INTRODUCTION TO WILD RAMIFICATION OF SCHEMES AND SHEAVES

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1. Brief summary on étale cohomology

In this section, k denotes a field, a scheme will mean a separated scheme of finite type over k and a morphism of schemes will mean a morphism over k. We put $p = \text{char } k$.

1.1. **Definition and examples of étale sheaves.** A morphism $X \to Y$ of schemes is said to be étale if $\Omega^1_{X/Y} = 0$ and if X is flat over Y.

Example 1.1. *An open immersion is étale.*

The morphism $\mathbf{G}_m = \text{Spec } k[T^{\pm 1}] \to \mathbf{G}_m$ *defined by* $T \mapsto T^m$ *is étale if and only if* m *is invertible in* k. *It is called the Kummer covering of degree* m.

The morphism $\mathbf{G}_a = \text{Spec } k[T] \to \mathbf{G}_a$ *defined by* $T \mapsto T^p - T$ *is étale if* $p > 0$. *It is called the Artin-Schreier covering.*

A family $(U_i \to X)_{i \in I}$ of étale morphisms is called an étale covering if $X = \bigcup_{i \in I} \text{Image}(U_i \to X)$.

A contravariant functor (= presheaf) $\mathcal{F}$: (Etale schemes over X) $\to$ (Sets) (or $\to$ (Abelian groups)) is called a sheaf if it satisfies the patching conditions

$$\mathcal{F}(U) \xrightarrow{\simeq} \text{Ker}\left(\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_{ij})\right)$$

for every étale covering $(U_i \to U)_{i \in I}$. Here $U_{ij} = U_i \times_U U_j$ and Ker denotes the set (or the abelian group)

$$\left\{ (s_i) \in \prod_{i \in I} \mathcal{F}(U_i) \;\middle|\; s_i|_{U_{ij}} = s_j|_{U_{ij}} \text{ for every } i, j \in I \right\}.$$

Example 1.2. *A representable functor* $\text{Hom}(-, Y)$ *is a sheaf. In particular, a constant sheaf, the additive sheaf* $\mathbf{G}_a$, *the mulitiplicative sheaf* $\mathbf{G}_m$ *etc. are actually sheaves.*

A morphism $f\colon X \to Y$ of schemes defines the push-forward functor (Etale sheaves$/X$) $\to$ (Etale sheaves$/Y$) by $f_*\mathcal{F}(V) = \mathcal{F}(V \times_Y X)$. It has a left adjoint functor f^*. For an open immersion $j\colon U \to X$, the extension by 0 is defined by $j_!\mathcal{F} = \mathrm{Ker}(j_*\mathcal{F} \to i_*i^*j_*\mathcal{F})$.

We will use the same notation for a scheme and for the corresponding representable functor. An étale sheaf representable by a finite étale scheme is called a locally constant constructible sheaf.

For a group scheme G over X, we say an étale sheaf T with G-action $G \times T \to T$ is an G-torsor if étale locally on X, T is isomorphic to G with the canonical action of G.

Example 1.3. *For an invertible $\mathcal{O}_X$-module $\mathcal{L}$, the sheaf $\mathcal{I}som(\mathcal{O}_X, \mathcal{L})$ is a $\mathbf{G}_m$-torsor. Conversely, a $\mathbf{G}_m$-torsor on X defines an invertible $\mathcal{O}_X$-module by flat descent.*

1.2. **Etale cohomology.** Etale sheaves of abelian groups on a scheme X form an abelian category with enough injectives and hence the right derived functor $H^i(X, -)$ of the left exact functor $\Gamma(X, -)$ is defined. The compact support cohomology $H^i_c(X, -)$ is defined as $H^i(\bar{X}, j_!-)$ by taking a compactification $j\colon X \to \bar{X}$. If X itself is proper, we have $H^i_c(X, -) = H^i(X, -)$.

If G is a commutative group scheme on X, the set of isomorphism classes of G-torsors on X is canonically idenitified with $H^1(X, G)$. In particular, $H^1(X, \mathbf{G}_m)$ is canonically identified with the Picard group $\mathrm{Pic}(X)$ defined as the group of isomorphism classes of invertible $\mathcal{O}_X$-modules. For an integer n invertible in k, the Kummer sequence $0 \to \mu_n \to \mathbf{G}_m \overset{t \mapsto t^n}{\to} \mathbf{G}_m \to 0$ induces an exact sequence

$$0 \to \Gamma(X, \mathcal{O}_X)^\times/(\Gamma(X, \mathcal{O}_X)^\times)^n \to H^1(X, \mu_n) \to \mathrm{Pic}(X)[n] \to 0. \tag{1}$$

Example 1.4. *If X is a proper smooth geometrically connected curve, it gives an isomorphism*

$$H^1(X_{\bar{k}}, \mu_n) \to \mathrm{Jac}_X(\bar{k})[n] \tag{2}$$

where Jac_X denotes the Jacobian variety of X. Further, we have an isomorphism

$$\mathrm{Pic}(X_{\bar{k}})/n\mathrm{Pic}(X_{\bar{k}}) = \mathbf{Z}/n\mathbf{Z} \to H^2(X_{\bar{k}}, \mu_n) \tag{3}$$

and vanishing $H^q(X_{\bar{k}}, \mu_n) = 0$ for $q > 2$.

1.3. **Fundamental group.** For a geometric point $\bar{x}$ of a connected scheme X, the fundamental group $\pi_1(X, \bar{x})$ is defined by requiring that the fiber functor

$$\begin{aligned}(4)\qquad &(\text{Finite étale schemes}/X)\\ &\to (\text{Finite sets with continuous action of } \pi_1(X,\bar{x}))\end{aligned}$$

defined by $X \to X(\bar{x})$ is an equivalence of categories. If we identify a commutative finite étale scheme A over X with a finite abelian group A with the action of $\pi_1(X,\bar{x})$, the étale cohomology $H^1(X,A)$ is identified with the cohomology $H^1(\pi_1(X,\bar{x}),A)$ of the profinite group $\pi_1(X,\bar{x})$.

If X is a normal integral scheme and K is the function field, the base change by the canonical map Spec $K \to X$ defines a fully faithful contravariant functor

$$(\text{Finite étale schemes } /X) \to (\text{Finite étale } K\text{-algebras}).$$

A quasi-inverse on the essential image is defined by the normalization of X in a finite étale K-algebra. The functor defines an surjection $G_K = \mathrm{Gal}(\bar{K}/K) \to \pi_1(X,\bar{\eta})$ where $\bar{\eta}$ denotes the generic geometric point defined by a separable closure $\bar{K}$.

We fix a prime number ℓ different from the characteristic of k. We call an inverse system $\mathcal{F} = (\mathcal{F}_n)$ of locally constant constructible sheaves of free $\mathbf{Z}/\ell^n\mathbf{Z}$-modules satisfying $\mathcal{F}_{n+1}/\ell^n\mathcal{F}_{n+1} \xrightarrow{\simeq} \mathcal{F}_n$ for every n a smooth $\mathbf{Z}_\ell$-sheaf. If X is connected, the equivalence (4) of categories induces an equivalence of categories

$$(\text{Smooth } \mathbf{Z}_\ell\text{-sheaves}/X) \to (\text{Cont. } \mathbf{Z}_\ell\text{-representations of } \pi_1(X,\bar{x})).$$

1.4. **The Euler number.** For a smooth $\mathbf{Q}_\ell$-sheaf $\mathcal{F}\otimes\mathbf{Q}_\ell$, its cohomology and the compact support cohomology are defined by $H^i(X_{\bar{k}},\mathcal{F}\otimes\mathbf{Q}_\ell) = \varprojlim_n H^i(X_{\bar{k}},\mathcal{F}_n)\otimes\mathbf{Q}_\ell$ and $H^i_c(X_{\bar{k}},\mathcal{F}\otimes\mathbf{Q}_\ell) = \varprojlim_n H^i_c(X_{\bar{k}},\mathcal{F}_n)\otimes\mathbf{Q}_\ell$. They are known to be a $\mathbf{Q}_\ell$-vector space of finite dimension and known to be 0 for $i > 2\dim X$. The Euler numbers are defined by

$$\chi(X_{\bar{k}},\mathcal{F}) = \sum_{i=0}^{2\dim X} (-1)^i \dim H^i(X_{\bar{k}},\mathcal{F}),$$

$$\chi_c(X_{\bar{k}},\mathcal{F}) = \sum_{i=0}^{2\dim X} (-1)^i \dim H^i_c(X_{\bar{k}},\mathcal{F})$$

respectively and known to be equal: $\chi(X_{\bar{k}},\mathcal{F}) = \chi_c(X_{\bar{k}},\mathcal{F})$.

Example 1.5. *If X is a proper smooth curve of genus g, we have $\dim H^0(X_{\bar{k}},\mathbf{Q}_\ell) = 1, \dim H^1(X_{\bar{k}},\mathbf{Q}_\ell) = 2g, \dim H^2(X_{\bar{k}},\mathbf{Q}_\ell) = 1$ and $\chi(X_{\bar{k}},\mathbf{Q}_\ell) = 2-2g$. If $U = X \setminus D$ is the complement of a finite étale divisor D of degree $d > 0$ in a proper smooth curve X of genus g, we* *have* $\chi_c(U_{\bar k}, \mathbf{Q}_\ell) = 2-2g-d$ *by the long exact sequence* $\to H^i_c(U_{\bar k}, \mathbf{Q}_\ell) \to H^i(X_{\bar k}, \mathbf{Q}_\ell) \to H^i(D_{\bar k}, \mathbf{Q}_\ell) \to$.

If X *is a proper smooth scheme of dimension* d*, we have an equality* $\chi(X_{\bar k}, \mathbf{Q}_\ell) = (-1)^d \deg c_d(\Omega^1_{X/k})$ *with the degree of the top Chern class. If* $U = X \setminus D$ *is the complement of a divisor* D *with normal crossings in a proper smooth scheme* X *of dimension* d*, we have* $\chi_c(U_{\bar k}, \mathbf{Q}_\ell) = \deg(-1)^d c_d(\Omega^1_{X/k}(\log D))$.

2. Formula for the Euler number

In this section, we keep the assumption that k is a field and schemes are separated of finite type over k. We discuss the following problem.

Problem. *How do we compute* $\chi_c(U_{\bar k}, \mathcal{F})$ *for a smooth* $\mathbf{Q}_\ell$*-sheaf on a smooth scheme* U*?*

In fact, we will be interested in computing the difference $\chi_c(U_{\bar k}, \mathcal{F}) - \text{rank } \mathcal{F} \cdot \chi_c(U_{\bar k}, \mathbf{Q}_\ell)$. If char $k = 0$, we have $\chi_c(U_{\bar k}, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U_{\bar k}, \mathbf{Q}_\ell)$. In the following, we assume k is a perfect field of characteristic $p > 0$.

2.1. The Grothendieck-Ogg-Shafarevich formula.

If $\dim U = 1$, we know a classical formula.

Theorem 2.1 (Grothendieck-Ogg-Shafarevich formula [11]). *Let* X *be a smooth proper curve and* U *be a dense open subscheme. Then, for a smooth* $\mathbf{Q}_\ell$*-sheaf on* U*, we have*

$$\chi_c(U_{\bar k}, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U_{\bar k}, \mathbf{Q}_\ell) - \sum_{x \in X \setminus U} \mathrm{Sw}_x \mathcal{F}. \tag{5}$$

Example 2.1. *Let* $\pi\colon V \to U$ *be a finite étale morphism and* $\mathcal{F} = \pi_* \mathbf{Q}_\ell$ *be the locally constant sheaf corresponding to the induced representation* $\mathrm{Ind}^{\pi_1(U)}_{\pi_1(V)} \mathbf{Q}_\ell$*. Let* X *and* Y *denote the smooth compactifications of* U *and* V *respectively and* $\bar\pi\colon Y \to X$ *be the induced morphism. let* $D \subset X$ *and* $E \subset Y$ *denote the reduced divisors such that the complements are* U *and* V *respectively. Then,* (5) *for* $\mathcal{F} = \pi_* \mathbf{Q}_\ell$ *gives*

$$\begin{aligned}\chi_c(V, \mathbf{Q}_\ell) =& [V : U] \cdot \chi_c(U, \mathbf{Q}_\ell) \\ & - \sum_{x \in X \setminus U} \sum_{y \in \pi^{-1}(x)} \mathrm{length}_{\mathcal{O}_{Y,y}} (\Omega^1_Y(\log E)/\bar\pi^* \Omega^1_X(\log D))_y.\end{aligned}$$

This gives a sheaf-theoretic reformulation of the Riemann-Hurwitz formula

$$2g_Y - 2 = [Y : X](2g_X - 2) + \sum_{y \in Y \setminus V} \mathrm{length}_{\mathcal{O}_{Y,y}} \Omega^1_{Y/X,y}.$$

2.2. **Swan conductor.** We recall the definition of the Swan conductor $\mathrm{Sw}_x\mathcal{F}$ from two points of view, corresponding to the upper and the lower ramification groups respectively (see Section 3).

Let K denote the local field at x. Namely the fraction field of the completion of the local ring $\mathcal{O}_{X,x}$. Let $\bar{K}$ be a separable closure of K and $G_K = \mathrm{Gal}(\bar{K}/K)$ be the absolute Galois group. Then, by the map $\mathrm{Spec}\ K \to U$, the pull-back of the ℓ-adic representation of $\pi_1(U, \mathrm{Spec}\ \bar{K})$ corresponding to $\mathcal{F}$ defines an ℓ-adic representation V of G_K.

The group G_K has the inertia subgroup $I_K = \mathrm{Gal}(\bar{K}/K^{\mathrm{ur}})$ and its pro-p Sylow subgroup $P_K = \mathrm{Gal}(\bar{K}/K^{\mathrm{tr}})$ corresponding to the maximal unramified extension K^{ur} and the maximal unramified extension $K^{\mathrm{tr}} = K^{\mathrm{ur}}(\pi^{1/m}; p \nmid m)$ respectively. (See also §3.1.)

The wild inertia group P_K has a decreasing filtration by ramification groups (G_K^r) indexed by positive rational numbers $r > 0$. Since the action of the pro-p-group P_K on V factors through a finite quotient, the filtration defines a decomposition $V = \bigoplus_{r\geq 0} V^{(r)}$ characterized by $(V^{(r)})^{G_K^r} = 0$ for $r > 0$, $(V^{(r)})^{G_K^s} = V^{(r)}$ for $0 < r < s$ and $V^{P_K} = V^{(0)}$. The Swan conductor $\mathrm{Sw}_x\mathcal{F} = \mathrm{Sw}_K V$ is defined to be $\sum_{r>0} r \cdot \dim V^{(r)}$. The equality $\mathrm{Sw}_K V = 0$ is equivalent to $V = V^{(0)}$ that means the action of P_K on V is trivial.

To explain another description of the definition, we make an extra assumption that the action of G_K on V factors through a finite quotient G corresponding to a finite Galois extension L over K. For $\sigma \neq 1, \in G$, we put

$$s_{L/K}(\sigma) = -\mathrm{length}\ \mathcal{O}_L \Big/ \left(\frac{\sigma(a)}{a} - 1; a \in \mathcal{O}_L, \neq 0\right) \tag{6}$$

The integer $s_{L/K}(\sigma)$ is 0 unless σ is not an element of the image $P \subset G$ of P_K. We define $s_{L/K}(1)$ by requiring $\sum_{\sigma\in G} s_{L/K}(\sigma) = 0$. Then, the Swan conductor $\mathrm{Sw}_K V$ is defined by

$$\mathrm{Sw}_K V = \frac{1}{|I|} \sum_{\sigma\in P} s_{L/K}(\sigma) \cdot \mathrm{Tr}(\sigma : V) \tag{7}$$

where $P \subset I \subset G$ denote the images of $P_K \subset I_K$.

2.3. **Log product.** We formulate a generalization of Theorem 2.1 to higher dimension by giving a geometric interpretation of the Swan character $s_{L/K}(\sigma)$.

Let U be a smooth separated scheme of finite type of dimension d over k. For a separated scheme S of finite type over k, the Chow group $CH_0(S)$ denotes the group of 0-cycles modulo rational equivalence.

We will give a definition of the *Swan class* Sw $\mathcal{F}$ as an element of $CH_0(X \setminus U)_{\mathbb{Q}(\zeta_{p^\infty})} = CH_0(X \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}(\zeta_{p^\infty})$ for a compactification X of U under some extra simplifying assumptions.

For a finite étale Galois covering $V \to U$ of Galois group G, we define the Swan character class

$$s_{V/U}(\sigma) \in CH_0(Y \setminus V)$$

for $\sigma \in G$ assuming that Y is a smooth compactification of V satisfying certain good properties. We refer to [14, Definition 4.1] for the definition in the general case that requires alteration.

Assume Y is a proper smooth scheme containing V as the complement of a divisor D with simple normal crossings. Let $D_1, \ldots, D_n$ be the irreducible components of D and let $(Y \times_k Y)' \to Y \times_k Y$ be the blow-up at $D_i \times_k D_i$ for every $i = 1, \ldots, n$. Namely the blow-up by the product of the ideal sheaves $\mathcal{I}_{D_i \times_k D_i} \subset \mathcal{O}_{Y \times_k Y}$. We call the complement $Y *_k Y \subset (Y \times_k Y)'$ of the proper transform of $(D \times_k Y) \cup (Y \times_k D)$ the log product. The diagonal map $\delta \colon Y \to Y \times_k Y$ is uniquely lifted to a closed immersion $\tilde{\delta} \colon Y \to Y *_k Y$ called the log diagonal. We introduce the log product in order to focus on the wild ramification.

Example 2.2. *Assume* $X = \operatorname{Spec} k[T_1, \ldots, T_d]$ *and* D *is defined by* $T_1 \cdots T_n$ *for* $0 \leq n \leq d$. *Then, the log product* $P = X *_k X$ *is the spectrum of*

(8)
$$A = k[T_1, \ldots, T_d, S_1, \ldots, S_d, U_1^{\pm 1}, \ldots, U_n^{\pm 1}]/(S_1 - U_1 T_1, \ldots, S_n - U_n T_n)$$

and the log diagonal $\tilde{\delta} \colon X \to P = X *_k X$ *is defined by* $U_1 = \cdots = U_n = 1$ *and* $T_{n+1} = S_{n+1}, \ \ldots, \ T_d = S_d$.

2.4. **Swan character class and an open Lefschetz trace formula.** Let $\sigma \in G$ be an element different from the identity and let Γ be a closed subscheme of $Y *_k Y$ of dimension $d = \dim Y$ such that the intersection $\Gamma \cap (V \times_k V)$ is equal to the graph Γ_σ of σ. By the assumption that V is etale over U, the intersection $\Gamma_\sigma \cap \Delta_V$ with the diagonal $\Delta_V = \delta(V) \subset V \times_k V$ is empty. Hence the intersection product $(\Gamma, \Delta_Y^{\log})_{Y *_k Y}$ with the log diagonal $\Delta_Y^{\log} = \tilde{\delta}(Y) \subset Y *_k Y$ is defined in $CH_0(Y \setminus V)$.

The intersection product $(\Gamma, \Delta_Y^{\log})_{Y *_k Y}$ is shown to be independent of the choice of Γ under the assumption that $V \to U$ is extended to a map $Y \to X$ to a proper scheme X over k containing U as the complement of a Cartier divisor B and that the image of Γ in the log product $X *_k X$ defined with respect to B is contained in the log diagonal $\Delta_X^{\log}$.

The Swan chararacter class $s_{V/U}(\sigma) \in CH_0(Y \setminus V)$ for $\sigma \neq 1$ is defined by

$$s_{V/U}(\sigma) = -(\Gamma, \Delta_Y^{\log})_{Y *_k Y}. \tag{9}$$

For $\sigma = 1$, it is defined by requiring $\sum_{\sigma\in G} s_{V/U}(\sigma) = 0$. For $\sigma \neq 1$, we have

$$\sum_{q=0}^{2\dim V} (-1)^q \mathrm{Tr}(\sigma^*\colon H^q_c(V_{\bar k}, \mathbb{Q}_\ell)) = -\deg_k s_{V/U}(\sigma) \tag{10}$$

by a Lefschetz trace formula for open varieties [14, Theorem 2.3.4] for a prime number ℓ different from the characteristic of k.

Example 2.3. *Assume that V is a curve and let Y be a smooth compactification. We have $CH_0(Y \setminus V) = \bigoplus_{y\in Y\setminus V} \mathbb{Z}$. For $\sigma \neq 1, \in G$, we have*

$$s_{V/U}(\sigma) = -\sum_{y\in\{y\in Y|\sigma(y)=y\}} \text{length } \mathcal{O}_y \Big/ \left(\frac{\sigma(a)}{a} - 1; a \in \mathcal{O}_y, \neq 0\right) \cdot [y]. \tag{11}$$

2.5. **Swan class and a generalization of the GOS formula.** Let ℓ be a prime number different from $p = \mathrm{char}\ k > 0$. We consider a smooth ℓ-adic sheaf $\mathcal{F}$ on U and define the Swan class $\mathrm{Sw}_U\mathcal{F} \in CH_0(X \setminus U)_{\mathbb{Q}(\zeta_{p^\infty})}$. Here we only give a definition assuming that there exist a finite etale Galois covering $f\colon V \to U$ trivializing $\mathcal{F}$ and a good compactification Y of V as above.

We refer to [14, Definition 4.2.2] for the definition in the general case that requires reduction modulo ℓ and Brauer traces. Let G denote the Galois group $\mathrm{Gal}(V/U)$ and M be the representation of G corresponding to $\mathcal{F}$. Then, the Swan class is defined by

$$\mathrm{Sw}_U\mathcal{F} = \frac{1}{|G|}\sum_{\sigma\in G} f_* s_{V/U}(\sigma) \cdot \mathrm{Tr}(\sigma : M). \tag{12}$$

By the equality (11), this is an immediate generalization of the classical definition (7).

The Lefschetz trace formula for open varieties (10) implies the following generalization of the Grothendieck-Ogg-Shafarevich formula:

Theorem 2.2 ([14, Theorem 4.2.9]). *Let U be a separated smooth scheme of finite type over k. For a smooth ℓ-adic sheaf $\mathcal{F}$ on U, we have*

$$\chi_c(U_{\bar k}, \mathcal{F}) = \mathrm{rank}\ \mathcal{F} \cdot \chi_c(U_{\bar k}, \mathbb{Q}_\ell) - \deg_k \mathrm{Sw}_U\mathcal{F}. \tag{13}$$

3. Ramification groups of a local field

We discuss a geometric definition of the filtration by ramification groups of Galois groups of local fields. Let K be a complete discrete valuation field with not necessarily perfect residue field $F = \mathcal{O}_K/\mathfrak{m}_K$.

3.1. **The lower and the upper ramification groups.** For a finite Galois extension L over K, the Galois group $G = \mathrm{Gal}(L/K)$ has two decreasing filtrations, the lower numbering filtration $(G_i)_{i\in\mathbb{N}}$ and the upper numbering filtration $(G^r)_{r\in\mathbb{Q},>0}$.

In the classical case where the residue field is perfect, they are the same up to renumbering by the Herbrand function [1, Chapitre IV Section 3]. However, their properties make good contrasts. The lower one has an elementary definition and is compatible with subgroups while the upper one has more sophisticated definition and is compatible with quotients.

The lower one and its logarithmic variant are simply defined by

$$\begin{aligned} G_i =& \mathrm{Ker}(G \to \mathrm{Aut}(\mathcal{O}_L/\mathfrak{m}_L^i)), \\ G_{i,\log} =& \mathrm{Ker}(G \to \mathrm{Aut}(L^\times/1+\mathfrak{m}_L^i)) \end{aligned}$$

for an integer $i \geq 1$. The satisfy $G_i \supset G_{i,\log} \supset G_{i+1}$. More precisely, we have

$$G_{i,\log} = \begin{cases} \mathrm{Ker}(G_i \to (\mathcal{O}/\mathfrak{m}_L)^\times) & \text{if } i = 1 \\ \mathrm{Ker}(G_i \to \mathfrak{m}_L^{i-1}/\mathfrak{m}_L^i) & \text{if } i > 1, \end{cases}$$

where the map is defined by $\sigma \mapsto \sigma(\pi_L)/\pi_L$ for $i = 1$ and $\sigma \mapsto \sigma(\pi_L)/\pi_L - 1$ for $i > 1$ and is independent of the choice of a uniformizer π_L of L. We also have

$$G_{i+1} = \mathrm{Ker}\big(G_{i,\log} \to \mathrm{Hom}_{\mathcal{O}_L/\mathfrak{m}_L}(\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}, \mathfrak{m}_L^i/\mathfrak{m}_L^{i+1})\big)$$

for $i \geq 1$ where the map sends σ to the map $da \mapsto \sigma(a) - a$. They show that $G_{1,\log}$ is the p-Sylow subgroup P of the inertia group $I = G_1$ and that I/P is a cyclic group of order prime to p.

3.2. **Rigid geometric picture.** We give a rigid geometric reinterpretation of the definition of the lower numbering filtration. Take a presentation

$$\mathcal{O}_K[X_1, \ldots, X_n]/(f_1, \ldots, f_n) \to \mathcal{O}_L$$

of the integer ring of L. We consider the n-dimensional closed disk D^n defined by $\|x\| \leq 1$ over K in the sense of rigid geometry and the morphism of disks $f\colon D^n \to D^n$ defined by $f_1, \ldots, f_n$. Then the Galois group G is identified with the inverse image $f^{-1}(0)$ of the origin $0 \in D^n$. In fact, a $\bar{K}$-rational point of D^n is identified with an $\mathcal{O}_K$-ring homomorphism $\mathcal{O}_K[X_1, \ldots, X_n] \to \mathcal{O}_{\bar{K}}$ and those in $f^{-1}(0)$ are identified with $\mathcal{O}_K$-ring homomorphisms $\mathcal{O}_K[X_1, \ldots, X_n]/(f_1, \ldots, f_n) \to \mathcal{O}_{\bar{K}}$. Thus, the inverse image $f^{-1}(0)$ is identified with $G = \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_L, \mathcal{O}_{\bar{K}})$.

In other words, we have a cartesian diagram

$$\begin{array}{ccc} G & \longrightarrow & D^n \\ \downarrow & & \downarrow f \\ \{0\} & \longrightarrow & D^n. \end{array} \tag{14}$$

The subgroups G_i and G^r are defined to consist of the points of G that are *close* to the identity in certain senses. For the lower one, the closeness is simply measured by the distance. Namely, the lower numbering subgroup $G_i \subset G$ consists of the points $\sigma \in G$ satisfying $d(\sigma, \mathrm{id}) \leq \|\pi_L^i\|$ for a prime element π_L of L.

To define the upper numbering filtration, we consider, for a rational number $r > 0$, the inverse image $V_r = \{x \in D^n \mid d(f(x), 0) \leq \|\pi_K\|^r\} \subset D^n$ of the closed subdisk of radius $\|\pi_K\|^r$, as an affinoid subdomain containing G. The upper numbering subgroup G^r consists of the points in G contained in the same geometric connected component of V_r as the identity.

In the case where the integer ring $\mathcal{O}_L$ is generated by one element, the inverse image V_r consists of finitely many subdisks of the unit disk $D = D^1$. The assumption is satisfied in the classical where the residue field $\mathcal{O}_L/\mathfrak{m}_L$ is a separable extension of F or in particular if the residue field F is perfect. The (logarithm of) the radius of the subdisk is a piecewise linear function of r and is nothing but the Herbrand function. This is verified easily by using the Newton polygon defined by the valuations of the roots of the minimal polynomial of a generator of $\mathcal{O}_L$.

3.3. **Interpretation via schemes.** In the following, we give a definition of a logarithmic variant of the upper numbering filtration only using schemes, under the following extra assumption:

(G) There exist a smooth scheme X over a perfect field k, a smooth irreducible divisor D of X and an isomorphism $\mathcal{O}_K \to \widehat{\mathcal{O}}_{X,\xi}$ to the completion of the local ring at the generic point ξ of D.

In the description using rigid geometry above, a key operation is shrinking of the radius. For schemes, the corresponding operation is the blow-up.

Let L be a finite Galois extension of K of Galois group G. Then, by replacing X by an étale neighborhood of ξ if necessary, there exist a

G-torsor V over the complement $U = X \setminus D$ and a cartesian diagram

$$\begin{array}{ccc} \text{Spec } L & \longrightarrow & V \\ \downarrow & & \downarrow \\ \text{Spec } K & \longrightarrow & U \end{array}$$

of G-torsors such that the normalization Y of X in $V \to U$ is smooth over k and finite flat over X and the reduced inverse image E of D is a smooth divisor of Y.

In the product $X \times_k S$, we have divisors $D \times_k S$ and $X \times_k D_S$ where $D_S = \text{Spec } F \subset S = \text{Spec } \mathcal{O}_K$ denotes the closed point. We consider the blow-up $(X \times_k S)'$ of $X \times_k S$ at their intersection $D \times_k D_S$ and define the log product $P = X *_k S \subset (X \times_k S)'$ to be the complement of the proper transforms of $D \times_k S$ and $X \times_k D_S$. The generic fiber $P \times_S \text{Spec } K$ is $U \times_k \text{Spec } K$. Let Q denote the normalization of P in the finite etale covering $V \times_k \text{Spec } K$ of $U \times_k \text{Spec } K$.

The canonical map $S \to X$ is uniquely lifted to a section $s\colon S \to P$. In the cartesian diagram

$$\begin{array}{ccc} T & \xrightarrow{i} & Q \\ \downarrow & & \downarrow f \\ S & \xrightarrow{s} & P \end{array} \tag{15}$$

we have $T = \text{Spec } \mathcal{O}_L$ and the vertical arrows are finite flat. This diagram should be regarded as a scheme theoretic counterpart of (14).

We consider a finite separable extension K' of K containing L as a subextension, in order to make a base change. We put $S' = \text{Spec } \mathcal{O}_{K'}$, $F' = \mathcal{O}_{K'}/\mathfrak{m}_{K'}$ and let $e = e_{K'/K}$ be the ramification index. Let $r > 0$ be a rational number and assume that $r' = e'r$ is an integer. We regard the divisor $R' = r'D_{S'} = \text{Spec } \mathcal{O}_{K'}/\mathfrak{m}_{K'}^{r'}$ of S' as a closed subscheme of $P_{S'} = P \times_S S'$ by the section $s'\colon S' \to P_{S'}$ induced by $s\colon S \to P$.

We consider the blow-up of $P_{S'}$ at the center R' and let $P_{S'}^{(r)}$ denote the complement of the proper transform of the closed fiber $P_{S'} \times_{S'} D_{S'}$. The scheme $P_{S'}^{(r)}$ is smooth over S' and the closed fiber $P_{S'}^{(r)} \times_{S'} D_{S'}$ is the vector bundle $\Theta_{F'}^{(r)}$ over F' such that the F'-vector space consisting of F'-valued points is canonically identified with $\Omega^1_{X/k}(\log D) \otimes \mathfrak{m}_{K'}^{-r'}/\mathfrak{m}_{K'}^{-r'+1}$.

Example 3.1. *Assume $X = \text{Spec } k[T_1, \ldots, T_d]$ and $D = (T_1)$. Then, we have $\mathcal{O}_K = k(T_2, \ldots, T_d)[[T_1]]$ and $X \times_k S = \text{Spec } \mathcal{O}_K[S_1, \ldots, S_d]$. The canonical map $S \to X$ induces a closed immersion $S \to X \times_k S$ defined by $S_i \mapsto T_i$.*

The log product $P = X *_k S$ *is* Spec $\mathcal{O}_K[U_1^{\pm 1}, S_2, \ldots, S_d]$ *with the canonical map* $P = X *_k S \to X \times_k S$ *defined by* $S_1 \mapsto U_1 T_1$. *If* π' *is a uniformizer of* K', *the scheme* $P_{S'}^{(r)} = X *_k S$ *is* Spec $\mathcal{O}_{K'}[V_1, \ldots, V_d]$ *with the canonical map* $P_{S'}^{(r)} \to P = X *_k S$ *defined by* $U_1 \mapsto 1 + \pi'^{r'} V_1$, $S_i \mapsto T_i + \pi'^{r'} V_i$ *for* $1 < i \leq d$.

We consider the normalizations $\bar{Q}_{S'}^{(r)}$ and $\bar{T}_{S'}$ of $Q \times_P P_{S'}^{(r)}$ and of $T \times_S S'$ respectively. Then, the diagram (15) induces a diagram

$$\begin{array}{ccc} \bar{T}_{S'} & \xrightarrow{i^{(r)}} & \bar{Q}_{S'}^{(r)} \\ \downarrow & & \downarrow f^{(r)} \\ S' & \xrightarrow{s^{(r)}} & P_{S'}^{(r)}. \end{array} \tag{16}$$

By the assumption that K' contains L, the scheme $\bar{T}_{S'}$ is isomorphic to the disjoint union of finitely many copies of S' and the geometric fiber $\bar{T}_{\bar{F}} = \bar{T}_{S'} \times_{S'} \bar{F}$ is identified with $\mathrm{Gal}(L/K)$.

3.4. **Definition of the upper ramification groups.** After replacing K' by some finite separable extension, the geometric closed fiber $\bar{Q}_{\bar{F}}^{(r)} = \bar{Q}_{S'}^{(r)} \times_{S'} \mathrm{Spec}\ \bar{F}$ is reduced and the formation of $\bar{Q}_{S'}^{(r)}$ commutes with further base change. We call such $\bar{Q}_{S'}^{(r)}$ a stable integral model. The finite map $i^{(r)} \colon \bar{T}_{S'} \to \bar{Q}_{S'}^{(r)}$ induces surjections

$$\begin{array}{ccc} \bar{T}_{\bar{F}} = \mathrm{Gal}(L/K) & \xrightarrow{i_*^{(r+)}} & f^{(r)-1}(0) \\ & \searrow^{i_*^{(r)}} & \downarrow \\ & & \pi_0(\bar{Q}_{\bar{F}}^{(r)}) \end{array} \tag{17}$$

of finite sets to the set of geometric connected components and to the inverse image of the origin $0 \in P_{\bar{F}}^{(r)} = \Theta_{\bar{F}}^{(r)}$.

Theorem 3.1 ([5, Theorems 3.3, 3.8], [15, Section 1.3])**.** *Let* L *be a finite Galois extension over* K *of Galois group* G *and we consider a diagram* (15) *as above.*

1. *For a rational number* $r > 0$, *we take a finite separable extension* K' *of* K *containing* L *such that* $e_{K'/K} r$ *is an integer and that* $Q_{\bar{S}'}^{(r)}$ *is a stable integral model.*

Then, the inverse image $i_*^{(r)-1}(i_*^{(r)}(1)) = G_{\log}^r \subset G$ *is independent of the choice of diagram* (15) *or an extension* K' *and is a normal subgroup* *of G. Further the surjection $i_*^{(r)}$ (17) induces a bijection $G/G^r_{\log} \to \pi_0(\bar{Q}^{(r)}_{\bar{F}})$.*

2. *Let the notation be as in* 1. *Then, there exist rational numbers $0 = r_0 < r_1 < \ldots < r_m$ such that $G^r_{\log} = G^{r_i}_{\log}$ for $r \in (r_{i-1}, r_i] \cap \mathbb{Q}$ and $i = 1, \ldots, m$ and $G^r_{\log} = 1$ for $r > r_m$.*

We put $G^{r+}_{\log} = G^{r_i}_{\log}$ for $r \in [r_{i-1}, r_i) \cap \mathbb{Q}$ and $i = 1, \ldots, m$ and $G^r_{\log} = 1$ for $r \geq r_m$. Then, the surjection $i_*^{(r+)}$ (17) induces a bijection $G/G^{r+}_{\log} \to f^{(r)-1}(0)$.

3. *For a subfield $M \subset L$ Galois over K and for a rational number $r > 0$, the subgroup $\mathrm{Gal}(M/K)^r_{\log} \subset \mathrm{Gal}(M/K)$ is the image of $G^r_{\log} = \mathrm{Gal}(L/K)^r_{\log}$.*

The proofs of 1 and 3 are rather straightforward. That of 2 requires some result from rigid geometry.

If L is an abelian extension of K, it is concretely described using the Artin-Schreier-Witt theory as follows.

Example 3.2 ([12], [8]). *A cyclic extension L of degree p^{m+1} is defined by a Witt vector by the isomorphism $W_{m+1}(K)/(F-1) \to H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ of Artin-Schreier-Witt theory. An increasing filtration on $W_{m+1}(K)$ is defined by*

$$F^n W_{m+1}(K)$$
$$= \{(a_0, \ldots, a_m) \in W_{m+1}(K) \mid p^{m-i} v_K(a_i) \geq -n \text{ for } i = 0, \ldots, m\}.$$

The filtration on $H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ induced by the canonical surjection $W_{m+1}(K) \to H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ is considered in [12]. *For $G = \mathrm{Gal}(L/K)$, the filtration $(G^n_{\log})_{n \geq 0}$ indexed by integers is the dual of the restriction to $Hom(\mathrm{Gal}(L/K), \mathbb{Z}/p^{m+1}\mathbb{Z}) \subset H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$. Namely, we have $G^n_{\log} = \{\sigma \in G \mid c(\sigma) = 0 \text{ if } c \in F^n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})\}$. Further, for a rational number $r \in (n-1, n] \cap \mathbb{Q}$, we have $G^r_{\log} = G^n_{\log}$.*

3.5. **Graded pieces.** We study the graded pieces. Let $\Omega^1_{\mathcal{O}_K}(\log)$ denote the free $\mathcal{O}_K$-module $\Omega^1_{X/k}(\log D)_\xi \otimes \mathcal{O}_K$ of rank $\dim X$. By abuse of notation, let $\Omega^1_F(\log)$ denote the F-vector space $\Omega^1_{\mathcal{O}_K}(\log) \otimes_{\mathcal{O}_K} F$. Then, we have an exact sequence $0 \to \Omega^1_F \to \Omega^1_F(\log) \xrightarrow{\mathrm{res}} F \to 0$ of F-vector spaces of finite dimension. We extend the normalized discrete valuation v of K to a separable closure $\bar{K}$ and, for a rational number r, we put $\mathfrak{m}^r_{\bar{K}} = \{a \in \bar{K} \mid v(a) \geq r\}$ and $\mathfrak{m}^{r+}_{\bar{K}} = \{a \in \bar{K} \mid v(a) > r\}$. The $\bar{F}$-vector space $\mathfrak{m}^r_{\bar{K}}/\mathfrak{m}^{r+}_{\bar{K}}$ is of dimension 1.

Corollary ([6, Theorem 2.15], [15, Theorem 1.24, Corollary 1.25]). *Let L be a finite Galois extension of Galois group G. Then, for a rational* *number $r > 0$, the graded quotient $\mathrm{Gr}^r_{\log} G = G^r_{\log}/G^{r+}_{\log}$ is abelian and killed by p.*

Further, there exists a canonical injection

$$Hom(\mathrm{Gr}^r_{\log} G, \mathbb{F}_p) \to Hom_{\bar F}(\mathfrak{m}^r_{\bar K}/\mathfrak{m}^{r+}_{\bar K}, \Omega^1_F(\log) \otimes_F \bar F). \tag{18}$$

The definition in a special case will be discussed in §4.2. For a non-trivial character $\chi \in Hom(\mathrm{Gr}^r_{\log} G, \mathbb{F}_p)$, we call the image $\mathrm{rsw}\chi \in Hom_{\bar F}(\mathfrak{m}^r_{\bar K}/\mathfrak{m}^{r+}_{\bar K}, \Omega^1_F(\log) \otimes_F \bar F)$ the refined Swan character of χ.

Example 3.3 ([12], [8])**.** *We keep the notation in Example 3.2. We define a canonical map $F^m d\colon W_{m+1}(K) \to \Omega^1_K$ by sending $(a_0, \ldots, a_m)$ to $a_0^{p^m-1} da_0 + \cdots + da_m$. It maps $F^n W_{m+1}(K)$ to $F^n \Omega^1_K = \mathfrak{m}_K^{-n} \Omega^1_{\mathcal{O}_K}(\log)$ for $n \in \mathbb{Z}$ and induces an injection*

$$\mathrm{Gr}^n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \mathrm{Gr}^n \Omega^1_K = Hom_F(\mathfrak{m}^n_K/\mathfrak{m}^{n+1}_K, \Omega^1_F(\log)) \tag{19}$$

for $n > 0$.

Let L be a cyclic extension of degree p^{m+1} corresponding to a character $\chi \in H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$. The smallest integer $n \geq 0$ such that $\chi \in F^n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ is called the conductor of χ and is equal to the smallest rational number r such that the ramification of L is bounded by $r+$. The character is ramified if and only if the conductor is > 0. For a ramified character χ of conductor $n > 0$, the image of the class of χ by the injection (19) in $Hom_F(\mathfrak{m}^n_K/\mathfrak{m}^{n+1}_K, \Omega^1_F(\log)) \subset Hom_{\bar F}(\mathfrak{m}^n_{\bar K}/\mathfrak{m}^{n+}_{\bar K}, \Omega^1_F(\log) \otimes_F \bar F)$ is the refined Swan character $\mathrm{rsw}\chi$.

4. Wild blow-up and differential forms

Let X be a smooth separated scheme of finite type over a perfect field k of characteristic $p > 0$ and $U = X \setminus D$ be the complement of a divisor D with simple normal crossings. We consider a finite étale G-torsor V over U for a finite group G and study the ramification of V along D.

4.1. **Wild blow-up and a groupoid structure.**

The ramification of V along D will be measured by linear combinations $R = \sum_i r_i D_i$ with rational coefficients $r_i \geq 0$ of irreducible components of D. In the following, we assume the coefficients of $R = \sum_i r_i D_i$ are integers, for simplicity.

We consider the log product $P = X *_k X \subset (X \times_k X)'$ and the log diagonal $\tilde\delta\colon X \to P = X *_k X$ as in Section 1.1. We define a relatively affine scheme $P^{(R)}$ over P. The scheme $P^{(R)}$ is the complement of the proper transforms of $P \times_X R$ in the blow-up of P at the center $R \subset X$ embedded by the log diagonal map $\tilde\delta\colon X \to P$. The log diagonal map is uniquely lifted to a closed immersion $\delta^{(R)}\colon X \to (X *_k X)^{(R)}$ and the open immersion $U\times U\to X*_k X$ is uniquely lifted to a closed immersion $j^{(R)}\colon U\times U\to (X*_k X)^{(R)}$.

Example 4.1. *We take the notation in Example 2.2. If we put $T^R=T_1^{r_1}\cdots T_n^{r_n}$, the scheme $(X*_k X)^{(R)}$ is the spectrum of*

$$\begin{aligned}&A[V_1,\ldots,V_d]/(U_1-1-V_1T^R,\ldots,U_n-1-V_nT^R,\\ &\qquad S_{n+1}-T_{n+1}-V_{n+1}T^R,\ldots,S_d-T_d-V_dT^R)\\ &=k[T_1,\ldots,T_d,V_1,\ldots,V_d,(1+V_1T^R)^{-1},\ldots,(1+V_nT^R)^{-1}].\end{aligned}\tag{20}$$

The immersion $\delta^{(R)}\colon X\to (X*_k X)^{(R)}$ is defined by $V_1=\cdots=V_d=0$.

The base change $P^{(R)}\times_X R$ with respect to the projection $P^{(R)}\to X\supset R$ is the twisted tangent bundle $\Theta^{(R)}=\mathbf{V}(\Omega^1_X(\log D)(R))\times_X R$ where $\mathbf{V}(\Omega^1_X(\log D)(R))$ denotes the vector bundle defined by the symmetric algebra of the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D)(R)$.

The projection $\mathrm{pr}_{13}\colon (X\times_k X)\times_X(X\times_k X)=X\times_k X\times_k X\to X\times_k X$ induces a morphism

$$\mu\colon (X*_k X)^{(R)}\times_X (X*_k X)^{(R)}\to (X*_k X)^{(R)}$$

and defines a groupoid structure on $(X*_k X)^{(R)}$. The group structure on the vector bundle $\Theta^{(R)}=P^{(R)}\times_X R$ is compatible with the groupoid structure.

4.2. **Isogeny and the graded piece.** Let V be a G-torsor over U for a finite group G. We consider the quotient $W=(V\times_k V)/\Delta G$ by the diagonal $\Delta G\subset G\times G$ as a finite étale covering of $U\times_k U$ and let Z be the normalization of $(X*_k X)^{(R)}$ in W. The diagonal map $V\to V\times_k V$ induces a closed immersion $U=V/G\to W=(V\times_k V)/\Delta G$ on the quotients and is extended to a closed immersion $e\colon X\to Z$.

Theorem 4.1. *Let X be a separated smooth scheme of finite type over k and $U=X\setminus D$ be the complement of a divisor with simple normal crossings. Let $R=\sum_i r_iD_i\geq 0$ be an effective Cartier divisor.*

Let V be a G-torsor over U for a finite group G. Let X be the normalization of $(X*_k X)^{(R)}$ in the quotient $W=(V\times_k V)/\Delta G$ and $e\colon X\to Z$ be the section induced by the diagonal.

Assume that Z is étale over $(X*_k X)^{(R)}$ on a neighborhood of the image of $e\colon X\to Z$. Let $Z_0\subset Z$ be the maximum open subscheme étale over $(X*_k X)^{(R)}$.

1. The base change $Z_{0,R}=Z_0\times_X R$ with respect to the projection $Z_0\to (X*_k X)^{(R)}\to X\supset R$ has a natural structure of smooth group scheme over R such that the map $e_R\colon X_R\to Z_{0,R}$ induced by $e\colon X\to Z$

is the unit. Further the étale map $Z_{0,R} \to \Theta^{(R)} = (X *_k X)^{(R)} \times_X R$ induced by the canonical map $Z \to (X *_k X)^{(R)}$ is a group homomorphism.

2. For every point $x \in R$, the connected component $Z^0_{0,x}$ of the fiber $Z_{0,x}$ is isomorphic to the product of finitely many copies of the additive group $\mathbf{G}_{a,x}$ and the map $Z^0_{0,x} \to \Theta^{(R)}_x$ is an étale isogeny.

Assume that D is irreducible and that $G^s_{\log} = 1$ if and only if $s > r$. Then, we have an exact sequence $0 \to G^r_{\log} \to Z^0_{0,\xi} \to \Theta^{(r)}_\xi \to 0$. Thus, $Z^0_{0,\xi}$ defines an extension of the vector space $\Theta^{(r)}_\xi$ by an elementary abelian p-group $G^r_{\log}$ and hence a class $[Z^0_{0,\xi}] \in \mathrm{Ext}(\Theta^{(r)}_\xi, G^r_{\log})$. For a vector space V, the pull-back of the Artin-Schreier covering (Example 1.1) defines a canonical isomorphism $\mathrm{Hom}(V, \mathbf{G}_a) \to \mathrm{Ext}(V, \mathbb{F}_p)$. Thus, the class $[Z^0_{0,\xi}] \in \mathrm{Ext}(\Theta^{(r)}_\xi, G^r_{\log})$ defines an element in $\mathrm{Hom}(\Theta^{(r)}_\xi, \mathbf{G}_a) \otimes G^r_{\log}$. Since the dual $\mathrm{Hom}(\Theta^{(r)}_\xi, \mathbf{G}_a)$ is canonically identified with the F-vector space $\Omega^1_X(\log D)(R)_\xi \otimes_{\mathcal{O}_{X,\xi}} F$, we obtain a canonical map

$$\begin{aligned} \mathrm{raw}\colon \mathrm{Hom}(G^r_{\log}, \mathbb{F}_p) \to & \Omega^1_X(\log D)(R)_\xi \otimes_{\mathcal{O}_{X,\xi}} F \\ & = \mathrm{Hom}_F(\mathfrak{m}^r_K/\mathfrak{m}^{r+1}_K, \Omega^1_X(\log D)(R)_\xi \otimes_{\mathcal{O}_{X,\xi}} F) \end{aligned}$$

called the refined Swan conductor.

4.3. **Ramification of a rank 1 sheaf.** As an application, we study the ramification of a rank 1 sheaf. Let $\mathcal{F}$ be a smooth sheaf of rank 1 corresponding to a character $\chi\colon \pi_1(U)^{\mathrm{ab}} \to \Lambda^\times$. For each irreducible component D_i, let K_i be the local field and n_i be the conductor of the p-part of the character $\chi_i\colon G^{\mathrm{ab}}_{K_i} \to \Lambda^\times$. We put $R = \sum_i n_i D_i$.

We consider a smooth sheaf $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})$ on $U \times U$. Then, the direct image $j^{(R)}_*\mathcal{H}$ by the open immersion $j^{(R)}\colon U \times_k U \to (X *_k X)^{(R)}$ is a smooth sheaf of Λ-modules of rank 1 on $(X *_k X)^{(R)}$. For a component with $r_i > 0$, the restriction of $j^{(R)}_*\mathcal{H}$ to the fiber $\Theta^{(R)}_{\xi_i}$ is the Artin-Schreier sheaf defined by the refined Swan character $\mathrm{rsw}\chi_i$ regarded as a linear form on $\Theta^{(R)}_{\xi_i}$ by [7, Proposition 4.2.2].

Further, we assume the following condition:

(C) For each irreducible component D_i of D such that $r_i > 0$, the refined Swan character $\mathrm{rsw}_i\,\chi$ defines a locally splitting injection

$$\mathrm{rsw}_i\,\chi\colon \mathcal{O}_X(-R) \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i} \to \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i}.$$

This condition says that for each irreducible component, the wild ramification of $\mathcal{F}$ is controlled at the generic point. It is called the cleanness condition and studied in [13].

Theorem 4.2 ([7, Theorem 4.2.6])**.** *Assume the condition* (C) *above is satisfied and* X *is proper. Then, we have*

$$\chi_c(U_{\bar{k}}, \mathcal{F}) = \deg(X, X)_{(X *_k X)^{(R)}}.$$

It is proved by showing that the characteristic class introduced in the following equals the intersection product $(X, X)_{(X *_k X)^{(R)}}$. A generalization to higher rank case is studied in [15, Theorem 3.4].

4.4. **Characteristic class.** Let X be a separated scheme of finite type over a field k. As a coefficient ring Λ, we consider a ring finite over $\mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ for a prime number $\ell \neq \operatorname{char} k$. Let $a\colon X \to \operatorname{Spec} k$ denote the structure map and $K_X = Ra^!\Lambda$ denote the dualizing complex. If X is smooth of dimension d over k, we have $K_X = \Lambda(d)[2d]$.

Let $\mathcal{F}$ be a constructible sheaf of flat Λ-modules on X and consider the object

$$\mathcal{H} = R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, R\mathrm{pr}_1^!\mathcal{F})$$

of the derived category $D_{\mathrm{ctf}}(X \times_k X, \Lambda)$ of constructible sheaves of Λ-modules of finite tor-dimension on the product $X \times_k X$. If X is smooth of dimension d over k and if $\mathcal{F}$ is smooth, we have a canonical isomorphism $\mathcal{H} \to \mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})(d)[2d]$.

A canonical isomorphism

$$End(\mathcal{F}) \to H^0_X(X \times_k X, \mathcal{H}) \tag{21}$$

is defined in [10]. Hence, we may regard the identity $\mathrm{id}_{\mathcal{F}}$ as a cohomology class $\mathrm{id}_{\mathcal{F}} \in H^0_X(X \times_k X, \mathcal{H})$ supported on the diagonal $X \subset X \times_k X$. Let $\delta\colon X \to X \times_k X$ be the diagonal map. Further in [10], a canonical map $\delta^*\mathcal{H} \to K_X$ is defined as the trace map. The characteristic class

$$C(\mathcal{F}) \in H^0(X, K_X)$$

is defined as the image of the pull-back $\delta^*\mathrm{id}_{\mathcal{F}} \in H^0(X, \delta^*\mathcal{H})$ by the induced map $H^0(X, \delta^*\mathcal{H}) \to H^0(X, K_X)$. If X is smooth and if $\mathcal{F}$ is smooth, we have $C(\mathcal{F}) = \operatorname{rank} \mathcal{F} \cdot (X, X)_{X \times_k X}$ where $(X, X)_{X \times_k X}$ denotes the self-intersection in the product $X \times_k X$. The Lefschetz trace formula [10] asserts that, if X is proper, the trace map $H^0(X, K_X) \to \Lambda$ sends the characteristic class $C(\mathcal{F})$ to the Euler number $\chi(X_{\bar{k}}, \mathcal{F})$. In other words, the characteristic class is a geometric refinement of the Euler number.

A relation of the canonical class with the Swan class is given in [7, Theorem 3.3.1]. In particular, under the assumption of Theorem 4.2, we have

$$\begin{aligned}\mathrm{Sw}_U\mathcal{F} &= (-1)^d c_d(\Omega^1_X(\log D)) - (-1)^d c_d(\Omega^1_X(\log D)(R)) \\ &= (-1)^{d-1}\big(c(\Omega^1_X(\log D))(1-R)^{-1}R\big)_{\dim 0}.\end{aligned}$$

References

Basics

[1] J-P. Serre, Corps Locaux, Hermann, Paris, France, 1962.

[2] —–, Représentations linéaires des groupes finis, Hermann, Paris, France, 1967.

[3] P. Deligne, rédigé par J. F. Boutot, *Cohomologie étale, les points de départ,* SGA $4\frac{1}{2}$, LNM 569, 4-75, Springer, (1977).

[4] A. Grothendieck, *Revêtements étales et groupe fondamental*, SGA 1, Springer LNM **224** (1971), Édition recomposée SMF (2003).

Articles

[5] A. Abbes, T. Saito, *Ramification of local fields with imperfect residue fields*, Amer. J. of Math. **124** (2002), 879-920

[6] —–, *Ramification of local fields with imperfect residue fields II*, Documenta Math., Extra Volume K. Kato (2003), 3-70.

[7] ——, *The characteristic class and ramification of an ℓ-adic étale sheaf*, Invent. Math., 168 (2007) 567-612.

[8] —–, *Analyse micro-locale ℓ-adique en caractéristique $p > 0$: Le cas d'un trait*, Publ. RIMS 45-1 (2009) 25-74.

[9] ——, *Ramification and cleanliness*, Tohoku Math. J. Centennial Issue, 63 No. 4 (2011), 775-853.

[10] A. Grothendieck, rédigé par L. Illusie, *Formule de Lefschetz,* exposé III, SGA 5, Springer LNM **589** (1977) 73-137.

[11] —–, rédigé par I. Bucur, *Formule d'Euler-Poincaré en cohomologie étale,* exposé X, SGA 5, Springer LNM **589** (1977) 372-406.

[12] K. Kato, *Swan conductors for characters of degree one in the imperfect residue field case*, Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), 101–131, Contemp. Math., 83, Amer. Math. Soc., Providence, RI, 1989.

[13] —–, *Class field theory, $\mathcal{D}$-modules, and ramification of higher dimensional schemes, Part I,* American J. of Math., 116 (1994), 757-784.

[14] K. Kato, T. Saito, *Ramification theory for varieties over a perfect field*, Ann. of Math., 168 (2008), 33-96.

[15] T. Saito, *Wild ramification and the characteristic cycle of an ℓ-adic sheaf*, Journal de l'Institut de Mathematiques de Jussieu, (2009) 8(4), 769-829

[16] —–, *Wild ramification of schemes and sheaves*, Proceedings of the international congress of mathematicians 2010 (ICM 2010) pp. 335-356.