Cohen-Macaulay representations, Singularity categories and Cluster categories

(joint work with Norihiro Hanihara)

Osamu Iyama

(The university of Tokyo)

R : comm. Noetherian ring, $\mathrm{mod}\, R$: category of fin. gen. R-mod.

① Assume R is local

$X \in \mathrm{mod}\, R \Longrightarrow \operatorname{depth} X \leq \dim X \leq \dim R$

X : (max.) Cohen-Macaulay (CM) $\overset{\text{def}}{\Longleftrightarrow} \operatorname{depth} X = \dim R$ or $X = 0$

② For general R,

X : CM $\overset{\text{def}}{\Longleftrightarrow} \forall\, \mathfrak{m}$: max. ideal of R, $X_{\mathfrak{m}} \in \mathrm{mod}\, R_{\mathfrak{m}}$ is CM

$\boxed{\text{Study the category } \mathrm{CM}\, R \text{ of CM } R\text{-modules}}$

③ Assume R is G-graded for an abelian group G

$\mathrm{CM}^G R = \{ X \in \mathrm{mod}^G R \mid X \in \mathrm{mod}\, R \text{ is CM} \}$

Ex R : regular $\Longrightarrow \mathrm{CM}\, R = \mathrm{proj}\, R$: cat. of fin. gen. proj. R-mod.

$\boxed{\text{Throughout. assume } R \text{ is Gorenstein}}$

$\Rightarrow$ ① $\mathrm{CM}R = \{ X \in \operatorname{mod} R \mid \operatorname{Ext}_R^{>0}(X, R) = 0 \}$

② $\mathrm{CM}R$ is a Frobenius category

closed under extensions, $\forall X \in \mathrm{CM}R, \exists Y, Z \in \mathrm{CM}R, \exists P, Q \in \operatorname{proj} R$
$\exists$ exact seq. $0 \to Y \to P \to X \to 0$, $0 \to X \to Q \to Z \to 0$

③ stable category $\underline{\mathrm{CM}}R$ is a triangulated category

object : same as $\mathrm{CM}R$

morph. : $\operatorname{Hom}_R(X, Y) / \{ X \to P \to Y \mid P \in \operatorname{proj} R \}$

$\Rightarrow$ $\boxed{\text{One can apply various methods in representation theory to study } \mathrm{CM}R}$

④ $K^b(\mathrm{proj}\,R)$: bounded homotopy category of $\mathrm{proj}\,R$

$D^b(\mathrm{mod}\,R)$: bounded derived category of $\mathrm{mod}\,R$

$D_{sg}(R) := D^b(\mathrm{mod}\,R)/K^b(\mathrm{proj}\,R)$: singularity category

$\Rightarrow$ $\underline{\mathrm{CM}}\,R \simeq D_{sg}(R)$ [Buchweitz, Orlov]

<u>Part I</u>: Study $\underline{\mathrm{CM}}^{\mathbb{Z}}R$ by constructing equivalences with derived categories (known results)

<u>Part II</u>: Study $\underline{\mathrm{CM}}\,R$ by constructing equivalences with cluster categories (ongoing joint work with Hanihara)

Part I Derived categories and $CM^G R$

$\mathcal{T}$: algebraic triangulated category . $[1]$: suspension functor

Ex $\mathcal{T} = \underline{CM}R$, $[1] = \Omega^{-}$: cosyzygy functor

Def $U \in \mathcal{T}$: tilting object $\overset{\text{def}}{\iff}$

① $\mathrm{Hom}_{\mathcal{T}}(U, U[i]) = 0 \quad 0 \neq {}^{\forall} i \in \mathbb{Z}$

② $\mathcal{T} = \text{thick}\, U$ (:= the smallest full subcat. of $\mathcal{T}$ containing U and closed under cones, $[\pm 1]$, direct summands)

Ex $K^b(\mathrm{proj}\, A) \ni A = [\cdots \to 0 \to \overset{0}{A} \to 0 \to \cdots]$ is a tilt. obj.

[Rickard, Keller] $U \in \mathcal{T}$: tilt. obj

$\Rightarrow {}^{\exists}$ equiv. $\mathcal{T} \simeq K^b(\mathrm{proj}\, \mathrm{End}_{\mathcal{T}}(U))$ up to direct summands

(I) dimension 0

$R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$- graded Artinian Gorenstein ring

$R_0 = k$: field. $a =$ degree of soc R

Thm (special case of [Yamaura 13])

$U := \bigoplus_{i=1}^{a} R(i)_{\geq 0} \in \underline{\mathrm{CM}}^{\mathbb{Z}} R$ is a tilt. obj

$\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq K^b(\mathrm{proj}\, A)$

for $A := \mathrm{End}_{\underline{\mathrm{CM}}^{\mathbb{Z}} R}(U) = \begin{bmatrix} R_0 & & & 0 \\ R_1 & R_0 & & \\ & & \ddots & \\ R_{a-1} & R_{a-2} & \cdots & R_0 \end{bmatrix}$

<u>Ex</u> ① $R = k[x]/(x^n)$ $\deg x = 1$

$$\Rightarrow A = \begin{bmatrix} k & & 0 \\ & \ddots & \\ k & & k \end{bmatrix} = kQ \qquad Q = [\overset{1}{\bullet} \to \overset{2}{\bullet} \to \overset{3}{\bullet} \cdots \bullet \to \overset{n}{\bullet}]$$

$$\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq K^b(\mathrm{proj}\, kQ)$$

② $R = k[x_1, \cdots, x_n]/(x_i x_j, x_i^2 - x_j^2 \mid 1 \le i < j \le n)$

$\deg x_i = 1$

$$\Rightarrow A = kQ \qquad Q = [\bullet \overset{n本}{\underset{\vdots}{\rightrightarrows}} \bullet]$$

$$\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq K^b(\mathrm{proj}\, kQ)$$

R is CM-finite if $n = 1$

tame $n = 2$

wild $n \ge 3$

(II) dimension 1

$R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$-graded reduced Gorenstein ring of dim 1

$R_0 = k$: field. a : a-invariant of R

$K = R[\{\text{homog. non-zerodivisors}\}^{-1}]$

${}^{\exists} p \geq 1$ s.t. $K(p) \simeq K$

Thm [Buchweitz - I - Yamaura 20] Assume $a \geq 0$

$$U := \bigoplus_{i=1}^{a+p} R(i)_{\geq 0} \in \underline{\mathrm{CM}}^{\mathbb{Z}} R \text{ is a tilt. obj}$$

$\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq K^b(\mathrm{proj} A)$ for

$$A := \mathrm{End}_{\underline{\mathrm{CM}}^{\mathbb{Z}} R}(U) \simeq \boxed{\begin{array}{ccccccc} R_0 & & & & & & \\ \vdots & \ddots & & 0 & & 0 & \\ R_{a-1} & \cdots & R_0 & & & & \\ K_a & \cdots & K_1 & K_0 & \cdots & K_{1-p} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ K_{a+p-1} & \cdots & K_p & K_{p-1} & \cdots & K_0 \end{array}}$$

simple singularity $\quad R = k[\![x, y, z_1, \cdots, z_d]\!]/(f)$

$A_n \quad x^{n+1} + y^2 + z_2^2 + \cdots + z_d^2$

$D_n \quad x^{n-1} + xy^2 + z_2^2 + \cdots + z_d^2$

$E_6 \quad x^4 + y^3 + z_2^2 + \cdots + z_d^2$

$E_7 \quad x^3y + y^3 + z_2^2 + \cdots + z_d^2$

$E_8 \quad x^5 + y^3 + z_2^2 + \cdots + z_d^2$

Thm (R, m) : complete local Gorenstein, containing $R/m =: k = \overline{k}$
char $k = 0$

simple singularity $\Longleftrightarrow$ CM-finite

$\Rightarrow$ $d=1$ Drozd-Roiter, Jacobinski, Greuel-Knörrer

$d=2$ Herzog, Auslander

$d \geq 3$ Knörrer

$\Leftarrow$ Buchweitz-Greuel-Schreyer

<u>Ex</u> $d=1$ (or odd) $\Longrightarrow$ $\underline{CM}^{\mathbb{Z}}R \simeq K^b(\mathrm{proj}\, kQ)$

R	A_{2n-1}	A_{2n}	D_{2n}	D_{2n+1}	E_6	E_7	E_8
$(\deg x, \deg y)$	$(1, n)$	$(2, 2n+1)$	$(1, n-1)$	$(2, 2n-1)$	$(3, 4)$	$(2, 3)$	$(3, 5)$
Q	D_{n+1}	A_{2n}	D_{2n}	A_{4n-1}	E_6	E_7	E_8

$\Longrightarrow$ $\widehat{R}$ is CM-finite (by Gabriel's Thm. for quivers)

- A_{2n} $\quad R \simeq k[t^2, t^{2n+1}] \subset k[t]$

$$A = \mathrm{End}(U) = k \left[\begin{tikzcd} R(1)_{\geq 0} \arrow[r, "t^2"] & R(3)_{\geq 0} \arrow[r, "t^2"] & \cdots \arrow[r, "t^2"] & R(2n-1)_{\geq 0} \arrow[d, "t"] \\ R(2)_{\geq 0} \arrow[r, "t^2"] & R(4)_{\geq 0} \arrow[r, "t^2"] & \cdots \arrow[r, "t^2"] & R(2n)_{\geq 0} \end{tikzcd} \right]$$

(III) quotient singularity

$\Bbbk$: field

$SL_d(\Bbbk) \supset G$: finite subgroup s.t. $\# G \neq 0$ in $\Bbbk$

$G \curvearrowright S := \Bbbk[x_1, \cdots, x_d]$

$R := S^G$ $\quad a = -d$

$\mathbb{Z}$-graded by $\deg x_i = 1$

Thm [I- Takahashi 13]

$U := \bigoplus_{i=1}^{d} (\text{max. CM summand of } \Omega_S^i(\Bbbk)) \in \underline{CM}^{\mathbb{Z}} R$ is tilt.

$\underline{CM}^{\mathbb{Z}} R \simeq K^b(\operatorname{proj} A)$ for $A := \operatorname{End}_{\underline{CM}^{\mathbb{Z}} R}(U)$

$\exists$ explicit description of A in terms of McKay quiver of G

Ex $d=2 \Rightarrow R$: simple singularity

$A = \Bbbk Q \times \Bbbk Q^{op}$ (Q : Dynkin quiver with alternating orientation)

$\Rightarrow \widehat{R}$ is CM-finite (by Gabriel's Thm. for quivers)

Ex $d=3$, $G = \frac{1}{3}(1,1,1)$ $R = S^{(3)}$

$A = (\Bbbk Q)^3$ for $Q = [\cdot \rightrightarrows\!\!\!\to \cdot]$

(IV) weighted proj. lines, Geigle–Lenzing compl. intersect.

$S := k[t_1, \cdots, t_d] \ni l_1, \cdots, l_n$: linear forms, as linear independent as possible

$\mathbb{Z}_{\geq 1} \ni p_1, \cdots, p_n$

$$R := S[x_1, \cdots, x_n] / (x_i^{p_i} - l_i \mid 1 \leq i \leq n)$$

$$\mathbb{L} := \bigoplus_{i=1}^{n} \mathbb{Z}\vec{x_i} \oplus \mathbb{Z}\vec{c} / (p_i\vec{x_i} - \vec{c} \mid 1 \leq i \leq n)$$

R is $\mathbb{L}$-graded by $\deg x_i = \vec{x_i}$, $\deg t_j = \vec{c}$

<u>Ex</u> $n = d+1$

$$R = k[x_1, \cdots, x_{d+1}] / (x_1^{p_1} + x_2^{p_2} + \cdots + x_{d+1}^{p_{d+1}})$$

Thm [Herschend-I-Minamoto-Oppermann)
($n = d+1$: Kussin-Lenzing-Meltzer, Futaki-Ueda)

$\exists\, U \in \underline{\mathrm{CM}}^{\mathbb{L}} R$: tilt. obj.

$\underline{\mathrm{CM}}^{\mathbb{L}} R \simeq K^b(\mathrm{proj}\, A)$ for $A := \mathrm{End}_{\underline{\mathrm{CM}}^{\mathbb{L}} R}(U)$

Ex $n = d+1 \Longrightarrow A = \bigotimes_{i=1}^{n} k[\overset{1}{\bullet} \to \overset{2}{\bullet} \to \cdots \overset{p_i-1}{\bullet} \to \bullet]$

Part II Cluster categories and CMR

[Auslander] R: compl. local Gorenstein isolated singularity

① CMR has Auslander-Reiten sequences

② triang. cat. $\underline{\mathrm{CM}}R$ is $(d-1)$-Calabi-Yau

$$\mathrm{Hom}(X,Y) \simeq D\mathrm{Hom}(Y, X[d-1]) \quad (\forall X,Y \in \underline{\mathrm{CM}}R)$$

- Cluster category $\mathcal{C}_n(A)$ of a fin. dim. k-alg.

(Buan-Marsh-Reineke-Reiten-Todorov for $A = kQ$
Amiot, Guo, Keller for general A)

Idea Replace $\boxed{(1) \circlearrowright \mathrm{CM}^{\mathbb{Z}}R \xrightarrow[\text{forget}]{} \mathrm{CM}R \quad \text{deg. shift}}$

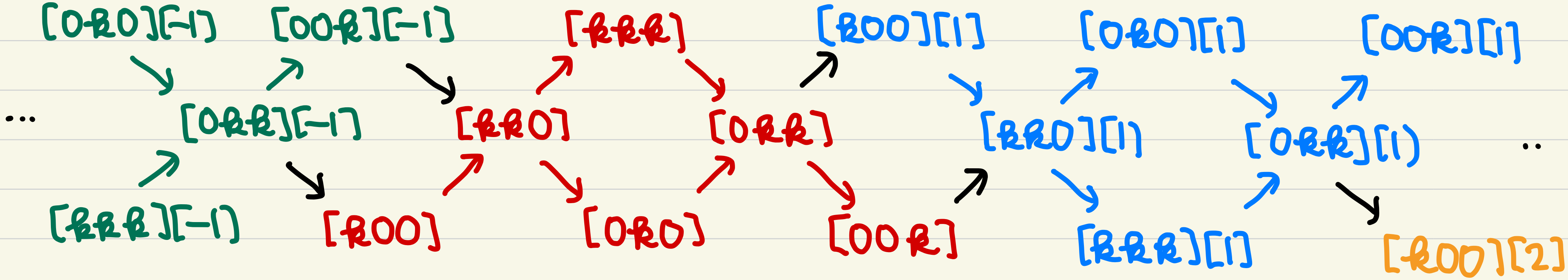
$$\mathrm{Hom}_R(X,Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{CM}^{\mathbb{Z}}R}(X, Y(i))$$

by $\boxed{\nu \circ [-n] \curvearrowright K^b(\mathrm{proj}\, A) \longrightarrow \mathcal{C}_n(A)}$

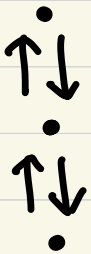
$\nu = - \overset{L}{\underset{A}{\otimes}} DA$: Nakayama functor

<u>Ex</u> $A = k[\cdot \to \cdot \to \cdot] = \begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{bmatrix}$

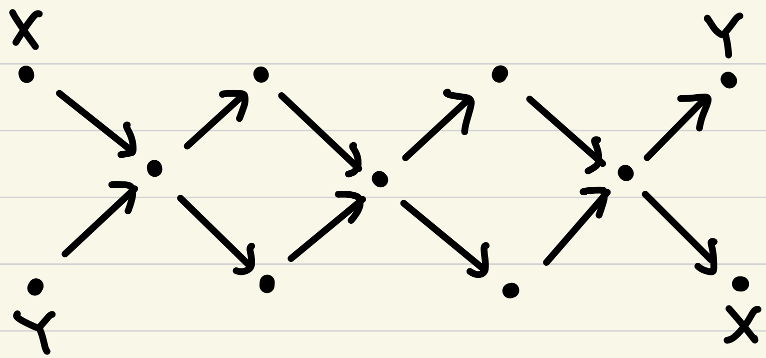
$K^b(\mathrm{proj}\, A) \simeq D^b(\mathrm{mod}\, A)$



$\mathcal{C}_1(A)$



$\mathcal{C}_2(A)$



For general A, we need dg enhancement

A : fin. dim k-alg. $A^e := A \otimes_k A^{op}$

$n \in \mathbb{Z}$ $\quad P \in D(\mathrm{Mod}\,A^e)$: K-projective

$$\Gamma(A,P) := \coprod_{i \in \mathbb{Z}} \operatorname*{colim}_{j \gg 0} \mathrm{Hom}_A\left(P^{\otimes_A i+j}, P^{\otimes_A j}\right)$$

dg algebra with canonical differential

$$\begin{array}{ccccccc}
(A,A) & & (A,P) & & (A,P^{\otimes 2}) & & \\
& \searrow & & \searrow & & \searrow & \\
(P,A) & & (P,P) & & (P,P^{\otimes 2}) & & \ddots \\
& \searrow & & \searrow & & \searrow & \\
(P^{\otimes 2},A) & & (P^{\otimes 2},P) & & (P^{\otimes 2},P^{\otimes 2}) & & \ddots \\
& \searrow & & \searrow & & \searrow &
\end{array}$$

<u>Def</u> ① $P :=$ (proj. resol. of $\underbrace{DA[-n]}_{\text{gives } \nu_0[-n]}$) ($DA[-n]$ の inverse $\mathrm{RHom}_A(DA, A)[n]$ を用いてもよい)

$$\mathcal{C}_n(A) := \operatorname{thick}\Gamma(A, P) \subset D(\Gamma(A, P))$$

n-cluster category

$$\boxed{\nu_0[-n] \circlearrowright K^b(\operatorname{proj} A) \longrightarrow \mathcal{C}_n(A)}$$

② More generally, when

$$P^{\otimes a} \simeq DA[-n] \text{ in } D(A^e) \text{ for } 0 \neq a \in \mathbb{Z}.$$

$$\mathcal{C}_n^{(1/a)}(A) := \operatorname{thick}\Gamma(A, P) \subset D(\Gamma(A, P))$$

Setting • $R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$- graded Gorenstein isolated sing.

dim d , $R_0 = k$: field. a- invariant $a \neq 0$

(i.e. $\mathrm{Ext}^d_R(k, R(a)) \simeq k$ in $\mathrm{mod}^{\mathbb{Z}} R$)

• A : fin. dim. k- alg

Main Thm [I-Hanihara]

If $\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq K^b(\mathrm{proj}\, A)$, then

① $\underline{\mathrm{CM}}^{\mathbb{Z}/a\mathbb{Z}} R \simeq \mathcal{C}_{d-1}(A)$

② $\underline{\mathrm{CM}} R \simeq \mathcal{C}^{(1/a)}_{d-1}(A)$

再掲 $\boxed{\text{(I) dimension 0}}$

$R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$-graded Artinian Gorenstein ring

$R_0 = k$: field. $a =$ degree of $\operatorname{soc} R$

Thm (special case of [Yamaura 13])

$U := \bigoplus_{i=1}^{a} R(i)_{\geq 0} \in \underline{\mathrm{CM}}^{\mathbb{Z}} R$ is a tilt. obj

$\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq K^b(\operatorname{proj} A)$

for $A := \operatorname{End}_{\underline{\mathrm{CM}}^{\mathbb{Z}} R}(U) = \begin{bmatrix} R_0 & & & 0 \\ R_1 & R_0 & & \\ & & \ddots & \\ R_{a-1} & R_{a-2} & \cdots & R_0 \end{bmatrix}$

By <u>Main Thm</u>

$$\boxed{\begin{aligned} \underline{\mathrm{CM}}^{\mathbb{Z}/a\mathbb{Z}} R &\simeq \mathcal{C}_{-1}(A) \\ \underline{\mathrm{CM}}\, R &\simeq \mathcal{C}_{-1}^{(1/a)}(A) \end{aligned}}$$

[Yamaura 13]

再掲 **(II) dimension 1**

$R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$-graded reduced Gorenstein ring of dim 1

$R_0 = k$: field. a : a-invariant of R

$K = R[\{\text{homog. non-zerodivisors}\}^{-1}]$

${}^{\exists} p \geq 1$ s.t. $K(p) \simeq K$

Thm [Buchweitz-I-Yamaura 20] Assume $a \geq 0$

$U := \bigoplus_{i=1}^{a+p} R(i)_{\geq 0} \in \underline{\mathrm{CM}}^{\mathbb{Z}} R$ is a tilt. obj

$\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq K^b(\operatorname{proj} A)$ for

$$A := \operatorname{End}_{\underline{\mathrm{CM}}^{\mathbb{Z}} R}(U) \simeq \boxed{\begin{matrix} R_0 & & 0 & & & 0 \\ \vdots & \ddots & & & & \\ R_{a-1} & \cdots & R_0 & & & \\ K_a & \cdots & K_1 & K_0 & \cdots & K_{1-p} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ K_{a+p-1} & \cdots & K_p & K_{p-1} & \cdots & K_0 \end{matrix}}$$

By <u>Main Thm</u>

$$\boxed{\begin{aligned} \underline{\mathrm{CM}}^{\mathbb{Z}/a\mathbb{Z}} R &\simeq \mathcal{C}_0(A) \\ \underline{\mathrm{CM}}\, R &\simeq \mathcal{C}_0^{(1/a)}(A) \end{aligned}}$$

再掲 $\boxed{\text{(III) quotient singularity}}$ k : field

$SL_d(k) \supset G$: finite subgroup s.t. $\#G \neq 0$ in k

$$\left.\begin{array}{l} G \curvearrowright S := k[x_1, \cdots, x_d] \\ R := S^G \qquad a = -d \end{array}\right\} \mathbb{Z}\text{-graded by } \deg x_i = 1$$

Thm [I- Takahashi 13]

$$U := \bigoplus_{i=1}^{d} (\text{max. CM summand of } \Omega_S^i(k)) \in \underline{\mathrm{CM}}^{\mathbb{Z}} R \text{ is tilt.}$$

$$\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq K^b(\mathrm{proj}\, A) \quad \text{for } A := \mathrm{End}_{\underline{\mathrm{CM}}^{\mathbb{Z}} R}(U)$$

$\exists$ explicit description of A in terms of McKay quiver of G

By <u>Main Thm</u>

$$\boxed{\begin{aligned} \underline{\mathrm{CM}}^{\mathbb{Z}/d\mathbb{Z}} R &\simeq \mathcal{C}_{d-1}(A) \\ \underline{\mathrm{CM}}\, R &\simeq \mathcal{C}^{(1/d)}_{d-1}(A) \end{aligned}}$$

再掲 (IV) weighted proj. lines, Geigle–Lenzing compl. intersect.

$S := k[t_1, \cdots, t_d] \ni l_1, \cdots, l_n$: linear forms, as linear independent as possible

$\mathbb{Z}_{\geq 1} \ni p_1, \cdots, p_n$

$$R := S[x_1, \cdots, x_n] / (x_i^{p_i} - l_i \mid 1 \leq i \leq n)$$

$$\mathbb{L} := \bigoplus_{i=1}^{n} \mathbb{Z}\vec{x_i} \oplus \mathbb{Z}\vec{c} / (p_i\vec{x_i} - \vec{c} \mid 1 \leq i \leq n)$$

R is $\mathbb{L}$-graded by $\deg x_i = \vec{x_i}$, $\deg t_j = \vec{c}$

<u>Ex</u> $n = d+1$

$$R = k[x_1, \cdots, x_{d+1}] / (x_1^{p_1} + x_2^{p_2} + \cdots + x_{d+1}^{p_{d+1}})$$

Thm [Herschend-I-Minamoto-Oppermann]
($n = d+1$: Kussin-Lenzing-Meltzer, Futaki-Ueda)

$\exists\, U \in \underline{\mathrm{CM}}^{\mathbb{L}} R$: tilt. obj.

$\underline{\mathrm{CM}}^{\mathbb{L}} R \simeq K^b(\mathrm{proj}\, A)$ for $A := \mathrm{End}_{\underline{\mathrm{CM}}^{\mathbb{L}} R}(U)$

Ex $n = d+1 \Rightarrow A = \bigotimes_{i=1}^{n} k[\bullet \xrightarrow{1} \bullet \xrightarrow{2} \cdots \bullet \xrightarrow{p_i - 1} \bullet]$

By $\mathbb{L}$-graded version of Main Thm

$$\boxed{\underline{\mathrm{CM}}^{\mathbb{L}/(\vec{\omega})} R \simeq \mathcal{C}_{d-1}(A)}$$

$\vec{\omega} \in \mathbb{L}$: a-invariant

First point of proof of Main Thm

Key Prop (enhanced AR duality)

$R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$-graded Gorenstein isolated sing. dim d

$R_0 = k$: field. a-invariant a, $(\ )^* = \mathrm{Hom}_R(-, R)$

$\mathcal{E} := C_{dg}^{acy}(\mathrm{proj}^{\mathbb{Z}} R)$ (dg enhancement of $\underline{\mathrm{CM}}^{\mathbb{Z}} R$)

$$\Rightarrow \exists \text{ iso. } \left.\begin{array}{l} \mathcal{E}^* \simeq \mathcal{E}[-1] \\ D\mathcal{E} \simeq \mathcal{E}(a)[d-1] \end{array}\right\} \text{ in } D^{\mathbb{Z}}(\mathcal{E}^e)$$

Taking H^0, we recover classical AR duality

$$\underline{\mathrm{Hom}}_R^{\mathbb{Z}}(X, Y) \simeq D\underline{\mathrm{Hom}}_R^{\mathbb{Z}}(Y, X(a)[d-1])$$

Second point of proof of Main Thm

- $\mathcal{C} = \mathcal{C}^{acy}_{dg}(\mathrm{proj}^{\mathbb{Z}} R)$ is a dg category with additional $\mathbb{Z}$-grading $\mathcal{C} = \coprod_{i \in \mathbb{Z}} \mathcal{C}_i$
- $\mathcal{C}_0$ is a dg category with $H^0(\mathcal{C}_0) \simeq \underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq K^b(\mathrm{proj} A)$
- $\mathcal{C}_a$ is a dg $\mathcal{C}_0^e$-module s.t. $\mathcal{C}_a \simeq (D\mathcal{C}_0)[1-d]$ (AR duality) which gives $\nu_0[1-d] \circlearrowright K^b(\mathrm{proj} A)$
- $\mathcal{C}_1$ is a dg $\mathcal{C}_0^e$-module s.t $\mathcal{C}_1^{\otimes_{\mathcal{C}_0} a} \simeq \mathcal{C}_a$

$$\boxed{\text{One can recover } \mathcal{C} \text{ from } \mathcal{C}_0 \text{ and } \mathcal{C}_1}$$

More details can be found in slide and video in

https://www.lancaster.ac.uk/maths/geometric-and-homological-methods/