RAMIFICATION THEORY AND RECIPROCITY SHEAVES
A LECTURE SERIES BY

THE FRENCH-JAPANESE LABORATORY OF MATHEMATICS
AND ITS INTERACTIONS
THE UNIVERSITY OF TOKYO, KOMABA

Introduction

The aim of these lectures is to give an overview of the two subjects in the title: the theory of ramification for l-adic sheaves, including recent developments involving the singular support and characteristic cycle introduced and developed by Alexander Beilinson, Takeshi Saito and others, and the theory of reciprocity sheaves and motives with modulus introduced and developed by Bruno Kahn, Hiroyasu Miyazaki, Shuji Saito, Takao Yamazaki and others. They are primarily aimed at graduate students, but other interested mathematicians are welcome to attend.

Programme

Ramification and characteristic cycles. The classical theory of ramification was applied to the theory of ℓ-adic sheaves in 1960es by Grothendieck with the help of Serre to prove a formula, called the Grothendieck–Ogg–Shafarevich formula, computing the Euler–Poincaré characteristic of an ℓ-adic sheaf on a curve. The formula is derived from a trace formula going back to Weil. A remarkable analogy between the theory of ℓ-adic sheaves and that of $\mathcal{D}$-modules was observed by Deligne and others. While they share the six functors formalism, the microlocal analysis on the cotangent bundle was first developed only on the $\mathcal{D}$-modules side.

The computation of the Euler–Poincaré characteristic of an ℓ-adic sheaf was generalized to higher dimensions as an index formula in the 2010es. The formula is expressed in terms of the *characteristic cycle* supported on the *singular support* defined in the cotangent bundle,

Date: 17–19/03/2025, *Room*: 117.

similarly as in microlocal analysis. The singular support admits a definition in terms of the six functor formalism, after slightly modifying the original definition due to Beilinson.

The characteristic cycles are also uniquely characterized by functoriaility. The compatibility with proper push-forward generalizes the index formula and was proved recently in a greater generality by Abe. The actual construction of characteristic cycles is based on a formula called the Milnor formula for the vanishing cycles and relies on the formalism of vanishing cycles over the general bases. The Milnor formula goes back to an exposé in SGA7 by Deligne in the constant coefficient case.

The characteristic cycle is determined by the wild ramification of the sheaf. In the case of curves, the contribution of the ramification is given by the conductor and the index formula specializes to the Grothendieck–Ogg–Shafarevich formula. In higher dimensions, the concrete description of the characteristic cycle remains to be investigated further in general.

Outline. These talks will be given by Takeshi Saito, Tomoyuki Abe and Yuri Yatagawa.

(1) Introduction. Transversality and the definition of singular support
(2) Radon transform and the existence of singular support
(3) Vanishing cycles over a general base and Fourier transform
(4) Construction of characteristic cycle
(5) Characteristic cycle and ramification

Reciprocity sheaves. In the 1990es, Voevodsky constructed a triangulated category of motives over a field and then over more general bases, realising part of a programme suggested by Beilinson to reach an abelian category of mixed motives by purely cycle-theoretic methods.[1] This construction has been very useful and rests largely on his theory of $\mathbf{A}^1$-invariant (pre)sheaves with transfers over a perfect field.

The condition of $\mathbf{A}^1$-invariance is the algebro-geometric analogue of homotopy invariance in algebraic topology, but is much more rigid. Recently, the necessity to relax it has appeared; a way to do this is to generalise the notion of *modulus* which originates in E. Artin's formulation of class field theory in the 1920es and was then geometrised by Rosenlicht, Lang and Serre in the late 1950es.

[1] This aim remains out of reach; Nori has constructed an abelian category of mixed motives in characteristic 0, but it has a cohomological ingredient.

When suitably abstracted, this notion of modulus leads to the definition of *reciprocity sheaves*, which generalise Voevodsky's $\mathbf{A}^1$-invariant sheaves (with transfers). Most of his results extend to this context. One can then use the notion of modulus again to define a triangulated category of *motives with modulus* enlarging that of Voevodsky. This category is related to reciprocity sheaves in an interesting way.

Here are two related topics which will only be mentioned in passing for lack of time: a) another enlargement called the triangulated category of *log-motives*, which was constructed by Binda, Park and Østvær; b) (higher) Chow groups with modulus.

Outline. These talks will be given by Bruno Kahn and Hiroyasu Miyazaki. The last talk, by Shuji Saito, will relate the two topics of the Lecture series.

(1) Introduction. Review of Voevodsky's $\mathbf{A}^1$-invariant theory.
(2) First definition and first properties of reciprocity sheaves.
(3) Topologies on modulus pairs.
(4) Construction of categories of motives with modulus; return to reciprocity sheaves.
(5) Reciprocity sheaves and ramification theory.

Schedule

The Lecture Series will take place at the room 117 at the Department of Mathematics, Komaba Campus of the University of Tokyo, on March 17–19 2025.

	Monday 17	Tuesday 18	Wednesday 19
10h00-11h00	Ramification 1	Ramification 3	Reciprocity 4
11h00-11h30	Tea break	Tea break	Tea break
11h30-12h30	Ramification 2	Ramification 4	Ramification 5
14h30-15h30	Reciprocity 1	Reciprocity 3	14h00-15h00 exceptionally
15h30-16h00	Coffee break		Reciprocity 5
16h00-17h00	Reciprocity 2		