Characteristic cycle of an ℓ-adic sheaf on an alg. surface [1]

k alg closed. char $p>0$. $\ell \neq p$.
X smooth alg var $/k$ $d = \dim X$
$\mathcal{F}$ ℓ-adic sheaf on X.
Analogy with micro local analysis on $\mathcal{D}$-mod $/\mathbb{C}$

<u>Expectation</u>. One can define the char cycle $\mathrm{Char}(\mathcal{F})$ as an alg cycle of dim d on the cotangent bundle T^*X satisfying

1) additivity $\mathrm{Char}(\mathcal{F}) = \mathrm{Char}(\mathcal{F}') + \mathrm{Char}(\mathcal{F}'')$ for an exact sequen $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$.
   ↳ suffices to consider $j_!\mathcal{F}$ for a dense openim $j: U \to X$, $\mathcal{F}$ smooth on U.

2) étale local
3) Euler number X proper (more generally $f: X \to Y$ proper)
$$\chi(X.\mathcal{F}) = (\mathrm{Char}\,\mathcal{F}.\, T^*_X X)_{T^*X}$$

4) vanishing cycle $f: X \to C$ flat morphism to smooth curve
$u \in X$ isolated characteristic pt. $(\mathrm{Char}\,\mathcal{F}.\, df)_{T^*X,u}$ def'd
$$-\dim\mathrm{tot}\,\phi_u(\mathcal{F}, f) = \quad =$$
(cx of vanishing cycles: coh sheaves = ℓ-adic rep of the abs Galois gp of the local field K_v of C at $v = f(u)$.
alternating sum of dim + Sw ← measure of wild ram'n Swan conductor.

<u>Example</u> 1). tamely ramified $D \subset X$ div. w/ S.N.C $U = X - D$
smooth $\mathcal{F}$ on U tamely ram'd along D $j: U \hookrightarrow X$
$$\mathrm{Char}(j_!\mathcal{F}) = (-1)^d\,\mathrm{rank}\,\mathcal{F} \cdot \sum_{I \subset \{1,\dots,n\}} [T^*_{X_I} X]$$
$D_1, \dots, D_n$ irred cpt of D $I \subset \{1,\dots,n\}$ $X_I = \bigcap_{i \in I} D_i$. conormal b'dle

2) X curve $D \subset X$ div
$$\mathrm{Char}(j_!\mathcal{F}) = -(\mathrm{rk}\,\mathcal{F} \cdot [T^*_X X] + \sum_{x \in D} \dim\mathrm{tot}_x \mathcal{F} \cdot [T^*_x X])$$
0-section fiber

<u>Theorem</u> X smooth surface $\mathcal{F}/U \overset{j}{\hookrightarrow} X$

$\mathrm{Char}(j_!\mathcal{F})$ is defined and satisfies the expected properties.

Define $\mathrm{Char}(j_!\mathcal{F})$ so that 4) is satisfied $\Rightarrow$ 3) is also satisfied.

How to define?

$\mathrm{Char}(j_!\mathcal{F}) = \mathrm{rank}\,\mathcal{F}\cdot[T^*_X X]$ + contribution of codim 1 + $\cdots$ codim 2.

$$\sum_{D_i}\sum \underline{\text{coeff}}\times\left[\frac{\text{sub line bundle of}}{T^*X\times_X D_i}\right] \qquad \sum_x \frac{n_x}{\uparrow}[T^*_x X]$$

determine coefficient

requires ramification gp of loc. field with imperfect residue field (Abbes – S.)

X smooth (arbitrary dim.) D smooth irr div. ξ gen pt of D

K local field at ξ = $\mathrm{Frac}(\hat{\mathcal{O}}_{X,\xi})$ — discrete valuation ring

F res. field $= \kappa(\xi)$ function field of D imperfect if $\dim X > 1$

$G_K = \mathrm{Gal}(K^{sep}/K) \supset G_K^r$ (non log) fil by ramification gps

dec fil'n by closed normal subgroups indexed by $r\in\mathbb{Q},\ r\geq 1$

$G_K^1 = I_K$ inertia $\qquad G_K^{r+} = \overline{\bigcup_{s>r} G_K^s} \subset G_K^r$

$\cup$

$G_K^{1+} = P_K$ wild inertia = pro p Sylow of I_K.

<u>Prop</u> $r>1$. $\mathrm{Gr}^r G_K = G_K^r/G_K^{r+}$ is abelian & killed by p.

A can. inj $\mathrm{Gr}^r G_K^\vee = \mathrm{Hom}(\mathrm{Gr}^r G_K, \mathbb{F}_p) \longrightarrow m_{K^{sep}}^{-r} \otimes_{\mathcal{O}_{K^{sep}}} \bar{F} \otimes_{\mathcal{O}_K} \Omega^1_{X,\xi}$.

$\mathcal{F}$ on $U = X - D$ — ℓ-adic repn of G_K — slope dec'n fil.; dec. by char abelian; $\downarrow$ ← inj; line in $\bar{F}\otimes_{\mathcal{O}_K}\Omega^1_{X,\xi}$ ⟹ contribution in codim 1

Radon transform X proj $\mathcal{L}$ invertible $\mathcal{O}_X$-module very ample

$X \hookrightarrow \mathbb{P} = \mathbb{P}(E^\vee) = E^\vee - \{0\}/k^\times$ $E = \Gamma(X, \mathcal{L})$

$\mathbb{P}^\vee = \mathbb{P}(E) = \{H \mid H \subset \mathbb{P} \text{ hyperplane}\}$ dual

$\mathbb{H} = \{(x, H) \mid x \in H\} \subset \mathbb{P} \times \mathbb{P}^\vee$ univ. hyperplane

$X \times_{\mathbb{P}} \mathbb{H}$ ← universal hyperplane section

$$\begin{array}{ccc} X \times_{\mathbb{P}} \mathbb{H} & \xrightarrow{q} & \mathbb{P}^\vee \\ {\scriptstyle p}\downarrow & & \\ X & & \end{array}$$

Fiber at $H \in \mathbb{P}^\vee$ $= X \cap H$

$R_{\mathcal{L}}(j_!\mathcal{F}) = Rq_*^{*} p^* j_!\mathcal{F}$ Radon transform.

$x \in X$ — $H_x \subset \mathbb{P}^\vee = \{H \mid x \in H\}$ dual hyperplane div of $\mathbb{P}^\vee$

use ramification of $R_{\mathcal{L}}(j_!\mathcal{F})$ along H_x

to define the coefficient of $[T_x^*X]$

Def'n $\Rightarrow$ Property 4) $\Rightarrow$ indep't of $\mathcal{L}$, étale local

↑ + stability of deformation nearby cycle ⇓ Grothendieck-Ogg-Shafarevich

Euler number