Iwasawa theory of de Rham (φ, Γ)-modules over the Robba ring

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Notation

K : a finite extension of $\mathbb{Q}_p$. $G_K := \mathrm{Gal}(\overline{K}/K)$. $\{\zeta_{p^n}\}_{n\geqq 0}$: a system of primitive p^n-th roots of unity such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. $K_n := K(\zeta_{p^n})$ $(n \geqq 1)$, $K_\infty := \cup_{n\geqq 1} K_n$. $\Gamma_K := \mathrm{Gal}(K_\infty/K)$. $\chi : \Gamma_K \hookrightarrow \mathbb{Z}_p^\times$: p-adic cyclotomic character. $\mathbf{B}_{\mathrm{cris}}, \mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}, \mathbf{B}_e := \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$: Fontaine's rings of p-adic periods. For a p-adic representation V of G_K, set $\mathbf{D}_{\mathrm{cris}}^K(V) := (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \mathbf{D}_{\mathrm{dR}}^K(V) := (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$.

Introduction

• Bloch-Kato's exponential map

Let V be a p-adic representation of G_K. Bloch-Kato ([BK90]) defined a $\mathbb{Q}_p$-linear map

$$\exp_{K,V} := \delta : \mathbf{D}_{\mathrm{dR}}^K(V) \to \mathrm{H}^1(K, V)$$

called Bloch-Kato's exponential map and is defined as the first boundary map δ of the following long exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(K,V) \to \mathrm{H}^0(K, \mathbf{B}_e \otimes V) \oplus\ & \mathrm{H}^0(K, \mathbf{B}_{\mathrm{dR}}^+ \otimes V) \\ & \to \mathrm{H}^0(K, \mathbf{B}_{\mathrm{dR}} \otimes V) \\ \xrightarrow{\delta} \mathrm{H}^1(K,V) \to \mathrm{H}^1(K, \mathbf{B}_e \otimes V) \oplus\ & \mathrm{H}^1(K, \mathbf{B}_{\mathrm{dR}}^+ \otimes V) \\ & \to \mathrm{H}^1(K, \mathbf{B}_{\mathrm{dR}} \otimes V) \\ \to \mathrm{H}^2(K,V) \to \mathrm{H}^2(K, \mathbf{B}_e \otimes V) \ & \to 0. \cdots (**)_V. \end{aligned}$$

associated to the short exact sequence obtained by tensoring V with Bloch-Kato's fundamental short exact sequence

$$0 \to \mathbb{Q}_p \xrightarrow{x \mapsto (x,x)} \mathbf{B}_e \oplus \mathbf{B}_{\mathrm{dR}}^+ \xrightarrow{(x,y)\mapsto x-y} \mathbf{B}_{\mathrm{dR}} \to 0.$$

When V is de Rham, we can define the dual exponential map

$$\exp^*_{K,V^*(1)} : \mathrm{H}^1(K,V) \to \mathbf{D}_{\mathrm{dR}}^K(V)$$

using the Tate paring $\mathrm{H}^1(K,V) \times \mathrm{H}^1(K, V^*(1)) \to \mathbb{Q}_p$ and the canonical paring $\mathbf{D}_{\mathrm{dR}}^K(V) \times \mathbf{D}_{\mathrm{dR}}^K(V^*(1)) \to K$, where $V^*(1)$ is the Tate dual of V.

• Perrin-Riou's exponential map

Perrin-Riou ([Per94]) constructed a system of maps which interpolate Bloch-Kato's exponential and the dual exponential maps. Let $\Lambda(\Gamma_K) := \mathbb{Z}_p[[\Gamma_K]]$ be the Iwasawa algebra of Γ_K and $\mathcal{R}^+(\Gamma_K) := \Gamma(\mathrm{Spf}(\Lambda(\Gamma_K))^{\mathrm{an}}, \mathcal{O})$ be the global section of the rigid analytic space $\mathrm{Spf}(\Lambda(\Gamma_K))^{\mathrm{an}}$over $\mathbb{Q}_p$ associated to the formal scheme $\mathrm{Spf}(\Lambda(\Gamma_K))$. For a p-adic representation V of G_K and $q \geqq 0$, define a $\Lambda(\Gamma_K)[1/p]$-module

$$\mathbf{H}^q_{\mathrm{Iw}}(K,V) := (\varprojlim_n \mathrm{H}^q(K_n, T)) \otimes_{\Lambda(\Gamma_K)} \Lambda(\Gamma_K)[1/p],$$

where T is a G_K-stable $\mathbb{Z}_p$-lattice of V and the limit is taken with respect to the corestriction maps. When K is unramified over $\mathbb{Q}_p$ and V is crystalline, Perrin-Riou constructed a system of $\mathcal{R}^+(\Gamma_K)$-linear maps $\{\Omega_{V,h}\}_{h>>0}$ called Perrin-Riou's exponential map

$$\begin{aligned}\Omega_{V,h} : & (\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}^K(V))^{\Delta=0} \\ & \to \mathcal{R}^+(\Gamma_K) \otimes_{\Lambda(\Gamma_K)[1/p]} (\mathbf{H}^1_{\mathrm{Iw}}(K,V)/\mathbf{H}^1_{\mathrm{iw}}(K,V)_{\mathrm{tor}})\end{aligned}$$

which interpolate $\exp_{L,V(k)}$ and $\exp^*_{L,V^*(1)}$ for any $L = K_n, K$ and for suitable $k \in \mathbb{Z}$, where $(\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}^K(V))^{\Delta=0}$ is a sub $\mathcal{R}^+(\Gamma_K)$-module of $\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}^K(V)$.

• (φ, Γ)-modules over the Robba ring

By the results of Fontaine, Cherbonnier-Colmez and Kedlaya, there exists a exact fully faithful functor

$$\begin{aligned}\mathbf{D}_{\mathrm{rig}} : & \{\text{the category of } p\text{-adic representations of } G_K\} \\ & \hookrightarrow \{\text{the category of } (\varphi,\Gamma)\text{-modules over } \mathcal{R}_K\} \\ & V \mapsto \mathbf{D}_{\mathrm{rig}}(V) := (\mathbf{B}^\dagger_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\overline{K}/K_\infty)}\end{aligned}$$

• Aim of this poster and of [Na12]

We study Bloch-Kato's and Perrin-Riou's exponential maps in the framework of (φ, Γ)-modules over the Robba ring. In particular,

(1) we define Bloch-Kato's exponential map for (φ,Γ)-modules without using Fontaine's rings of p-adic periods,

(2) we generalize Perrin-Riou's exponential maps for all the de Rham (φ,Γ)-modules.

Bloch-Kato's exponential map for (φ, Γ)-modules

• Recall of basic definitions

$\mathcal{R}_K := \cup_{n>>0} \mathcal{R}_K^{(n)}$: the Robba ring of K, i.e. the ring of rigid analytic functions on some annulus with φ, Γ_K-actions, more precisely, when K is unramified, $\mathcal{R}_K^{(n)}$ and the actions of φ, Γ_K are defined by

$$\mathcal{R}_K^{(n)} := \{f(T) := \textstyle\sum_{m\in\mathbb{Z}} a_m T^m | a_m \in K, f(T) \text{ is convergent on } |\zeta_{p^n} - 1| \leqq |T| < 1\},$$

$$\varphi(f(T)) := \sum_{m\in\mathbb{Z}} \varphi(a_m)((1+T)^p - 1)^m,$$

$$\gamma(f(T)) := \sum_{m\in\mathbb{Z}} a_m((1+T)^{\chi(\gamma)} - 1)^m \ (\gamma \in \Gamma_K).$$

Set $t := \log(1+T) \in \mathcal{R}_K$ ($\varphi(t) = pt, \gamma(t) = \chi(\gamma)t$). For any sufficiently large n, there exists a Γ_K-equivariant injection

$$\iota_n : \mathcal{R}_K^{(n)} \hookrightarrow K_n[[t]]$$

such that

$$\iota_n(T) = \zeta_{p^n}\exp(t/p^n) - 1.$$

Definition 0.1. We call D a (φ,Γ)-module over $\mathcal{R}_K$ if

(1) D is a φ-module over $\mathcal{R}_K$, i,e, a finite free $\mathcal{R}_K$-module with a φ-semi-linear action $\varphi : D \hookrightarrow D$ such that the map $\mathcal{R}_K \otimes_{\varphi, \mathcal{R}_K} D \to D : a \otimes x \mapsto a\varphi(x)$ is isomorphism,

(2) D is equipped with a continuous semi-linear action of Γ_K which commutes with φ.

For each $k \in \mathbb{Z}$, we define the k-th Tate twist $\mathcal{R}_K(k) := \mathcal{R}_K e_k$ by $\varphi(e_k) := e_k$ and $\gamma(e_k) := \chi(\gamma)^k e_k$ $(\gamma \in \Gamma_K)$.

Remark 0.2. It is known that D can be written uniquely as $D = \cup_{n>>0} D^{(n)}$ such that $D^{(n)}$ is a finite free $\mathcal{R}_K^{(n)}$-module, $\mathcal{R}_K \otimes_{\mathcal{R}_K^{(n)}} D^{(n)} = D$, $\mathcal{R}_K^{(n+1)} \otimes_{\varphi, \mathcal{R}_K^{(n)}} D^{(n)} \to D^{(n+1)} : a \otimes x \mapsto a\varphi(x)$ is isomorphism.

Set $\mathbf{D}^+_{\mathrm{dif},n}(D) := K_n[[t]] \otimes_{\iota_n, R_K^{(n)}} D^{(n)}, \mathbf{D}_{\mathrm{dif},n}(D) := \mathbf{D}^+_{\mathrm{dif},n}(D)[1/t], \mathbf{D}^+_{\mathrm{dif}}(D) := K_\infty[[t]] \otimes_{K_n[[t]]} \mathbf{D}^+_{\mathrm{dif},n}(D)$, $\mathbf{D}_{\mathrm{dif}}(D) := \mathbf{D}^+_{\mathrm{dif}}(D)[1/t]$, where $K_\infty[[t]] := \cup_{n\geqq 1} K_n[[t]]$. For simplicity, we assume that there exists a topological generator $\gamma \in \Gamma_K$. For $D_0 := D, D[1/t]$, we define a complex

$$C^\bullet_{\varphi,\gamma}(D_0) : [D_0 \xrightarrow{d_1} D_0 \oplus D_0 \xrightarrow{d_2} D_0]$$

with $d_1(x) := ((\gamma-1)x, (\varphi-1)x), d_2(x,y) := (\varphi-1)x - (\gamma-1)y$. For $D_1 := \mathbf{D}^+_{\mathrm{dif}}(D), \mathbf{D}_{\mathrm{dif}}(D)$, we define

$$C^\bullet_\gamma(D_1) : [D_1 \xrightarrow{x \mapsto (\gamma-1)x} D_1].$$

Set $\mathrm{H}^q(K, D_0) := \mathrm{H}^q(C^\bullet_{\varphi,\gamma}(D_0))$, $\mathrm{H}^q(K, D_1) := \mathrm{H}^q(C^\bullet_\gamma(D_1))$, $\mathbf{D}_{\mathrm{dR}}^K(D) := \mathrm{H}^0(K, \mathbf{D}_{\mathrm{dif}}(D))$, $\mathbf{D}_{\mathrm{cris}}^K(D) := D[1/t]^{\gamma=1}$.

• Bloch-Kato's exponential map for (φ,Γ)-modules

Our first theorem is the following.

Theorem 0.3. (1) *There exists the following functorial exact sequence*

$$\begin{aligned} 0 \to \mathrm{H}^0(K,D) \to \mathrm{H}^0(K, D[1/t]) \oplus\ & \mathrm{H}^0(K, \mathbf{D}^+_{\mathrm{dif}}(D)) \\ & \to \mathrm{H}^0(K, \mathbf{D}_{\mathrm{dif}}(D)) \\ \xrightarrow{\delta} \mathrm{H}^1(K,D) \to \mathrm{H}^1(K, D[1/t]) \oplus\ & \mathrm{H}^1(K, \mathbf{D}^+_{\mathrm{dif}}(D)) \\ & \to \mathrm{H}^1(K, \mathbf{D}_{\mathrm{dif}}(D)) \\ \to \mathrm{H}^2(K,D) \to \mathrm{H}^2(K, D[1/t]) \to 0. & \cdots (**)_D \end{aligned}$$

(2) *For any p-adic representation V of G_K, we have canonical isomorphisms*

(i) $\mathrm{H}^q(K,V) \xrightarrow{\sim} \mathrm{H}^q(K, \mathbf{D}_{\mathrm{rig}}(D))$,

(ii) $\mathrm{H}^q(K, \mathbf{B}_e \otimes V) \xrightarrow{\sim} \mathrm{H}^q(K, \mathbf{D}_{\mathrm{rig}}(V)[1/t])$,

(iii) $\mathrm{H}^q(K, \mathbf{B}_{\mathrm{dR}}^{(+)} \otimes V) \xrightarrow{\sim} \mathrm{H}^q(K, \mathbf{D}^{(+)}_{\mathrm{dif}}(\mathbf{D}_{\mathrm{rig}}(V)))$,

and these isomorphisms induce an isomorphism between $(**)_V$ and $(**)_{\mathbf{D}_{\mathrm{rig}}(V)}$.

Remark 0.4. The isomorphism (i) was proved by Liu and (iii) was proved by Fontaine.

Definition 0.5. We define Bloch-Kato's exponential map of D as the first boundary map δ of the above exact sequence

$$\exp_{K,D} := \delta : \mathbf{D}_{\mathrm{dR}}^K(D) \to \mathrm{H}^1(K,D).$$

Remark 0.6. $\exp_{K,D}$ is explicitly defined as follows. For any $x \in \mathbf{D}_{\mathrm{dR}}^K(D)$, we can take $\tilde{x} \in D^{(n)}[1/t]$ $(n >> 0)$ such that $\iota_m(\tilde{x}) - x \in \mathbf{D}^+_{\mathrm{dif},m}(D)$ for any $m \geqq n$, then we can show that $(\gamma-1)\tilde{x}, (\varphi-1)\tilde{x} \in D$ and that

$$\exp_{K,D} := [((\gamma-1)\tilde{x}, (\varphi-1)\tilde{x})] \in \mathrm{H}^1(K,D).$$

When D is de Rham, i.e. when the equality $\dim_K \mathbf{D}_{\mathrm{dR}}^K(D) = \mathrm{rank}(D)$ holds, we can define the dual exponential map

$$\exp^*_{K,D^*(1)} : \mathrm{H}^1(K,D) \to \mathbf{D}_{\mathrm{dR}}^K(D)$$

using the Liu's paring $\mathrm{H}^1(K,D) \times \mathrm{H}^1(K, D^*(1)) \to \mathbb{Q}_p$ and the canonical paring $\mathbf{D}_{\mathrm{dR}}^K(D) \times \mathbf{D}_{\mathrm{dR}}^K(D^*(1)) \to K$.

Perrin-Riou's exponential map for de Rham (φ,Γ)-modules

To generalize Perrin-Riou's exponential map for de Rham (φ,Γ)-modules, we need to generalize

(i) the definition of $\mathbf{H}^q_{\mathrm{Iw}}(K,V)$ for D,

(ii) $\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}_{\mathrm{cris}}^K(V)$ for de Rham D.

• Analytic Iwasawa cohomology after Pottharst

As a generalization of Iwasawa cohomology $\mathbf{H}^q_{\mathrm{Iw}}(K,V)$, for any (φ,Γ)-module D over $\mathcal{R}_K$, Pottharst ([Po10]) defined $\mathcal{R}^+(\Gamma_K)$-module $\mathbf{H}^q_{\mathrm{Iw}}(K,D)$ called analytic Iwasawa cohomology (c.f. Xiao's talk), which satisfies the following properties.

Theorem 0.7. *([Po10])*

(1) $\mathbf{H}^q_{\mathrm{Iw}}(K,D) = 0$ *if* $q \neq 1,2$,

(2) $\mathbf{H}^1_{\mathrm{Iw}}(K,D)$ *is a finite $\mathcal{R}^+(\Gamma_K)$-module such that $\mathbf{H}^1_{\mathrm{Iw}}(K,D)_{\mathrm{tor}}$ is finite dimensional over $\mathbb{Q}_p$ and $\mathbf{H}^1_{\mathrm{Iw}}(K,D)/\mathbf{H}^1_{\mathrm{Iw}}(K,D)_{\mathrm{tor}}$ is free of rank $[K:\mathbb{Q}_p]\mathrm{rank}(D)$,*

(3) $\mathbf{H}^2_{\mathrm{Iw}}(K,D)$ *is finite dimensional over $\mathbb{Q}_p$.*

(4) *For any p-adic representation V of G_K, we have a canonical isomorphism*

$$\mathbf{H}^q_{\mathrm{Iw}}(K, \mathbf{D}_{\mathrm{rig}}(V)) \xrightarrow{\sim} \mathcal{R}^+(\Gamma_K) \otimes_{\Lambda(\Gamma_K)[1/p]} \mathbf{H}^q_{\mathrm{Iw}}(K,V).$$

(5) $\mathbf{H}^q_{\mathrm{Iw}}(K,D)$ *can be defined as the cohomology of the following complex*

$$C^\bullet_\psi(D) : [0 \to D \xrightarrow{\psi-1} D],$$

i.e. we have the following canonical isomorphisms of $\mathcal{R}^+(\Gamma_K)$-modules

$$\mathbf{H}^1_{\mathrm{Iw}}(K,D) \xrightarrow{\sim} D^{\psi=1}, \mathbf{H}^2_{\mathrm{Iw}}(K,D) \xrightarrow{\sim} D/(\psi-1)D.$$

Remark 0.8. $\psi : D \to D$ is defined by

$$\psi : D = \oplus_{i=0}^{p-1}(1+T)^i\varphi(D) \to D : x = \sum_{i=0}^{p-1}(1+T)^i\varphi(x_i) \mapsto x_0.$$

For each $L = K_n, K$ and $k \in \mathbb{Z}$, we have a canonical projection map

$$\mathrm{pr}_{L,k} : \mathbf{H}^q_{\mathrm{Iw}}(K,D) \to \mathrm{H}^q(L, D(k)).$$

• p-adic differantial equations associated to de Rham (φ,Γ)-modules after Berger

As a generalization of $\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}_{\mathrm{cris}}^K(V)$ for general de Rham D, we propose to use the p-adic differential equation $\mathbf{N}_{\mathrm{rig}}(D)$ associated to D, which was defined by Berger ([Ber02], [Ber08]). From now on, we assume that D is de Rham, then we have a canonical isomorphism $K_n((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}^K(D) \xrightarrow{\sim} \mathbf{D}_{\mathrm{dif},n}(D)$ for any $n >> 0$, hence $K_n[[t]] \otimes_K \mathbf{D}_{\mathrm{dR}}^K(D)$ becomes a Γ_K-stable $K_n[[t]]$-lattice of $\mathbf{D}_{\mathrm{dif},n}(D)$. Set $\nabla_0 := \frac{\log(\gamma)}{\log(\chi(\gamma))} \in \mathcal{R}^+(\Gamma_K)$ and $\nabla_i := \nabla_0 - i$ $(i \in \mathbb{Z})$. ∇_0 acts on $\mathcal{R}_K$ by the formula $\nabla_0(f) := t(1+T)\frac{df}{dT}$.

Theorem 0.9. *([Ber02],[Ber08]) For n sufficiently large, set*

$$\mathbf{N}^{(n)}_{\mathrm{rig}}(D) := \{x \in D^{(n)}[1/t] | \iota_m(x) \in K_m[[t]] \otimes_K \mathbf{D}_{\mathrm{dR}}^K(D) \text{ for any } m \geqq n\},$$

then $\mathbf{N}_{\mathrm{rig}}(D) := \cup_{n>>0}\mathbf{N}^{(n)}_{\mathrm{rig}}(D)$ is a (φ,Γ)-module over $\mathcal{R}_K$ which satisfies the following,

(1) $\mathbf{N}_{\mathrm{rig}}(D)[1/t] = D[1/t]$,

(2) $\mathbf{D}^+_{\mathrm{dif},n}(\mathbf{N}_{\mathrm{rig}}(D)) = K_n[[t]] \otimes_K \mathbf{D}_{\mathrm{dR}}^K(D)$,

(3) $\nabla_0(\mathbf{N}_{\mathrm{rig}}(D)) \subseteq t\mathbf{N}_{\mathrm{rig}}(D)$.

By the condition (3), we can define a differential operator

$$\partial : \mathbf{N}_{\mathrm{rig}}(D) \to \mathbf{N}_{\mathrm{rig}}(D(-1)) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig}}(D) \otimes te_{-1}$$

by

$$\partial(x) := \frac{\nabla_0(x)}{t} \otimes te_{-1},$$

where e_{-1} is the base of $\mathcal{R}_K(-1)$.
For each $L = K_n, K$, we define a map

$$T_{L,D} : \mathbf{H}^1_{\mathrm{Iw}}(K, \mathbf{N}_{\mathrm{rig}}(D)) \to \mathbf{D}^L_{\mathrm{dR}}(D)$$

as the composite of $\mathbf{H}^1_{\mathrm{Iw}}(K, \mathbf{N}_{\mathrm{rig}}(D)) \xrightarrow{\sim} \mathbf{N}^{(m)}_{\mathrm{rig}}(D)^{\psi=1}$ ($m >> n$), $\iota_m : \mathbf{N}^{(m)}_{\mathrm{rig}}(D)^{\psi=1} \hookrightarrow K_m[[t]] \otimes_{K_m} \mathbf{D}^{K_m}_{\mathrm{dR}}(D)$ (use (2)), $K_m[[t]] \otimes_{K_m} \mathbf{D}^{K_m}_{\mathrm{dR}}(D) \to \mathbf{D}^{K_m}_{\mathrm{dR}}(D) : \sum_{k=0}^{\infty} a_k t^k \mapsto a_0$, $\frac{1}{[K_m:L]}\mathrm{Tr}_{K_m/L} : \mathbf{D}^{K_m}_{\mathrm{dR}}(D) \to \mathbf{D}^L_{\mathrm{dR}}(D)$.

• Perrin-Riou's exponential map

Using (2) and (3), we can easily show the following lemma.

Lemma 0.10. *Let $h \in \mathbb{Z}_{\geqq 1}$ such that $\mathrm{Fil}^{-h}\mathbf{D}^K_{\mathrm{dR}}(D) = \mathbf{D}^K_{\mathrm{dR}}(D)$, then $t^h\mathbf{N}_{\mathrm{rig}}(D) \subseteq D$, in particular, $\nabla_{h-1}\cdots\nabla_0(\mathbf{N}_{\mathrm{rig}}(D)) \subseteq D$.*

Using this lemma, we define the following map, which is a generalization of Perrin-Riou's exponential map for de Rham case. The following definition is strongly influenced by the work of Berger ([Ber03]).

Definition 0.11. For any $h \in \mathbb{Z}_{\geqq 1}$ such that $\mathrm{Fil}^{-h}\mathbf{D}^K_{\mathrm{dR}}(D) = \mathbf{D}^K_{\mathrm{dR}}(D)$, then we define $\mathcal{R}^+(\Gamma_K)$-linear map

$$\mathrm{Exp}_{D,h} : \mathbf{H}^1_{\mathrm{Iw}}(K, \mathbf{N}_{\mathrm{rig}}(D)) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig}}(D)^{\psi=1} \xrightarrow{\nabla_{h-1}\cdots\nabla_0} D^{\psi=1} \xrightarrow{\sim} \mathbf{H}^1_{\mathrm{Iw}}(K, D).$$

Our main theorem is the following, which states that $\mathrm{Exp}_{D,h}$ interpolates $\exp_{L,D(k)}$ and $\exp^*_{L,D^*(1-k)}$.

Theorem 0.12. *For any $L = K_n, K$,*

(1) (i) if $k \geqq 1$ and if there exists $x_k \in \mathbf{H}^1_{\mathrm{Iw}}(K, \mathbf{N}_{\mathrm{rig}}(D(k)))$ such that $\partial^k(x_k) = x$,

or

(ii) if $0 \geqq k \geqq -(h-1)$ and $x_k := \partial^{-k}(x)$,

then we have

$$\mathrm{pr}_{L,k}\ (\mathrm{Exp}_{D,h}(x)) = \frac{(-1)^{h+k-1}(h+k-1)!|\Gamma_{L,\mathrm{tor}}|}{p^{n(L)}}\exp_{L,D(k)}(T_{L,D(k)}(x_k)),$$

(2) if $-h \geqq k$, then we have

$$\exp^*_{L,D^*(1-k)}(\ \mathrm{pr}_{L,k}(\mathrm{Exp}_{D,h}(x))) = \frac{|\Gamma_{L,\mathrm{tor}}|}{(-h-k)!p^{n(L)}}T_{L,D(k)}(\partial^{-k}(x))$$

here $|\Gamma_{L,\mathrm{tor}}|$ is the order of $\Gamma_{L,\mathrm{tor}}$ and $n(L) := \min\{\mathrm{ord}_p(\log(\chi(\gamma))|\gamma \in \Gamma_L\}$.

• Determinant of $\mathrm{Exp}_{D,h}$

To state the theorem, we need to recall the notion of characteristic ideal of torsion co-admissible $\mathcal{R}^+(\Gamma_K)$-modules. Let $\mathfrak{m}$ be the Jacpbson radical of $\Lambda(\Gamma_K)$. Set $\Lambda_n := \widehat{\Lambda(\Gamma_K)[\frac{\mathfrak{m}^n}{p}]}[1/p]$ for each $n \geqq 1$ (here $\widehat{A}$ is the p-adic completion of A),then $\mathcal{R}^+(\Gamma_K) \xrightarrow{\sim} \varprojlim_n \Lambda_n$ and Λ_n is a finite product of P.I.D. For a $\mathcal{R}^+(\Gamma_K)$-module M which is finite dimensional over $\mathbb{Q}_p$ (more generally torsion co-admissible), then we can define unique principal ideal $\mathrm{char}(M) = (f_M) \subseteq \mathcal{R}^+(\Gamma_K)$ such that $f_M\Lambda_n = \mathrm{char}_{\Lambda_n}(\Lambda_n \otimes_{\mathcal{R}^+(\Gamma_K)} M)$ for any n. Set $\mathbf{H}^1(K, D)_{\mathrm{fr}} := \mathbf{H}^1(K, D)/\mathbf{H}^1(K, D)_{\mathrm{tor}}$.

Theorem 0.13. $(\delta'(D))$
Let $d := \mathrm{rank}(D)$ and $\{h_1, h_2, \cdots, h_d\}$ be the Hodge-Tate weight of D (we take Hodge-Tate weight of $\mathbb{Q}_p(1)$ is 1). Then we have the following equality of fractional principal ideals of $\mathcal{R}^+(\Gamma_K)$.

$$\begin{aligned}\det\ &(\mathrm{Exp}_{D,h} : \mathbf{H}^1(K, \mathbf{N}_{\mathrm{rig}}(D))_{\mathrm{fr}} \to \mathbf{H}^1(K, D)_{\mathrm{fr}})\\ &= (\textstyle\prod_{i=1}^{d} \nabla_{h_i}\nabla_{h_i+1}\cdots\nabla_{h-1})^{[K:\mathbb{Q}_p]}\\ &\quad \mathrm{char}(\mathbf{H}^1_{\mathrm{Iw}}(K, \mathbf{N}_{\mathrm{rig}}(D))_{\mathrm{tor}})\mathrm{char}(\mathbf{H}^1_{\mathrm{Iw}}(K, D)_{\mathrm{tor}})^{-1}\\ &\quad \mathrm{char}(\mathbf{H}^2_{\mathrm{Iw}}(K, \mathbf{N}_{\mathrm{rig}}(D)))^{-1}\mathrm{char}(\mathbf{H}^2_{\mathrm{Iw}}(K, D)).\end{aligned}$$

Crystalline case

Here, we assume that K is unramified over $\mathbb{Q}_p$ and that D is crystalline (i.e. $\dim_K\mathbf{D}^K_{\mathrm{cris}}(D) = \mathrm{rank}(D)$). Under this assumption, we compare our results with Perrin-Riou's results.
Define a φ, Γ_K-stable subring $\mathcal{R}^+_K$ of $\mathcal{R}_K$ by

$$\mathcal{R}^+_K := \{f(T) = \textstyle\sum_{n=0}^{\infty} a_nT^n | a_n \in K,\ f(T) \text{ is convergent on } 0 \leqq |T| < 1\}.$$

Remark 0.14. The following facts are important for comparing our results with Perrion-Riou's ones.

(1) We have $\mathbf{N}_{\mathrm{rig}}(D) = \mathcal{R}_K \otimes_K \mathbf{D}^K_{\mathrm{cris}}(D)$,

(2) $K \otimes_{\mathbb{Q}_p} \mathcal{R}^+(\Gamma_K) \xrightarrow{\sim} (\mathcal{R}^+_K)^{\psi=0} : a \otimes \lambda \mapsto a(\lambda(1+T))$ is isomorphism.

By (2), we obtain an isomorphism $\iota : \mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D) \xrightarrow{\sim} (\mathcal{R}^+_K)^{\psi=0} \otimes_K \mathbf{D}^K_{\mathrm{cris}}(D) = (\mathcal{R}^+_K \otimes_K \mathbf{D}^K_{\mathrm{cris}}(D))^{\psi=0}$
Using this ι, we define a sub $\mathcal{R}^+(\Gamma_K)$-module $(\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D))^{\Delta=0} := \iota^{-1}((\varphi-1)(\mathcal{R}^+_K \otimes_K \mathbf{D}^K_{\mathrm{cris}}(D))^{\psi=1})$ of $\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D)$. By definition, we obtain an isomorphism

$$\iota^{-1}(\varphi-1) : (\mathcal{R}_K \otimes_K \mathbf{D}^K_{\mathrm{cris}}(D))^{\psi=1}/(\mathcal{R}_K \otimes_K \mathbf{D}^K_{\mathrm{cris}}(D))^{\varphi=1} \xrightarrow{\sim} (\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D))^{\Delta=0}.$$

Berger ([Ber03]) proved the following theorem.

Theorem 0.15. ([*Ber03*]) *Perrin-Riou's exponential map*

$$\Omega_{D,h} : (\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D))^{\Delta=0} \to \mathbf{H}^1_{\mathrm{Iw}}(K, D)_{\mathrm{fr}}$$

is equal to the composite of the maps

$$\begin{aligned}&(\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D))^{\Delta=0}\\ &\xrightarrow{(\iota^{-1}(\varphi-1))^{-1}} (\mathcal{R}^+_K \otimes_K \mathbf{D}^K_{\mathrm{cris}}(D))^{\psi=1}/(\mathcal{R}^+_K \otimes_K \mathbf{D}^K_{\mathrm{cris}}(D))^{\varphi=1}\\ &\hookrightarrow (\mathcal{R}_K \otimes_K \mathbf{D}^K_{\mathrm{cris}}(D))^{\psi=1}/(\mathcal{R}_K \otimes_K \mathbf{D}^K_{\mathrm{cris}}(D))^{\varphi=1}\\ &= \mathbf{N}_{\mathrm{rig}}(D)^{\psi=1}/\mathbf{N}_{\mathrm{rig}}(D)^{\varphi=1} \xrightarrow{\sim} \mathbf{H}^1_{\mathrm{Iw}}(K, \mathbf{N}_{\mathrm{rig}}(D)))_{\mathrm{fr}}\\ &\xrightarrow{\mathrm{Exp}_{D,h}} \mathbf{H}^1_{\mathrm{Iw}}(K, D)_{\mathrm{fr}}.\end{aligned}$$

For the determinant of $\Omega_{D,h}$, Perrin-Riou ([Per94]) conjectured (when $D = \mathbf{D}_{\mathrm{rig}}(V)$) the following formula which she called $\delta(V)$-conjecture. She proved that this conjecture is a consequence of her another conjecture $\mathrm{Rec}(V)$. $\mathrm{Rec}(V)$ conjecture was proved by Colmez ([Col98]), Kato-Kurihara-Tsuji ([KKT96]), etc. Potttharst ([Po11]) generalized $\delta(V)$-conjecture for general crystalline D and proved $\delta(D)$ using the slope filtration argument and reducing to étale case.

Theorem 0.16. *($\delta(D)$) Let $\{h_1, \cdots, h_d\}$ be the Hodge-Tate weight of D, then we have the following equality of fractional principal ideal of $\mathcal{R}^+(\Gamma_K)$.*

$$\begin{aligned}\det(\ &\Omega_{D,h} : (\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D))^{\Delta=0} \to \mathbf{H}^1_{\mathrm{Iw}}(K, D)_{\mathrm{fr}})\\ &= (\textstyle\prod_{i=1} \nabla_{h_i}\nabla_{h_i+1}\cdots\nabla_{h-1})^{[K:\mathbb{Q}_p]}\\ &\quad \mathrm{char}(\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D)/(\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D))^{\Delta=0})\\ &\quad \mathrm{char}(\mathbf{H}^1_{\mathrm{Iw}}(K, D)_{\mathrm{tor}})^{-1}\mathrm{char}(\mathbf{H}^2_{\mathrm{Iw}}(K, D)).\end{aligned}$$

Our result in this section is the following, which compare our $\delta'(D)$ with the above $\delta(D)$. In particular, when K is unramified and D is crystalline, our $\delta'(D)$ is equivalent to $\delta(D)$ and gives a new and more direct proof of $\delta(D)$.

Proposition 0.17. *We have the following equality of fractional principal ideals of $\mathcal{R}^+(\Gamma_K)$,*

$$\begin{aligned}\det(\ &\mathrm{Exp}_{D,h} : \mathbf{H}^1(K, \mathbf{N}_{\mathrm{rig}}(D))_{\mathrm{fr}} \to \mathbf{H}^1(K, D)_{\mathrm{fr}})\\ &\mathrm{char}(\mathbf{H}^1_{\mathrm{Iw}}(K, \mathbf{N}_{\mathrm{rig}}(D))_{\mathrm{tor}})^{-1}\mathrm{char}(\mathbf{H}^2_{\mathrm{Iw}}(K, \mathbf{N}_{\mathrm{rig}}(D)))\\ =\ &\det(\Omega_{D,h} : (\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D))^{\Delta=0} \to \mathbf{H}^1_{\mathrm{Iw}}(K, D)_{\mathrm{fr}})\\ &\mathrm{char}(\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D)/(\mathcal{R}^+(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{cris}}(D))^{\Delta=0})^{-1}.\end{aligned}$$

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