SS in mixed char.

First we recall what is known in geom case.
and compare the situation in mixed char

$\Lambda/\mathbb{F}_\ell$ finite ext'n ℓ inv. on sch X.

$\mathcal{F}$ cx of sheaves of Λ-mod on $X_{\acute{e}t}$
$H^q(\mathcal{F})$ const. $\forall q$ $=0$ except fin many

geom case
X smooth / perf. fd k char $p>0$
$p \neq \ell$

T^*X vector bdle $\Omega^1_X = \Omega^1_{X/k}$
C closed conical subset
micro support G_m-action ~~defined~~ later

SS = smallest C on which $\mathcal{F}$ is microsupp'd

exists. $\dim SS = \dim X$

CC defined. index formula
Milnor formula

mixed char.

X regular sch / $\mathbb{Z}_{(p)}$ $\quad p \neq \ell$

assume $(X_{\mathbb{F}_p})_{red}$ is of finite type

over a field k s.t $[k : k^p]$ is finite.

$F\Omega_X$ locally free $\mathcal{O}_{X_{\mathbb{F}_p}}$ - mod of finite rk

modify the definition of derivation

$$d(a+b) = da + db$$
$$d(ab) = adb + bda$$

by

$$d(a+b) = da + db + \text{"} \frac{(a+b)^p - a^p - b^p}{p} \text{"}$$

$$dab = \underline{a}^p db + b^p da$$

Frobenius

Witt

$x \in X_{\mathbb{F}_p}$

$$0 \to F^* m_x/m_x^2 \to F\Omega_X \otimes k(x) \to F^*\Omega^1_{k(x)} \to 0$$

micro supp defined

shortly

SS smallest existence ?

CC ? Milnor formula ?

index formula conductor formula.

$X/\mathcal{O}_K$ ref field of k perfect

$\dim X = 2$

$j : \mathcal{F}$ $\mathcal{F}$ · rk 1. $\mathcal{O}_K$.

Beilinson's def of micro supp.

$$X \xleftarrow{h} W \xrightarrow{f} Y$$

transversality local acyclicity

does not work in mixed char.

not enough f.

1. absolute work with h only

$\to$ def. of m.s

Beilinson's proof of existence of SS

After reduction to $X = \mathbb{P}^n$. use

Radon fm.

$$\begin{tikzcd} \mathbb{P}^n & Q = \text{univ. hyperplane} \arrow[l, "p = h"'] \arrow[d, "p^\vee = f"] \\ & \mathbb{P}^{n\vee} \end{tikzcd}$$

2 relative. = fix S reg. $/\mathbb{Z}_{(p)}$ satisfying fin. cond $\mathcal{E}$

$X \to S$.

modify Beilinson's def of m.s

use $(h. f)$ by modify cond on f. using 1.

Our def $\neq$ Hu-Yang

$\uparrow$ already for $S = \mathbb{F}_p$ & k imperfect

doesn't commute with b.c

commute with b.c.

essentially k perfect

1. Absolute case.

micro supp

$C \subset FT^*X$ closed conical subset stable under $\mathbb{G}_m$-action

supp of $\mathcal{F} \subset$ base of C &

$\forall\ h: W \to X$ C-transversal $\Rightarrow h$ $\mathcal{F}$-transversal

on a nbd of $X_{\mathbb{F}_p}$ (or at $X_{\mathbb{F}_p}$)

C-transversal

$h^*C \cap \ker(FT^*W \times_W X \longrightarrow FT^*X)$

$\subset$ 0-section

$\mathcal{F}$ - trans.

$$(h^{*}\mathcal{F} \otimes h^{!}\Lambda) \to h^{!}\mathcal{F}$$

ism

on a nbd of $W_{\mathbb{F}_p}$

If X smooth / k perfect

equiv. to Beilinson's def.

2. relative
local acyclicity
(Braverman - Gaitsgory)

Lemma. Let $f : X \to Y$ morph / k field
Y smooth / k. Then f A E
$\mathcal{F}$ shf on X

(1) f is $\dfrac{\text{loc. acyclic rel to } \mathcal{F}}{\mathcal{F}\text{-acyclic}}$

(2) $\forall$ shf on Y
$\gamma = (1, f) : X \to X \times Y$ is

$\mathcal{F} \boxtimes \mathcal{G}$ - transversal
$\quad \shortparallel$
$p_1^{*}\mathcal{F} \otimes p_2^{*}\mathcal{G}$

The assumption k field is used to assume that

$X \to \mathrm{Spec}\, k$ is univ. $\exists$-acyclic

heuristic observation. suppose

$S \to \mathrm{Spec}\, k$. smooth

Y_0 smooth $/k$ (virtual)

$X \to Y_0$ $\exists$-acyclic

$\Leftrightarrow \forall\, \mathcal{G}_0$ on Y_0 $\quad X \to X \times_k Y_0 \quad \exists \otimes \mathcal{G}_0$-trans

eliminate k. $\qquad\qquad\qquad\qquad \Vert$

work $/S$ $\quad Y = Y_0 \times_k S \quad X \times_S Y \quad \exists \otimes p_1^* \mathcal{G}_0$-trans

Among sheaves $\mathcal{G}$ on Y which $\mathcal{G}$ comes from Y_0

We use absolute case

Def We say that a shf $\mathcal{G}$ on Y is S-acyclic if

$\exists\, C' \subset FT^*Y$ s.t.

- $\mathcal{G}$ ms on C'
- $C' \cap \mathrm{Im}(FT^*S \times_S Y \to FT^*Y) \subset$ 0-section

($\Rightarrow$ $Y \to S$ $\mathcal{G}$-acyclic)

Def We say (h,f): $h : W \to X$

$f : W \to Y$, Y smooth $/S$ is

$\mathcal{F}$-acyclic over S if

$\forall\, \mathcal{G}$ S-acyclic on Y

$(h,f) : W \longrightarrow X \times_S Y$ is $\mathcal{F} \boxtimes \mathcal{G}$-trans

(h, f) $\quad C$-acyclic $/S$

$$h^{\circ} C \times_W (FT^{*}Y \times_Y W) \subset |h| (FT^{*}X \times_X W) \times_W (FT^{*}Y \times_Y W) \to FT^{*}W)$$

$$\subset |h| (\quad = \quad \to FT^{*}(X \times_S Y) \times_{X \times_S Y} W)$$

$\mathcal{F}$ $\quad S$-m.c on C

$\forall$ (h, f) $\; C$-acyclic $/S \Rightarrow \mathcal{F}$-acyclic $/S$.

If $S = S_{pt}$ the point, recover Beilinson's def

Def $\quad SS_S \mathcal{F}$ $\quad$ smallest C on which

$\mathcal{F}$ is S-m.s.

$SS_S^{sat} \mathcal{F}$ $\quad$ smallest S-sat C

on which — — -

abs rel.

Conj. $\exists$ ms on $C \Rightarrow \exists$ S-ms on C

true if S sm. over k perfect

Converse

C. S-saturated stable under the $FT^*S \times_S X$-action

Prop $\exists$ S-ms on C, C S-sat $\Rightarrow \exists$ ms on C

too small

$$SS_S \mathcal{F} \subset SS\mathcal{F} \subset SS_S^{sat}\mathcal{F}$$

not yet known to exists exists

Existence of $SS_S\mathcal{F}$? reduced to X smooth$/S$, $X = \mathbb{P}^n_S$.

Thm If X is smooth$/S$.

$SS_S^{sat}\mathcal{F}$ exists

Proof $0 \to FT^*S \times_S X \to FT^*X \to FT^*X/S|_{X_F} \to 0$

$\cup$ $\cup$ exact

saturated $C^{(1)}$

Beilinson's pf. works

Questions

- dim = dox ?
- rel'n abs vs rel ?
- $SS^{sat}_{S} \longrightarrow SS$ how to recover ?