1 Singular support

1.1 Closed conical subsets and the transversality

Definition 1.1.1. *Let C be a closed conical subset of the cotangent bundle T^*X and let $h\colon W \to X$ be a morphism of smooth schemes over k.*

We say that h is C-transversal if the intersection of the subsets $h^*C = W \times_X C$ and $\mathrm{Ker}(W \times_X T^*X \to T^*W)$ of $W \times_X T^*X$ is a subset of the 0-section.

The intersection $C \cap T^*_X X$ with the 0-section $X = T^*_X X$ is called the base of C.

If h is smooth, then h is C-transversal for any C.

If C is a subset of the 0-section, any h is C-transversal.

If $C \subset C'$, the C'-transversality implies the C-transversality.

The transversality is an open condition.

Lemma 1.1.2. *Assume that $h\colon W \to X$ is C-transversal. Then, $W \times_X T^*X \to T^*W$ is finite on h^*C.*

Lemma 1.1.3. $\dim h^*C \geqq \dim C + \dim W - \dim X$.

Lemma 1.1.4. *Assume that $h\colon W \to X$ is C-transversal. For a morphism $g\colon V \to W$ of smooth schemes over k, the following conditions are equivalent:*

(1) *g is $h^\circ C$-transversal.*

(2) *$h \circ g$ is C-transversal.*

Definition 1.1.5. *Let C be a closed conical subset of the cotangent bundle T^*X and C' be a closed conical subset of the cotangent bundle T^*Y. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k.*

1. *We say that (h, f) is (C, C')-transversal if $(h, f)\colon W \to X \times Y$ is $C \times C'$-transversal.*

2. *If $h = 1_X$ and $C' = T^*Y$, we say that f is C-transversal if $(1_X, f)$ is (C, T^*Y)-transversal.*

Lemma 1.1.6. 1. *The following conditions are equivalent:*

(1) *$h\colon W \to X$ is C-transversal.*

(2) *$(h, 1_W)$ is $(C, T^*_W W)$-transversal.*

1. *The following conditions are equivalent:*

(1) *$f\colon X \to Y$ is C-transversal.*

(2) *The inverse image of C by $X \times_Y T^*Y \to T^*X$ is a subset of the 0-section.*

2. *The following conditions are equivalent:*

(1) *(h, f) is (C, T^*Y)-transversal.*

(2) *$h\colon W \to X$ is C-transversal and $f\colon W \to X$ is $h^\circ C$-transversal.*

$f\colon X \to Y$ is $T^*_X X$-transversal if and only if f is smooth.

If $f\colon X \to Y$ is C-transversal, then f is smooth on a neighborhood of the base of C.

Definition 1.1.7. *Let $C \subset T^*X$ be a closed conical subset and $f\colon X \to Y$ be a morphism of smooth schemes over k. Assume that f is proper on the base of C. Then, we define a closed conical subset $f_\circ C \subset T^*Y$ by the algebraic correspondence $T^*X \leftarrow X \times_Y T^*Y \to T^*Y$.*

Proposition 1.1.8. *Let $g\colon X' \to X$ be a morphism of smooth schemes over k and let $C \subset T^*X'$ be a closed conical subset. Assume that g is proper on the basis B' of C' and define $C = g_\circ C' \subset T^*X$.*

1. *Let $h\colon W \to X$ be a morphism of smooth schemes over k and*

$$
\begin{array}{ccc}
X' & \xleftarrow{h'} & W' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\
X & \xleftarrow{h} & W
\end{array}
$$

be a cartesian diagram. Assume that h is C-transversal. Then, there exists an open neighborhood U' of the inverse image $B'_{W'} = h'^{-1}(B') \subset W'$ smooth over W.

2. *For a morphism $f\colon W \to Y$ of smooth schemes over k, the following conditions are equivalent:*

(1) *(h, f) is C-transversal.*

(2) *$(h'|_{U'}, f \circ g'|_{U'})$ is C'-transversal.*

1.2 Legendre transform

Let $\mathbf{P}$ be a projective space, $\mathbf{P}^\vee$ be the dual projective space and $Q \subset \mathbf{P} \times \mathbf{P}^\vee$ be the universal hyperplane. The kernel $\mathrm{Ker}((T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P}\times\mathbf{P}^\vee} Q \to T^*Q$ equals the conormal bundle $T^*_Q(\mathbf{P} \times \mathbf{P}^\vee)$.

We identify Q as the projective space bundle $\mathbf{P}(T^*\mathbf{P})$ associated to the vector bundle $T^*\mathbf{P}$. Symmetrically, Q is identified with $\mathbf{P}(T^*\mathbf{P}^\vee)$.

Definition 1.2.1. *Let C be a closed conical subset $C \subset T^*\mathbf{P}$. We consider the projectivization $E = \mathbf{P}(C) \subset \mathbf{P}(T^*\mathbf{P}) = Q$ as a closed subset of Q. Define the Legendre transform $C^\vee = LC$ by $C^\vee = p^\vee_\circ p^\circ C$.*

Lemma 1.2.2. *The intersection of $C \times T^*\mathbf{P}^\vee$ with $\mathrm{Ker}((T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P}\times\mathbf{P}^\vee} Q \to T^*Q = T^*_Q(\mathbf{P} \times \mathbf{P}^\vee)$ equals the union of $T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \times_Q E$ with the 0-section on $p^{-1}B$.*

Proof. Since the image of the conormal bundle $T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \subset (T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P}\times\mathbf{P}^\vee} Q$ in $T^*\mathbf{P} \times_{\mathbf{P}} Q$ by the first projection is the tautological line bundle, the assertion follows.

Proposition 1.2.3. 1. *The complement $Q - E$ is the largest open subset where $(p, p^\vee)$ is C-transversal.*

2. *C is equal to the image of the intersection of $(C \times T^*\mathbf{P}^\vee) \cap T^*_Q(\mathbf{P} \times \mathbf{P}^\vee)$ by the composition $(T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P}\times\mathbf{P}^\vee} Q \to T^*\mathbf{P} \times_{\mathbf{P}} Q \to T^*\mathbf{P}$.*

Proof. 1. Clear from Lemma.

2.

Corollary 1.2.4. $\mathbf{P}(C) = \mathbf{P}(C^\vee)$.

Proof. Since $C^\vee$ is equal to the image of the intersection of $(C \times T^*\mathbf{P}^\vee) \cap T^*_Q(\mathbf{P} \times \mathbf{P}^\vee)$ by the composition $(T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P}\times\mathbf{P}^\vee} Q \to T^*\mathbf{P}^\vee \times_{\mathbf{P}^\vee} Q \to T^*\mathbf{P}^\vee$, it follows from Lemma and Proposition.

Proposition 1.2.5. *Let $C^+ = C \subset T^*_{\mathbf{P}}\mathbf{P}$ be the union with the 0-section. Then, we have*

$$C^+ = p_\circ(p^{\vee\circ} T^*\mathbf{P}^\vee \times_Q E) \cup T^*_{\mathbf{P}}\mathbf{P}.$$

Proof. By Lemma and Proposition, we have $C \subset p_\circ(p^{\vee\circ} T^*\mathbf{P}^\vee \times_Q E) \cup T^*_{\mathbf{P}}\mathbf{P} \subset C^+$.

Corollary 1.2.6. *We consider a cartesian diagram*

$$\begin{array}{ccccccc} \mathbf{P}^\vee & \xleftarrow{p^\vee} & Q & \xleftarrow{h_Q} & Q_W & & \\ & & {\scriptstyle p}\big\downarrow & \square & \big\downarrow{\scriptstyle p_W} & & \\ & & \mathbf{P} & \xleftarrow{h} & W & \xrightarrow{f} & Y \end{array}$$

of smooth schemes over k. For a closed conical subset $C \subset T^*\mathbf{P}$ and its Legendre transform $C^\vee \subset T^*\mathbf{P}^\vee$ and the union $C^+ = C \cup T^*_{\mathbf{P}}\mathbf{P}$ with the 0-section, the following conditions are equivalent:

(1) *(h, f) is C^+-transversal.*

(2) *$f\colon W \to Y$ is smooth and $Q_W \to \mathbf{P}^\vee \times Y$ is smooth of the inverse image $E_W = E \times_Q Q_W$.*

Proof. Since $C^+ = p_\circ(p^{\vee\circ} T^*\mathbf{P}^\vee \times_Q E) \cup T^*_{\mathbf{P}}\mathbf{P}$ by Lemma, the condition (1) is equivalent to the combination of the following conditions.

($1'$) (h, f) is $T^*_{\mathbf{P}}\mathbf{P}$-transversal.

($1''$) (h, f) is $p_\circ(p^{\vee\circ} T^*\mathbf{P}^\vee \times_Q E)$-transversal.

The condition ($1'$) is equivalent to that $f\colon W \to Y$ is smooth. Since p is proper and smooth, by Lemma, the condition ($1''$) is equivalent to $(h_Q, f \circ p_W)$ is $p^{\vee\circ} T^*\mathbf{P}^\vee \times_Q E$-transversal. Since the transversality is an open condition, this is equivalent to that $(h_Q, f \circ p_W)$ is $p^{\vee\circ} T^*\mathbf{P}^\vee$-transversal on a neighborhood of E_W. By Lemma, this is further equivalent to that $(p \vee \circ h_Q, f \circ p_W)$ is $T^*\mathbf{P}^\vee$-transversal on a neighborhood of E_W. This means that $Q_W \to \mathbf{P}^\vee \times Y$ is smooth of the inverse image $E_W = E \times_Q Q_W$.

Let $h\colon W \to \mathbf{P}$ be an immersion and $f\colon W \to Y$ be a smooth morphism. Define sub vector bundles $C_W \subset C_f \subset T^*\mathbf{P} \times_{\mathbf{P}} W$ by $C_W = T^*_W\mathbf{P}$ and C_f as the inverse image of $W \times_Y T^*Y \subset T^*W$ by the surjection $T^*\mathbf{P} \times_{\mathbf{P}} W \to T^*W$.

Lemma 1.2.7. *Let $C^\vee \subset T^*\mathbf{P}^\vee$ be a closed conical subset and let $C = L^\vee C^\vee \subset T^*\mathbf{P}$ be the inverse Legendre transform.*

1. *The following conditions are equivalent:*

(1) *h is C-transversal.*

(2) *The intersection of $\mathbf{P}(C) \subset \mathbf{P}(T^*\mathbf{P}) = Q$ and $\mathbf{P}(C_W) \subset \mathbf{P}(T^*\mathbf{P} \times_{\mathbf{P}} W) = Q \times_{\mathbf{P}} W \subset Q$ is empty.*

2. *Assume that $h\colon W \to \mathbf{P}$ is C-transversal. Then $Q \times_{\mathbf{P}} W \to \mathbf{P}^\vee$ is $C^\vee$-transversal. The complement $Q \times_{\mathbf{P}} W - \mathbf{P}(C \cap C_f)$ equals the largest open subset $U \subset Q \times_{\mathbf{P}} W$ where $(p^\vee\colon Q \times_{\mathbf{P}} W \to \mathbf{P}^\vee, fp\colon Q \times_{\mathbf{P}} W \to W \to Y)$ is $C^\vee$-transversal. Further $\mathbf{P}(C \cap C_f)$ is a subset of the inverse image of the complement of the largest open subset where f is $h^\circ C$-transversal.*

3. *Further if $\dim Y = 1$, the closed subset $\mathbf{P}(C \cap C_f) \subset Q \times_{\mathbf{P}} W$ is finite over W.*

Proof. 1. (1) means $C \cap C_W$ is a closed subset of the zero-section and is equivalent to (2).

2. By Proposition 1.1.8, the C-transversality of $h\colon W \to \mathbf{P}$ implies the $C^\vee$-transversality of $Q \times_{\mathbf{P}} W \to Q$. Since $p^\vee\colon Q \to \mathbf{P}^\vee$ is smooth, the first assertion follows.

The largest open subset $U \subset Q \times_{\mathbf{P}} W$ is the same as that where $(p^\vee, p)$ is $C^\vee \times C_f$-transversal. Hence, it equals the complement of $\mathbf{P}(C^\vee) \cap \mathbf{P}(C_f) = \mathbf{P}(C) \cap \mathbf{P}(C_f) = \mathbf{P}(C \cap C_f)$.

If f is $h^\circ C$-transversal, then $(p^\vee, fp)$ is $C^\vee$-transversal and the last assertion follows.

3. Since $\dim Y = 1$, the subvector bundle $C_W \subset C_f$ is of codimension 1 and the complement $\mathbf{P}(C_f) - \mathbf{P}(C_W)$ is a vector bundle over W. Since $\mathbf{P}(C \cap C_W)$ is empty by 1, the intersection $\mathbf{P}(C \cap C_f)$ is a closed subset of $\mathbf{P}(C_f - C_W)$. Hence its closed subset $\mathbf{P}(C \cap C_f)$ proper over W is finite over W.

1.3 Local acyclicity

Let $f: X \to S$ be a morphism of schemes. Let $x \to X$ and $t \to S$ be geometric points and let $S_{(s)}$ be the strict localization at the image $s = f(x) \to S$ of x. Then a specialization $x \leftarrow t$ is a lifting of $t \to S$ to $t \to S_{(s)}$.

Definition 1.3.1. *Let $f: X \to S$ be a morphism of schemes and $\mathcal{F}$ be a complex of torsion sheaves on X. We say that f is locally acyclic relatively to $\mathcal{F}$ if for each geometric points $x \to X$ and $t \to S$ and each specialization $x \leftarrow t$, the canonical morphism $\mathcal{F}_x \to R(X_{(x)} \times_{S_{(s)}} t, \mathcal{F})$ is an isomorphism.*

We say that f is universally locally acyclic relatively to $\mathcal{F}$, if for every morphism $S' \to S$, the base change of f is locally acyclic relatively to the pull-back of $\mathcal{F}$.

For geometric points s, t of S and a specialization $t \to S_{(s)}$, let $i: X_s \to X \times_S S_{(s)}$ and $j: X_t \to X \times_S S_{(s)}$ denote the canonical morphisms. Then, the local acyclity is equivalent to that the canonical morphism $i^*\mathcal{F} \to i^*Rj_*\mathcal{F}$ is an isomorphism for each s, t and $s \leftarrow t$.

If $\mathcal{F}$ is a constructible sheaf on X, $\mathcal{F}$ is locally constant if and only if 1_X is locally acyclic relatively to $\mathcal{F}$.

The local acyclicity is preserved by quasi-finite base change $S' \to S$. Hence for constructible $\mathcal{F}$, the universal local acyclicity is reduced to smooth base change.

Theorem 1.3.2. 1. (local acyclicity of smooth morphism) *Assume that $f: X \to S$ is smooth and that $\mathcal{F}$ is locally constant killed by an integer invertible on S. Then f is ula relatively to $\mathcal{F}$.*

2. (generic local acyclicity) *Assume that $f: X \to S$ is of finite type and that $\mathcal{F}$ is constructible. Then, there exists a dense open subscheme $U \subset S$ such that the base change of f to U is ula relatively to the restriction of $\mathcal{F}$.*

Corollary 1.3.3. *Assume that $g: Y \to S$ is smooth, that $f: X \to Y$ is la relatively to $\mathcal{F}$ and $\mathcal{F}$ is killed by an integer invertible on S. Then, gf is locally acyclic relatively to $\mathcal{F}$.*

Lemma 1.3.4. *Let $f: X \to Y$ be a proper morphism of schemes over S and assume that $X \to S$ is locally acyclic relatively to $\mathcal{F}$. Then $Y \to S$ is locally acyclic relatively to $Rf_*\mathcal{F}$.*

Proof. Proper base change theorem.

1.4 Micro support

Definition 1.4.1. *Let $\mathcal{F}$ be a constructible complex on X and $C \subset T^*X$ be a closed conical subset. We say that $\mathcal{F}$ is micro supported on C, if for every C-transversal pair (h, f) of $h: W \to X$ and $f: W \to Y$, f is (universally) locally acyclic relatively to $h^*\mathcal{F}$.*

If $\mathcal{F}$ is micro supported on $C \subset C'$, then $\mathcal{F}$ is micro supported on C'.

Lemma 1.4.2. *If $\mathcal{F}$ is micro supported on C, then the support of $\mathcal{F}$ is a subset of the base B of C.*

Proof. Let $U = X - B$. It suffices to show that $\mathcal{F}|_U = 0$. The pair $U \to X, U \to 0 \subset \mathbf{A}^1$ is C-transversal. Hence $U \to \mathbf{A}^1$ is locally acyclic relatively to $\mathcal{F}|_U$ and $\mathcal{F}|_U = 0$.

Lemma 1.4.3. *Let $U \subset X$ be an open subscheme and A be the complement. Assume that $\mathcal{F}$ is micro supported on C and assume that $\mathcal{F}|_U$ is micro supported on C'_U. Then $\mathcal{F}$ is micro supported on the union of $C|_A$ and the closure C' of C'_U.*

Lemma 1.4.4. *Let $\to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ be a distinguished triangle and suppose that $\mathcal{F}'$ and $\mathcal{F}''$ are micro supported on C' and on C'' respectively. Then $\mathcal{F}$ is micro supported on $C = C' \cup C''$.*

Lemma 1.4.5. *The following conditions are equivalent:*

(1) *$\mathcal{F}$ is locally constant.*

(2) *$\mathcal{F}$ is micro supported on the 0-section $T^*_X X$.*

Proof. (h, f) is $T^*_X X$-transversal if and only if f is smooth.

(1)⇒(2): f is universally locally acyclic relatively to locally constant $h^*\mathcal{F}$.

(2)⇒(1): $(1_X, 1_X)$ is $T^*_X X$-transversal. Hence, 1_X is locally acyclic relatively to $\mathcal{F}$ and $\mathcal{F}$ is locally constant.

Lemma 1.4.6. *Any constructible $\mathcal{F}$ is micro supported on T^*X.*

Proof. Suppose (h, f) is T^*X-transversal. Then $W \to X \times Y$ is smooth. Locally, $W \to Y$ is the composition of an étale morphism $W \to X \times \mathbf{A}^n \times Y$ with the projection $X \times \mathbf{A}^n \times Y \to Y$. Hence the local acyclicity follows from the generic local acyclicity and Corollary 1.3.3.

Lemma 1.4.7. *Assume that $\mathcal{F}$ is micro supported on C.*

1. *If $h\colon W \to X$ is C-transversal, then $h^*\mathcal{F}$ is micro supported on $h^\circ C$.*
2. *If $f\colon X \to Y$ is proper on the base of C, then $Rf_*\mathcal{F}$ is micro supported on $f_\circ C$.*

Proof. 1. Suppose $g\colon V \to W, f\colon V \to Y$ is $h^\circ C$-transversal. Then, (hg, f) is C-transversal and f is locally acyclic relatively to $(hg)^*\mathcal{F}$.

2. Suppose $h\colon W \to Y, g\colon W \to Z$ is $f_\circ C$-transversal. Then, $h_X\colon W \times_Y X \to X, g \circ f_W\colon W \times_Y X \to W \to Z$ is C-transversal and $h_X^*\mathcal{F}$ is locally acyclic relatively to $g \circ f_W$. Hence $h^* Rf_*\mathcal{F} = Rf_{W*} h_X^*\mathcal{F}$ is locally acyclic relatively to g.

1.5 Singular support

Definition 1.5.1. *We say that $C \subset T^*X$ is the singular support of $\mathcal{F}$ if for $C' \subset T^*X$, the inclusion $C \subset C'$ is equivalent to the condition that $\mathcal{F}$ is micro supported on C.*

Lemma 1.5.2. *Let $\mathcal{F}$ be a constructible sheaf on X.*

1. *Let $U \subset X$ be an open subscheme. Assume that $C \subset T^*X$ is the singular support of $\mathcal{F}$. Then, $C|_U$ is the singular support of $\mathcal{F}|_U$.*
2. *Let (U_i) be an open covering of X and C_i be the singular support of $\mathcal{F}|_{U_i}$. Then, $C = \bigcup_i C_i$ is the singular support of $\mathcal{F}$.*

Lemma 1.5.3. *Let $i\colon X \to P$ be a closed immersion. Assume that $C_P \subset T^*P$ is the singular support of $i_*\mathcal{F}$.*

1. *C_P is a subset of $T^*P|_X$ and its image $C \subset T^*X$ is the singular support of $\mathcal{F}$.*
2. *We have $C_P = i_\circ C$.*

Proof. 1. By Lemma 1.4.3, C_P is a subset of $T^*P|_X$.

To show $C = SS\mathcal{F}$, it suffices to show the following:

(1) If $\mathcal{F}$ is micro supported on C', we have $C \subset C'$.

(2) C is closed and $\mathcal{F}$ is micro supported on C.

We show (1). Suppose $\mathcal{F}$ is micro supported on C'. Then by Lemma ??, $i_*\mathcal{F}$ is micro supported on $i_\circ C'$. Since C_P is the smallest, we have $C_P \subset i_\circ C'$ and hence $C \subset C'$.

We show (2). Since the assertion is local, we may assume that there exists a cartesian diagram

$$\begin{array}{ccc} P & \stackrel{i}{\longleftarrow} & X \\ \big\downarrow & & \big\downarrow \\ \mathbf{A}^n_k & \longleftarrow & \mathbf{A}^m_k \end{array}$$

such that the vertical arrows are isomorphism. Then, by choosing a projection $\mathbf{A}^n_k \to \mathbf{A}^m_k$ inducing the identity on $\mathbf{A}^m_k$, we obtain a cartesian diagram

$$\begin{array}{ccc} P & \longleftarrow & Q \\ \big\downarrow & & \big\downarrow{\scriptstyle r} \\ \mathbf{A}^m_k & \longleftarrow & X \end{array}$$

where the horizontal arrows are étale. The immersion $X \to P$ induces a section $i' \colon X \to Q$. Since $h \colon Q \to P$ is étale, $i'_*\mathcal{F}$ is micro supported on $h^\circ C_P$. By Lemma ??, $\mathcal{F} = r_*j_*\mathcal{F}$ is micro supported on $C_r = r_\circ h^\circ C_P$. Hence by (1), we have $C \subset C_r$. Since $C_r \subset C$, we have $C_r = C$ and C is closed and $\mathcal{F}$ is micro supported on $C = C_r$.

2. By the proof of (2), we have $C = C_{r'}$ for any projection r'. If k is infinite, this implies $C_P = i_\circ C$.

Theorem 1.5.4. (Beilinson) *$SS\mathcal{F}$ exists.*

Proof will be given at the end of next section.

Theorem 1.5.5. (Beilinson) 1. $\dim E \leqq \dim \mathbf{P} - 1$.

2. *Every irreducible component of E has* $\dim \mathbf{P} - 1$.

1.6 Radon transform

We define the naive Radon transform $R\mathcal{F}$ to be $Rp^\vee_* p^*\mathcal{F}$ and the naive inverse Radon transform $R^\vee\mathcal{G}$ to be $Rp_*p^{\vee *}\mathcal{G}$.

Proposition 1.6.1. *There exists a distinguished triangle*

$$\to \bigoplus_{q=0}^{n-2} R\Gamma(\mathbf{P}_{\bar{k}}, \mathcal{F})(q)[2q] \to R^\vee R\mathcal{F} \to \mathcal{F}(n-1)[2(n-1)] \to .$$

Proof. By the cartesian diagram

$$
\begin{array}{ccccc}
\mathbf{P} & \xleftarrow{\ p\ } & Q & \xleftarrow{\ pr_1\ } & Q \times_{\mathbf{P}^\vee} Q \\
 & & {\scriptstyle p^\vee}\big\downarrow & & \big\downarrow{\scriptstyle pr_2} \\
 & & \mathbf{P}^\vee & \xleftarrow{\ p^\vee\ } & Q \\
 & & & & \big\downarrow{\scriptstyle p} \\
 & & & & \mathbf{P}
\end{array}
$$

and the proper base change theorem, we have a canonical isomorphism

$$R^\vee R\mathcal{F} \to Rpr_{2*}\big(pr_1^*\mathcal{F} \otimes R(p \times p)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q}\big)$$

for $p \times p\colon Q \times_{\mathbf{P}^\vee} Q \to \mathbf{P} \times \mathbf{P}$.

We compute $R(p \times p)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q}$. The closed scheme $Q \times_{\mathbf{P}^\vee} Q \subset \mathbf{P} \times \mathbf{P} \times \mathbf{P}^\vee$ is the $\mathbf{P}^{n-1}$-bundle Q on the diagonal $\mathbf{P} \subset \mathbf{P} \times \mathbf{P}$. On the complement $\mathbf{P} \times \mathbf{P} - \mathbf{P}$, it is a sub $\mathbf{P}^{n-2}$-bundle. Hence, we have a distinguished triangle

$$\to \tau_{\leqq 2(n-2)} R\Gamma(\mathbf{P}^\vee_{\bar{k}}, \Lambda) \otimes \Lambda_{\mathbf{P}\times\mathbf{P}} \to R(p \times p)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q} \to \Lambda_{\mathbf{P}}(n-1)[2(n-1)] \to .$$

Proposition 1.6.2. *For $\mathcal{G}$ on $\mathbf{P}^\vee$ and $C^\vee \subset T^*\mathbf{P}^\vee$, we have implications* (1)⇒(2)⇒(3)*.*

(1) *$\mathcal{G}$ is micro supported on $C^\vee$.*

(2) *p is universally locally acyclic relatively to $p^{\vee *}\mathcal{G}$ outside $E = \mathbf{P}(C^\vee)$.*

(3) *$R^\vee\mathcal{G}$ is micro supported on C^+.*

Proof. (1)⇒(2): Since $p^\vee\colon Q \to \mathbf{P}^\vee, p\colon Q \to \mathbf{P}$ is $C^\vee$-transversal outside $E = \mathbf{P}(C^\vee)$, p is universally locally acyclic relatively to $p^{\vee *}\mathcal{G}$ outside E.

(2)⇒(3): Assume $h\colon W \to \mathbf{P}, f\colon W \to Y$ is C^+-transversal. We consider the cartesian diagram

$$
\begin{array}{ccccc}
\mathbf{P}^\vee & \xleftarrow{\ p^\vee\ } & Q & \xleftarrow{\ h'\ } & Q_W \\
 & & {\scriptstyle p}\big\downarrow & \square & \big\downarrow{\scriptstyle p'} \\
 & & \mathbf{P} & \xleftarrow{\ h\ } & W \\
 & & & & \big\downarrow{\scriptstyle f} \\
 & & & & Y.
\end{array}
$$

We first show that $fp'\colon Q_W \to Y$ is locally acyclic relatively to $\mathcal{G}_{Q_W} = h'^* p^{\vee *}\mathcal{G}$. By (2), $p'\colon Q_W \to W$ is locally acyclic relatively to $\mathcal{G}_{Q_W}$ outside the inverse image $E_W \subset Q_W$ of E. By Corollary 1.2.6, $f\colon W \to Y$ is smooth and $Q_W \to \mathbf{P}^\vee \times Y$ is smooth on the inverse image E_W.

Hence by Corollary 1.3.3, $fp'\colon Q_W \to Y$ is locally acyclic relatively to $\mathcal{G}_{Q_W}$ outside E_W. Further by the generic local acyclicity and Corollary 1.3.3, $fp'\colon Q_W \to Y$ is locally acyclic relatively to $\mathcal{G}_{Q_W}$ on a neighborhood of E_W. Thus, $fp'\colon Q_W \to Y$ is locally acyclic relatively to $\mathcal{G}_{Q_W}$. Hence by Lemma, $f\colon W \to Y$ is locally acyclic relatively to $Rp'_*\mathcal{G}_{Q_W} = h^* R^\vee \mathcal{G}$.

Proof of Theorem 1.5.4. It is reduced to the case X is affine, an affine space and then a projective space.

Let $E \subset Q$ be the smallest closed subset such that $p\colon Q \to \mathbf{P}$ is universally locally acyclic relatively to $p^{\vee *}R\mathcal{F}$ on the complement $Q - E$. Let $C \subset T^*\mathbf{P}$ be the closed conical subset defined by E. Then, by ??, $R^\vee R\mathcal{F}$ is micro supported on C^+. Hence by ??, $\mathcal{F}$ is also micro supported on C^+.

Let $U = \mathbf{P} - B$ be the complement of the base of C. Then, since $C^+ \cap T^*U = T^*_U U$, the restriction $\mathcal{F}|_U$ is locally constant. If $\mathcal{F}|_U = 0$, $\mathcal{F}$ is micro supported on C. We show that C is the singular support of $\mathcal{F}$ if $\mathcal{F}|_U = 0$ and that C^+ is the singular support of $\mathcal{F}$ if otherwise.

Suppose $\mathcal{F}$ is micro supported on C'. Then by (1)$\Rightarrow$(3), $\mathcal{G} = R\mathcal{F}$ is micro supported on $C'^{\vee +}$. Hence by (1)$\Rightarrow$(2), $p\colon Q \to \mathbf{P}$ is universally locally acyclic relatively to $p^{\vee *}\mathcal{G}$ outside $E' = \mathbf{P}(C'^{\vee}) = \mathbf{P}(C')$. Since E is the smallest, we have $E \subset E'$ and hence $C \subset C'$. If $\mathcal{F}|_U \neq 0$, we have $\operatorname{supp}\mathcal{F} = \mathbf{P}$ and hence $T^*_{\mathbf{P}}\mathbf{P} \subset C'$ and $C^+ \subset C'$.