curvearrowright L^2(X, \mu)$ is almost $L^{q(G;X)}$.

Recall $q(G;X)$ is the optimal constant q for which $\mathrm{vol}(gS \cap S)$ is almost L^q for all compact subset $S \subset X$.

Harish-Chandra's tempered representation — Definition

Let G be a locally compact group.

Def A unitary rep π of G is called tempered if $\pi \prec L^2(G)$.

$\prec$ $\cdots$ weakly contained

i.e., every matrix coefficient of π is a uniform limit on every compacta of G by a sequence of sum of coefficients of $L^2(G)$.

Almost L^2 representation *vs* tempered representations

Definition A unitary representation π of G is called tempered if $\pi \prec L^2(G)$.

- For a semisimple Lie group G, one has

Fact C (Cowling–Haagerup–Howe)* One has the equivalence:

$$\pi \text{ is tempered} \iff \pi \text{ is almost } L^2.$$

* M. Cowling–M. Haagerup–R. Howe, Almost L^2 matrix coefficients, J. Reine Angew. Math. **387**, (1988), 97–110.

Almost L^2 representation *vs* tempered representations

Definition A unitary representation π of G is called tempered if $\pi \prec L^2(G)$.

- For a solvable Lie group G,

all unitary reps π are tempered (Hulanicki, Reiter), but are not always almost L^2.

E.g. the trivial one-dimensional rep is not almost L^p $(1 \leq p < \infty)$ if G is non-compact.

- For a semisimple Lie group G, one has

Fact C (Cowling–Haagerup–Howe)* One has the equivalence:

π is tempered $\iff$ π is almost L^2.

* M. Cowling–M. Haagerup–R. Howe, Almost L^2 matrix coefficients, J. Reine Angew. Math. **387**, (1988), 97–110.

Temperedness under disintegration

Mautner: Any unitary rep Π can be decomposed into irreducibles:

$$\Pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi \, \pi \, d\mu(\pi) \qquad \text{(direct integral).}$$

Fact Π is tempered $\Leftrightarrow$ irreducible reps π are tempered for μ-a.e.

$$\begin{aligned} \widehat{G} \quad &= \{\text{irreducible unitary reps}\} \\ \cup \quad & \\ \widehat{G}_{\text{temp}} \quad &:= \{\text{irreducible tempered reps}\}. \end{aligned}$$

That is,

$$\Pi \text{ is tempered} \iff \int_{\widehat{G}_{\text{temp}}}^{\oplus} m_\pi \pi d\mu(\pi).$$

Classification theory of the unitary dual $\widehat{G}$

<u>Fact</u> (Kirillov, Duflo) Classification of the unitary dual $\widehat{G}$ for real algebraic groups G is reduced to that for real reductive Lie groups.

Suppose G is a real reductive Lie group (*e.g.*, $GL(n,\mathbb{R})$, $O(p,q)$).

irred. admissible		irred. unitary		irred. tempered
$\mathrm{Irr}(G)$	$\supset$	$\widehat{G}$	$\supset$	$\widehat{G}_{\mathrm{temp}}$
classified		(still mysterious)		classified
Langlands		over 70 years		Knapp–Zuckerman

reduce ($\mathrm{Irr}(G)$ $\to$ $\widehat{G}_{\mathrm{temp}}$)

Tempered representations (warming up)

V. Bargmann (1947): Irreducible unitary reps of $SL(2,\mathbb{R})$

$= \{\mathbf{1}\} \amalg \{\text{principal series}\} \amalg \{\text{complementary series}\}$

$\amalg \{\text{discrete series}\} \amalg \{\text{limit of discrete series}\}$

Tempered representations (warming up)

V. Bargmann (1947): Irreducible unitary reps of $SL(2,\mathbb{R})$

$= \{ \mathbf{1} \} \amalg \{$ principal series $\} \amalg \{$ complementary series $\}$

$\amalg \{$ discrete series $\} \amalg \{$ limit of discrete series $\}$

$-\frac{1}{2}$ Casimir operator acts on them as scalars

$\{0\}$, $[\frac{1}{4}, \infty)$, $(0, \frac{1}{4})$, $\{\frac{1}{4}(n^2 - 1) : n \in \mathbb{N}_+\}$, $\{\frac{1}{4}\}$

Γ: congruence subgroup of $G = SL(2,\mathbb{R})$

Selberg's $\frac{1}{4}$ eigenvalue conjecture *:

All eigenvalues of Δ on Maas wave forms for $\Gamma \geq \frac{1}{4}$.

$\Longleftrightarrow$ The unitary rep of $G \curvearrowright L^2_{\mathrm{cusp}}(\Gamma \backslash G)$ is tempered.

Just one irred non-tempered rep would deny the conjecture.

* A. Selberg, On the estimate of Fourier coefficients of modular forms, Proc. Symp. Pure Math. 1965.

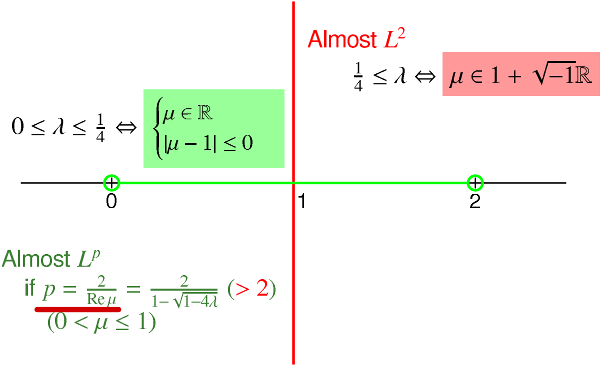
Example 1. L^p-estimate of K-finite eigenfunctions

$D = \{z \in \mathbb{C} : |z| < 1\}$ $\quad ds^2 = \frac{4(dx^2+dy^2)}{(1-|z|^2)^2}$ $\quad$ (Poincaré disc)

Any K-finite function f satisfying $\Delta f = \lambda f$ is almost $L^{p(\lambda)}$ $(\lambda > 0)$,

where $p(\lambda) := \frac{2}{1-\sqrt{1-4\lambda}}$ $(0 \le \lambda \le \frac{1}{4})$; $= 2$ $(\frac{1}{4} \le \lambda)$.

Figure in the μ-plane with $\lambda = -\frac{1}{4}(\mu^2 - 2\mu)$.



Irreducible tempered reps — semisimple Lie groups

Def A unitary representation π of G is called tempered if $\pi \prec L^2(G)$.

- For a semisimple Lie group G and irreducible $\pi \in \widehat{G}$, tempered representations π have been studied extensively.

Known results on tempered reps and beyond ...

- Many equivalent definitions, *e.g.*, $L^{2+\varepsilon}(G)$,
- Harish-Chandra's theory towards Plancherel formula,
- Knapp–Zuckerman's classification *,
- A cornerstone of Langlands' classification,
- Selberg $\frac{1}{4}$ eigenvalue conjecture (1965-),
- Gan–Gross–Prasad conjecture, $\cdots$

$*$ A. W. Knapp–G. Zuckerman, Classification of irreducible tempered representations of semisimple Lie groups, Ann. Math., (1980), 389–455; 457–501.

Tempered homogeneous spaces and tempered subgroups

$G \supset H$ Lie groups

- Induction

Definition We say G/H is a tempered homogeneous space if $L^2(G/H)$ is a tempered rep of G.

- Restriction

Definition We say H is a G-tempered subgroup if $\pi|_H$ is tempered for any $\pi \in \widehat{G} \setminus \{\mathbf{1}\}$.

cf. Margulis used the terminology "G-tempered subgroup" in a stronger sense by using an L^1 -estimate rather than an $L^{2+\varepsilon}$ -estimate.

Basic questions on Harish-Chandra's tempered representations

$G \supset H$ Lie groups

Problem 2 (induction) Find a criterion for (G, H) such that $L^2(G/H)$ is a tempered rep of G.

Problem 3 (restriction) Find a criterion for (G, H) such that the restriction $\pi|_H$ is a tempered rep of H $\forall \pi \in \widehat{G} \setminus \{\mathbf{1}\}$.

Problem 3 is related to the existence problem of cocompact discontinuous groups Γ for G/H.

Tempered homogeneous space $X = G/H$, *i.e.*, $L^2(X) \prec L^2(G)$

Problem 2 When is the unitary rep on $L^2(X)$ tempered?

$\Downarrow\!\!\!\!/$

cf. $L^2(X)$ can be disintegrated by irred X-tempered reps (this is almost 'tautology'). (Harish-Chandra, Oshima, Bernstein ~ 80s).

Towards a temperedness criterion

Problem 2 For which pair $G \supset H$, is the unitary rep of G on $L^2(G/H)$ tempered?

For semisimple Lie groups G, we have already discussed a refinement of Problem 2 as below:

Problem 1 Find the optional constant $q(G; G/H)$ for which $\mathrm{vol}(gS \cap S)$ is almost L^q for all compact subset $S \subset G/H$.

$$q(G; G/H) \leq 2 \Longleftrightarrow L^2(G/H) \text{ is tempered.}$$

Temperedness criterion in the reductive case

G semisimple Lie group,
H any reductive subgroup.

Since we know from Theorem B that

$$q(G; G/H) = p_{\mathfrak{g}/\mathfrak{h}} + 1$$

analysis — combinatorics

where $p_V = \max_{\mathfrak{h} \ni Y \neq 0} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$ is defined for a linear action $H \curvearrowright V$, one obtains from the volume estimate:

Theorem B′* For a pair of real reductive Lie groups, one has $L^2(G/H)$ is G-tempered $\Longleftrightarrow$ $p_{\mathfrak{g}/\mathfrak{h}} \leq 1$.

Remark. $p_{\mathfrak{g}/\mathfrak{h}} \leq 1 \iff 2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ on $\mathfrak{h}$.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

Easy example

$p_V = \max_{\mathfrak{h} \ni Y \neq 0} \dfrac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$ is defined for a linear action $H \curvearrowright V$.

Obviously, $p_V = 1$ if we take $V = \mathfrak{h}$.

The temperedness criterion

$$L^2(G/H) \text{ is tempered } \iff p_{\mathfrak{g}/\mathfrak{h}} \leq 1$$

applied to the very special case where $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{h}$ implies that $L^2(G \times G/\Delta G)$ and $L^2(G_{\mathbb{C}}/G)$ are tempered representations.

Example $L^2(GL(n,\mathbb{C})/GL(n,\mathbb{R}))$, $L^2(GL(n,\mathbb{C})/U(p,n-p))$ and $L^2(GL(2m,\mathbb{C})/GL(m,\mathbb{H}))$ are tempered.

Plan of Lectures

- Talk 1: (February 17, 2025)
  Tempered homogeneous spaces
  —Dynamical approach

- Talk 2: (February 19, 2025)
  Classification theory of tempered G/H
  —Combinatorics of convex polyhedra

- Talk 3: (February 21, 2025)
  Tempered homogeneous spaces
  —Interaction with topology and geometry

Thank you for your attention!

References

The theme of the mini-course is joint with Yves Benoist.

Tempered Homogeneous Spaces:

——— I. (J. Euro Math., 2015)

Method (Dynamical System)

——— II. (Margulis Festschrift, 2022, Chicago Univ. Press)

Representation Theory

——— III. (J. Lie Theory, 2021)

Classification Theory (Combinatorics)

——— IV. (J. Inst. Math. Jussieu, 2023)

Limit algebra, geometric quantization

Tensor product of GL_n (J. Algebra, 2023)