$SS\mathcal{F} \subset T^*X$. $CC\mathcal{F}$

no direct construction on T^*X is known.

transcendental setting Kashiwara - Schapira
micro localization
μhom.

negative $\longrightarrow$ at least neutral.

Cisinski . Khan . Yang . .
same ⊗ . no final goal.

1. SS & CC.
2. ~~....~~ specialization & microlocalization
3. construction.
4. functoriality

1 k field ch $p > 0$.
X smooth / k dim n
T^*X cotangent bdle rk n.
Λ finite local ring ch $\ell \neq p$.
$D^b_{ctf}(X, \Lambda)$ bounded constructible
finite tor dim. on X_{et}.

$\mathcal{F} \in \mathrm{Ob}\, D^b_{ctf}$

stable under $\mathbb{G}_m$-action

$SS\mathcal{F} \subset T^*X$. closed conical subset
$= \bigcup C_a$ irred cpt dim $C_a = n$
$CC\mathcal{F} = \sum m_a C_a$. $m_a \in \mathbb{Z}$. if k perfect

Example

1. $\mathcal{F}$ l.c.c $\Leftarrow$ $SS\mathcal{F} \subset T^*_X X$ 0-section

$$CC\mathcal{F} = (-1)^n \cdot \mathrm{rk}\,\mathcal{F} \cdot [T^*_X X]$$

2. $\dim X = 1$ $U \subset X$ largest open s.t. $\mathcal{F}|_U$ l.c.c

$$SS\mathcal{F} = T^*_X X \cup \bigcup_{x \in X-U} (T^*_x X)$$

if $\mathcal{F}|_U \neq 0$

$$CC\mathcal{F} = (-1)\left(\mathrm{rk}\,\mathcal{F}|_U \cdot [T^*_X X] + \sum_{x \in X-U} a_x \mathcal{F} \cdot [T^*_x X]\right)$$

cf. the proof

$$a_x \mathcal{F} = \mathrm{rk}\,\mathcal{F}|_U - \mathrm{rk}\,\mathcal{F}_{\bar{x}} + \mathrm{Sw}_x \mathcal{F}.$$

2 def to normal cone

X/k smooth $Z \subset X$ closed subsch smooth/k

$$X \times \mathbb{A}^1 \leftarrow (X \times \mathbb{A}^1)' \supset (X \times \mathbb{A}^1)^\sim$$

bl up at $Z \times 0$ compl't of p.t. of $X \times 0$

normal bdle

$$\begin{array}{ccccc} T_Z X & \to & (X \times \mathbb{A}^1)^\sim & \leftarrow & X \times \mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathbb{A}^1 & \leftarrow & \mathbb{G}_m \end{array}$$

$\mathcal{F}$ on X specialization $\nu_{Z/X}\mathcal{F}$ on $T_Z X$

$$= \psi(pr_1^* \mathcal{F})$$

nearby cycles on $T_Z X$ on $X \times \mathbb{G}_m$

Fourier transfm. $\psi : \mathbb{F}_p \to \Lambda^\times$ faithful character

$\mathbb{A}^1 \to \mathbb{A}^1$ $t \mapsto t^p - t = x$.   Artin Schreier Gal $= \mathbb{F}_p$

$\mathcal{L}_\psi$ rk 1 l.c.c on $\mathbb{A}^1$.

E v-bon X $\quad E^\vee$ dual

$\mu: E \times_X E^\vee \to \mathbb{A}^1$ can. pairing

$$\begin{array}{ccc} & pr_1 \swarrow \quad \searrow pr_2 & \\ \mathcal{F} \quad E & & E^\vee \end{array} \qquad F_\psi \mathcal{F} = pr_{2!}(pr_1^* \mathcal{F} \otimes \mu^* \mathcal{L}_\psi)$$

e.g. $F_\psi(e^! \Lambda) = 0^\vee_* \Lambda \qquad e: E \to X \quad 0^\vee: X \to E^\vee$

$F_\psi(0_* \Lambda) = \Lambda. \qquad 0: X \to E$

microlocalization $\mu_{Z/X} \mathcal{F} = F_\psi \nu_{Z/X} \mathcal{F}$

3

$X \subset X \times X$ diagonal

$T_X X \times X = TX$ tangent b'dle

$\mathcal{F}_1, \mathcal{F}_2$ on X

$\nu\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2) = \nu_{X/X\times X} \mathcal{H}om(pr_1^* \mathcal{F}_1, pr_2^! \mathcal{F}_2)$ on TX

$\mu\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2) = \mu_{X/X\times X} \quad " \quad$ on T^*X

Def'n 1. $SS_\mu \mathcal{F} = \operatorname{supp} \mu\mathcal{H}om(\mathcal{F}, \mathcal{F}) \subset T^*X$

Quest 1. $SS_\mu \mathcal{F} \subset SS\mathcal{F}$?

$$\delta^! \overset{\mathcal{H}}{\mathcal{H}om}(pr_1^* \mathcal{F}_1, pr_2^! \mathcal{F}_2) \simeq \mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$$

$$\delta^* \underline{\qquad\qquad} \simeq D_X \mathcal{F}_1 \otimes \mathcal{F}_2$$

$\mathcal{H}om(\mathcal{F}, \mathcal{K}_X)$

$\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2 \qquad \delta^! \mathcal{H} \leftarrow \Lambda \qquad \mathcal{H}om(\mathcal{F}, \mathcal{F}) \leftarrow \Lambda \qquad a^! \Lambda$

$\delta^* \mathcal{H} \simeq D_X \mathcal{F} \otimes \mathcal{F} \to \mathcal{K}_X$ .ev

$$\Lambda \xrightarrow{c} \delta^! \mathcal{H} \to \delta^* \mathcal{H} \xrightarrow{ev} \mathcal{K}_X \qquad H^0(X, \mathcal{K}_X) \ni C(\mathcal{F})$$

can. class

adj $\quad \mathcal{O}_!\Lambda \to \mathcal{K} \to \mathcal{O}_* K_X$ on $X \times X$

$\nu_X / K_{X\times X}$

$$\mathcal{O}_!\Lambda \to \nu\mathcal{K}_{en} \to \mathcal{O}_* K_X \quad \text{on } TX$$

F_4

$$\Lambda \to \mu\mathcal{K}_{en} \to e^* K_X$$

Def'n $\quad C(\mu) \in H^0_{SS}(T^*X, e^{\nu*} K_X)$

Qn 2 $\quad C(\mu) = d\, C(\ )$?

Prop 1. If $\dim X \leq 1$. Yes for Qn 1 & 2.

Prop 2. Yes for Qn 1 $\Rightarrow$ Yes for Qn 2.

4. functoriality

$f: X \to Y$ proper $\quad T^*X \xleftarrow{a} T^*_{f,X} \xrightarrow{b} T^*Y$

$f_* = b_* \circ a^*$ $\qquad b_* a^* e^{\nu*}_X K_X \to e^{\nu*}_Y K_Y$

Prop 3. 1. $\quad SS_\mu(f_*) \subset f_\circ SS_\mu$

2. $\quad CC_\mu(f_*) = f_* CC_\mu$

in $H^0_{f(SS)}(T^*Y, e^{\nu*} K_Y)$

Cor X proper / k $\Rightarrow$ $\chi(X_{\bar k}, \ ) = (CC_\mu, T^*_X X)_{T^*X}$

$h: W \to X$ $\mathcal{F}$-transversal

$h^* \mathcal{F} \otimes h^! \Lambda \to h^! \mathcal{F}$ isom

$SS_\mu \mathcal{F}$ - transversal

$b^{-1}(SS_\mu \mathcal{F}) = h^* SS\mathcal{F} \subset T^*X \times_X W \to T^*W$

finite $h^\circ SS\mathcal{F}$ ...

$$T^*X \xleftarrow{b} T^*X \times_X W \xrightarrow{a} T^*W \qquad a b^{-1}$$

$$a_! b^* e_X^{vk} k_X \to e_W^{vk} k_W$$

$$h^! CC_\mu \mathcal{F} = (-1)^{\dim X - \dim W} a_* b^* CC_\mu \mathcal{F}$$

$\in H^0_{h^\circ(SS_\mu \mathcal{F})}(W, e_W^{vk} k_W)$

Prop 4 h. $\mathcal{F}$-trans $SS_\mu \mathcal{F}$-tr.

1. $SS_\mu h^* \mathcal{F} \subset h^\circ(SS_\mu \mathcal{F})$

2 $CC_\mu h^* \mathcal{F} = h^! CC_\mu \mathcal{F}$.

Proof of Prop 1. 1. $\mathcal{F}$. l.c.c $\Rightarrow SS_\mu \mathcal{F} \subset T^*_X X$

$\overset{\dim X = 1}{\Rightarrow}$ $SS_\mu \mathcal{F} \subset SS\mathcal{F}$

2. vk Prop 4 skyscraper Prop 3.

family ~~...~~. compute Milnor.

general. Katz-Gabber. GOS + μ-index.

Prop 2. reduce to X projective.

suffices to compare coeff of dim n. cpt.

take Lefschetz pencil.

$CC\mathcal{F}$ characterized by Milnor fml

$CC_\mu \mathcal{F}$ satisfies Milnor fml by Prop 1. & 3.