Ramification groups of local fields (joint work with A. Abbes)

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Abstract

We recall the definition and basic properties of ramification groups. We compute it in the most elementary cases. One is the classical case, i.e., the case where the residue field is perfect. The other is the abelian and the equal characterictic case.

Plan

1. Ramification groups.
2. Classical case.
3. Rank 1 case.

(Possible) Applications.

1. Grothendieck-Ogg-Shafarevich formula, conductor formula (in progress).
2. Ramification of finite flat group schemes, by S. Hattori.

1 Ramification groups.

Let K be a complete discrete valuation field and L be a finite Galois extension of Galois group G. The lower numbering filtration $G_i \subset G, (i \in \mathbb{N})$ is defined by

$$G_i = \mathrm{Ker}(G \to \mathrm{Aut}(O_L/m_L^i)).$$

They have the properties:

1. Easy to define.
2. Not compatible with quotients.

The upper numbering filtration has the opposite properties:

1. Definition is sophisticated.
2. Compatible with quotients.

Geometric interpretation of the definition of the lower numbering filtration. Take a presentation $O_L = O_K[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$. Then $G = Hom_{O_K\text{-alg}}(O_L, O_{\bar{K}})$ is identified with the subset

$$G = \{x = (x_1, \ldots, x_n) \in D^n | f_1(x) = \ldots = f_m(x) = 0\}$$

of the n-dimensional disk D^n of radius 1. Then G_i is defined by

$$G_i = G \cap D(i, z)$$

where z corresponds to the unit of G, $D(i, z) = \{x \in D^n | d(x, z) \geq i\}$ is a subdisk and $d(x, z) = \min_k \operatorname{ord}_L(x_k - z_k)$. Idea of the definition is that

$x \in G$ is close to $e \in G$ if the distance $p^{-d(x,e)}$ is small.

A defect in the definition is that the way G is defined and the way the distance is measured are unrelated.

To remedy it, we make the following definition of the upper numbering filtration. For a rational number $j \geq 0$, we define an affinoid subdomain $X^j \subset D^n$ by requiring that the underlying set is given by

$$\{x \in D^n | \operatorname{ord}_K f_k(x) \geq j \text{ for } j = 1, \ldots, m\}.$$

We have $G = X^\infty = \bigcap_j X^j$. Now we can define the upper numbering filtration by following the idea

$x \in G$ is close to $e \in G$ if they are in the same connected component.

Definition 1 *For a rational number $j \geq 0$, we put*

$$G^j = \{x \in G | x \text{ is in the same geometric connected component of } X^j \text{ as } e \in G\}.$$

Here, if $A^j = K\langle X_1, \ldots, X_n\rangle\langle f_1/\pi^j, \ldots, f_m/\pi^j\rangle$ denotes the affinoid algebra over K defining X^j, the geometric connected component means that in $\operatorname{Spec} A^j \otimes_K \bar{K}$.

With this definition, we have

Theorem 2 *1. For a rational number $j \geq 0$, $G^j \subset G$ is a normal subgroup.*

2. The filtration is compatible with quotient.

3. There exist a finite number of rational numbers $0 = j_0 \leq j_1 \leq \ldots \leq j_m$ such that G^j is constant for $j \in (j_{k-1}, j_k]$ and is 1 for $j \in (j_m, \infty)$.

4. $G^{j_k}/G^{j_{k+1}}$ is an abelian p-group for $j_k > 1$ at least if p is not a prime element.

Variants.

1. logarithmic version. Replace X^j by an affinoid subdomain $X^j_{\log}$ defined as follows. Assume z_1 is a prime element and put $e = e_{L/K}$ and $e_i = \operatorname{ord}_L z_i$ for $i \geq 2$. Take polynomials $u(x)$ and $v_i(x)$ satisfying $z_1^e/\pi = u(z)$ and $z_1^{e_i}/z_i = v_i(z)$. Then we define $X^j_{\log}$ by further imposing the conditions $\operatorname{ord}_K(x_1^e/\pi - u(x)) \geq j$ and $\operatorname{ord}_K(x_1^{e_i}/x_i - v_i(x)) \geq j$.

2. We may replace the surjection $O_K[X_1, \ldots, X_n] \to O_L$ by an arbitrary surjection $A \to O_L$ from a smooth O_K-algebra A.

2 Classical case

Assume L is totally ramified and $O_L = O_K[x]/f(x)$. We assume that the image $z = z_n$ of x is a prime element of L and hence $f(x)$ is an Eisenstein polynomial of degree n. Then the affinoid variety X^j is described as follows using the Newton polygon of $h(x) = f(x+z)$ as follows. The NP of $h(x) = x^n + b_1 x^{n-1} + \cdots + b_{n-1}x$ is defined as

the convex hull of of $(0,0), (1, \mathrm{ord}_L b_1), \ldots, (n-1, \mathrm{ord}_L b_{n-1})$.

Figure 2.

If we arrange the solution $z_1, \ldots, z_n = z$ of $f(x)$ in such a way that

$$\mathrm{ord}_L(z_1 - z_n) \leq \mathrm{ord}_L(z_2 - z_n) \leq \cdots \leq \mathrm{ord}_L(z_{n-1} - z_n),$$

the slope on the interval $[t-1, t]$ is given by $\mathrm{ord}_L(z_t - z_n)$.

For $\mathrm{ord}_L(z_t - z_n) < i \leq \mathrm{ord}_L(z_{t+1} - z_n)$, we have

$$\mathrm{ord}_K f(x+z) = \mathrm{ord}_K h(x) = \frac{1}{e_{L/K}} \left(\sum_{s \leq t} \mathrm{ord}_L(z_s - z_n) + (n-t)i \right)$$

generically on $z + D_i$. Thus if we put

$$\varphi(i) = \frac{1}{e_{L/K}} \left(\sum_{s \leq t} \mathrm{ord}_L(z_s - z_n) + (n-t)i \right),$$

one can show that the inverse image $X^{\varphi(i)}$ of the disk of radius $\varphi(i)$ is a disjoint union of disks of radius $i/e_{L/K}$. Further, we have the $G_i = G^{\varphi(i)}$.

The function $\varphi(i)$ is called the Herbrand function. It is equal to the minimum of the difference:

$$\varphi(i) = \frac{1}{e_{L/K}} \min_{0 \leq t \leq n-1} (p(t) - i(t-n))$$

where $p(t)$ is the defining function of the Newton polygon. In other words,

$$\begin{aligned} \varphi(i) &= \frac{1}{e_{L/K}} \min_t (i(n-t) + \mathrm{ord}_L b_t) \\ &= \mathrm{ord}_K h(\pi^{i/e_{L/K}} x). \end{aligned}$$

Thus, if

NP has slope i_{k-1} on the left of $n - g_k$ and i_k on the right of $n - g_k$

then

Hf has slope g_k on the interval $[i_{k-1}, i_k]$.

In this case, we have $g_k = |G^j|$ for $j \in (\varphi(i_{k-1}), \varphi(i_k)]$. More precisely,

$$G^j = \{x \in G | \mathrm{ord}(x - z) \geq i_k\}.$$

The fact that g_k is a power of p is a consequence of the following elementary lemma + no cancellation.

Lemma 3 *The vertices of the Newton polygon of $(x+1)^n - 1$ are*

$$(0,0), (n - p^e, 0), (n - p^{e-1}, \mathrm{ord}p), \ldots, (n-1, e \cdot \mathrm{ord}p)$$

where e is the p-adic valuation of n.

3 Abelian and equal characteristic case

Assume K is of characteristic $p > 0$. Then then Artin-Schreier-Witt theory gives a surjection $W_{m+1}(K) \to H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$. Brylinski defined an increasing filtration on $W_{m+1}(K)$ by

$$F_r W_{m+1}(K) = \{(x_0, \ldots, x_m) | p^{m-i} \mathrm{ord} x_i \geq -r \text{ for } i = 0, \ldots, m\}.$$

Theorem 4 *The image of $F_\bullet$ on $H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) = Hom(G_K^{\mathrm{ab}}, \mathbb{Z}/p^{m+1}\mathbb{Z})$ is the dual of the logarithmic upper numbering filtration. More precisely, for an integer $r \geq 1$, we have $G_{K,\log}^{\mathrm{ab},j} = G_{K,\log}^{\mathrm{ab},r}$ for $j \in (r-1, r]$ and*

$$F_r H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) = Hom(G_{K,\log}^{\mathrm{ab},r}, \mathbb{Z}/p^{m+1}\mathbb{Z}).$$

Remark. The image of $F_\bullet$ on $H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ is known to coincide with the filtration defined by Kato.

Let F be the residue field. We put

$$\Omega_F(\log) = (\Omega_F \oplus F \otimes K^\times)/(da - a \otimes a, a \in O_K \cap K^\times).$$

We have an exact sequence $0 \to \Omega_F \to \Omega_F(\log) \xrightarrow{\mathrm{res}} F \to 0$. We define a map

$$R_r : Gr_r^F W_{m+1}(K) \to Hom(m_K^r / m_K^{r+1}, \Omega_F(\log))$$

by

$$R_r(x_0, \ldots, x_m) = x_0^{p^m} d\log x_0 + \cdots + x_m d\log x_m.$$

It is not obvious that the map is well-defined. However, one can prove it in the following way.

Assume F is a function field over a perfect subfield F_0. Let t_1 be a prime element of K, $\bar{t}_2, \ldots, \bar{t}_d$ be a separating transcendental basis of F over F_0 and $t_2, \ldots, t_d$ be their liftings. Let $\mathcal{K}$ be the fraction field of the completion of the localization of $O_K[u_1, \ldots, u_d]$ at (t_1) and $\alpha : K \to \mathcal{K}$ be the inclusion. We define another map $\beta : K \to \mathcal{K}$ by $t_i \mapsto t_i(1 + u_i t_1^r)$ and requiring that the induced map on the residue field is the canonical map. Then, some elementary computations show the following.

Proposition 5 *The map $\beta - \alpha : W_{m+1}(K) \to W_{m+1}(\mathcal{K})$ sends $F_r W_{m+1}(K)$ to $F_0\mathcal{K} = O_{\mathcal{K}}$. Further the following diagram is commutative*

$$\begin{array}{ccc} F_r W_{m+1}(K) & \xrightarrow{\beta-\alpha} & F_0\mathcal{K} \\ \downarrow & & \downarrow \\ Gr_r^F W_{m+1}(K) & & O_{\mathcal{K}}/m_{\mathcal{K}} \\ {\scriptstyle R_r}\downarrow & & \uparrow\cup \\ Hom(m_K^r/m_K^{r+1}, \Omega_F(\log)) & \xrightarrow{T_r} & \sum F u_i. \end{array}$$

The bottom horizontal arrow T_r is an isomorphism sending the basis $t_1^r \mapsto d\log t_i$ to u_i.

Proposition 5 forces R_r to be well-defined.

Corollary 6 (Kato) *The map $R_r : Gr_r^F W_{m+1}(K) \to Hom(m_K^r/m_K^{r+1}, \Omega_F(\log))$ induces an injection* $\mathrm{rsw}_r : Gr_r^F H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to Hom(m_K^r/m_K^{r+1}, \Omega_F(\log))$.

Sketch of Proof of Theorem 4. Let X be a smooth scheme over F_0 and D be a smooth divisor of X. Let ξ be a generic point of D and assume O_K is identified with the completion of the discrete valuation ring $O_{X,\xi}$.

Let $Y \to X$ be a cyclic covering of degree p^{m+1} étale on the complement $U = X \setminus D$. Assume it gives a cyclic extention L over K. Then, a surjection $A \to O_L$ from a smooth O_K-algebra A is given by the immersion $Y \to Y \times X$ to a smooth X-scheme. Since the diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y \times X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

is cartesian, the upper ramification group of $G = \mathrm{Gal}(L/K)$ is computed by the map $Y \times X \to X \times X$.

Let $\mathcal{F}$ be a smooth rank 1 sheaf on U defined by a faithful character of the Galois group of $\mathrm{Gal}(Y/X)$. We consider a smooth sheaf $\mathcal{H} = \mathcal{H}om(p_2^*\mathcal{F}, p_1^*\mathcal{F})$ on $U \times U$. The restriction of $\mathcal{H}$ to $U \times V$ is the same as the sheaf defined by the upper right vertical arrow in the cartesian diagram

$$\begin{array}{ccc} V \times U & \longleftarrow & V \times V \\ \downarrow & & \downarrow \\ U \times U & \longleftarrow & U \times V \\ \downarrow & & \downarrow \\ U & \longleftarrow & V. \end{array}$$

Then, Proposition 5 has the following geometric interpretation. Let $(X \times X)^{(0)} \to X \times X$ be the blow-up at $D \times D \subset X \times X$. The diagonal map $X \to X \times X$ induce the log diagonal map $X \to (X \times X)^{(0)}$. Let $(X \times X)^{(1)} \to (X \times X)^{(0)}$ at the image of $D \subset X$ by the log diagonal map $X \to (X \times X)^{(0)}$. Applying the construction inductively, we obtain a sequence of blow-ups

$$(X \times X)^{(r)} \to \cdots \to (X \times X)^{(1)} \to (X \times X)^{(0)} \to X \times X.$$

Figure 3.

The local field $\mathcal{K}$ is identified with the local field of $(X \times X)^{(r)}$ at the generic point of the exceptional divisor $E^{(r)}$ of the last blow-up. The maps α and β are identified with the maps induced by the first and the second projections respectively. Then Proposition 5 implies that the sheaf $\mathcal{H}$ has a smooth extension along $E^{(r)}$ (after shrinking X). Further Corollary 6 implies that its restriction on $E^{(r)}$ is an Artin-Schreier sheaf defined by a non-trivial linear form. Thus Theorem 4 follows.