j-operatrors and naive lifting of dg modules

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References

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strongly commutative dg R-algebra

$R =$ a commutative Noetherian ring (e.g. a complete regular local ring)
$A =$ a (non-negatively graded) strongly commutative dg R-algebra
(dg = differential graded) , i.e.

- $A = \bigoplus_{n \geq 0} A_n$ is a non-negatively graded R-algebra.
- $d^A \colon A \to A(-1)$ such that $(d^A)^2 = 0$ and
  $d^A(ab) = d^A(a)b + (-1)^{|a|} a d^A(b)$ for $a, b \in A$.
- $ab = (-1)^{|a||b|} ba$ for all $a, b \in A$, and $a^2 = 0$ if $|a|$ is odd.

strongly commutative dg R-algebra; Examples

- $R =$ dg algebra concentrated in degree 0, $d^R = 0$
- [Koszul complex] For $t \in R$,

$$K(t;R) = R + XR,$$

where $|X| = 1$, $X^2 = 0$, $d^K(X) = t$.
$K(t;R) = R\langle X \rangle$ is a free extension of R with X.

- [More generally] For A a strongly comm. dg R-algebra, if $|X|$ is odd (> 0),

$$A\langle X \rangle = A + XA$$

where $X^2 = 0$, $d^{A\langle X \rangle}(X) = t \in A$.

strongly commutative dg R-algebra; Examples

Let A be a strongly comm. dg R-algebra. If $|X|$ is even,

- [Polynomial extension]

$$A[X] = A \oplus XA \oplus X^2 A \oplus \cdots ,$$

  where $d^{A[X]}(X) = t \in A$.
  Note: $d^{A[X]}(X^n) = nX^{n-1}t$.

- [Free extension]

$$A\langle X \rangle = A \oplus XA \oplus X^{(2)} A \oplus X^{(3)} A \oplus \cdots ,$$

  where $d(X^{(n)}) = X^{(n-1)}t$, $X^{(n)} X^{(m)} = \binom{n+m}{n} X^{(n+m)}$.
  On the supposition that $X^{(n)} = \frac{1}{n!} X^n$.

- $\mathbb{Q} \subset R \;\Rightarrow\; A\langle X \rangle = A[X]$

dg modules

Let A be a (strongly comm. non-negatively graded) dg R-algebra.
(M, ∂^M) is a dg A-module iff

- M is a graded (right) A-module.
- $\partial^M : M \to M(-1)$ satisfies $(\partial^M)^2 = 0$ and the Leibniz rule:

$$\partial^M(xa) = \partial^M(x)a + (-1)^{|x|}xd^A(a)$$

$D(A) =$ the derived category of all dg A-modules.

Tate resolution

$R \to R/I$ a ring homomorphism of commutative Noetherian rings.
Then there is a free extension of R with at most countably infinite variables $X_1, X_2, \cdots$ such that the dg R-algebra map

$$R\langle X_1, X_2, \cdots \rangle \to R/I$$

is a quasi-isomorphism.
This is called a Tate resolution of R/I over R.
In general, Keller's theorem says that $D(A) \cong D(B)$ if $\exists$ a quasi-isom. dg R-algebra map $A \to B$.
In particular,

$$D(R\langle X_1, X_2, \cdots \rangle) \cong D(R/I)$$

Avramov resolution

$R \to R/I$ a ring homomorphism of commutative Noetherian rings.
Then there is a polynomial extension of R with at most countably infinite variables $X_1, X_2, \cdots$ such that the dg R-algebra map

$$R[X_1, X_2, \cdots] \to R/I$$

is a quasi-isomorphism.
We call this an Avramov's polynomial resolution of R/I over R.

$$D(R[X_1, X_2, \cdots]) \cong D(R/I)$$

Original j-operators

In the case $|X|$ is odd, $A\langle X\rangle = A + XA$, and

$$j_X : A\langle X\rangle \to A\langle X\rangle(-|X|); \quad a_0 + Xa_1 \mapsto a_1 = \frac{d}{dX}(a_0 + Xa_1)$$

is a dg A-module homo. whose image is $A(-|X|)$, hence an exact sequence

$$0 \longrightarrow A \longrightarrow A\langle X\rangle \xrightarrow{j_X} A(-|X|) \longrightarrow 0$$

In the case $|X|$ is even,

$$j_X : A\langle X\rangle \to A\langle X\rangle(-|X|);$$

where

$$j_X(a_0+Xa_1+X^{(2)}a_2+\cdots) = a_1+Xa_2+\cdots = \frac{d}{dX}(a_0+Xa_1+X^{(2)}a_2+\cdots)$$

$$0 \longrightarrow A \longrightarrow A\langle X\rangle \xrightarrow{j_X} A\langle X\rangle(-|X|) \longrightarrow 0$$

Generalization of j-operators

$A =$ a (strongly comm. non-negatively graded) dg R-algebra,
$X=$ a variable of $|X| > 0$
$B = A\langle X\rangle$
$N =$ a graded free (right) B-module, bounded below (not a dg module)

$$\mathcal{E} := \mathrm{End}_B^*(N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\text{graded } B\text{-mod}}(N, N(n)) \ \subset \mathrm{End}_R^*(N)$$

$$\mathrm{Diff}_B(N) := \{\delta : N \to N(-1) | \delta(xb) = \delta(x)b + (-1)^{|x|} x db \quad (x \in N, b \in B)\}$$

Definition

$\mathcal{D} = \mathcal{E} + \mathcal{E} \circ \mathrm{Diff}_B(N)$

Lemma

$\mathcal{D} = \mathcal{E} + \mathrm{Diff}_B(N) \circ \mathcal{E}$ *and it is a subring of* $\mathrm{End}_R^*(N)$.

Generalization of j-operators

$\mathcal{B} = \{e_\lambda\}_\lambda :=$ a free basis of the graded free B-module
Matrix representation

$$\delta \in \mathcal{E} \cup \mathrm{Diff}_B(N) \Rightarrow \delta(e_\lambda) = \sum_{e_\mu} e_\mu b_{\mu\lambda} \quad (b_{\mu\lambda} \in B)$$

$$\delta \quad \leftrightarrow \quad (b_{\mu\lambda})$$

Define $j_X^{\mathcal{B}}$ by

$$\mathcal{E} \cup \mathrm{Diff}_B(N) \ni \delta \mapsto \text{the } B\text{-linear map defined by the matrix } \left(\frac{db_{\mu\lambda}}{dX}\right) \in \mathcal{E}$$

Theorem (Nasseh-Ono-Yoshino)

$j_X^{\mathcal{B}}$ extends to the map $\mathcal{D} \to \mathcal{D}$ that satisfies the Leibniz rule:

$$j_X^{\mathcal{B}}(\alpha\beta) = j_X^{\mathcal{B}}(\alpha)\beta + (-1)^{|\alpha|}\alpha j_X^{\mathcal{B}}(\beta) \quad (\alpha, \beta \in \mathcal{D})$$

Liftability

- By definition, $\delta \in \mathcal{E} \cup \mathrm{Diff}_B(N)$ is liftable to A iff $j_X^{\mathcal{B}}(\delta) = 0$ for $\exists \mathcal{B}$.
- A dg B-module (N, ∂) $(\partial \in \mathrm{Diff}_B(N), \partial^2 = 0)$ is said to be liftable to A if ∂ is liftable to A.
- By Leibniz rule,

$$0 = j_X^{\mathcal{B}}(\partial^2) = j_X^{\mathcal{B}}(\partial)\partial + (-1)^{|X|}\partial j_X^{\mathcal{B}}(\partial),$$

  hence $j_X^{\mathcal{B}}(\partial)$ is a cycle in $\mathrm{Hom}_B(N, N(-|X| - 1))$
- $[j_X^{\mathcal{B}}(\partial)] \in \mathrm{Ext}_B^{|X|+1}(N, N)$ is defined.
- If (N, ∂) is liftable, then $[j_X^{\mathcal{B}}(\partial)] = 0$.

Liftability Theorem

Theorem

- *The class* $[j_X^{\mathcal{B}}(\partial)] \in \mathrm{Ext}_B^{|X|+1}(N, N)$ *is independent of* $\mathcal{B}$*.*
- (Ono-Yoshino)
  If $|X|$ *is even, then* $[j_X(\partial)] = 0 \iff (N, \partial)$ *is liftable.*
- (Nasseh-Yoshino)
  If $|X|$ *is odd, then* $[j_X(\partial)] = 0 \iff (N, \partial) \oplus (N(-|X|), -\partial)$ *is liftable.*

INTERMISSION

Naïve lifting

Let $A \to B$ be a free extension or a polynomial extension of dg R-algebra, and (N, ∂) a semi-free dg B-module, e.g. bounded below and free as an underlying graded B-module.
Restrict the action to A, and we get a dg A-module $(N|_A, \partial)$.
$\pi_N : N|_A \otimes_A B \to N;\ x \otimes b \mapsto xb$ is a (right) dg B-module homomorphism.

Definition

We say N is naïvely liftable to A if π_N splits, i.e. π_N has a right inverse as a dg B-module homomorphism.

Naïve lifting for simple extensions

Theorem (Nasseh-Ono-Yoshino)

Let $B = A\langle X \rangle$, and (N, ∂) a dg B-module that is free and bounded below as an underlying graded B-module. Then TFAE:

1. *(N, ∂) is naïvely liftable to A.*
2. *If $|X|$ is even then (N, ∂) is liftable to A. If $|X|$ is odd then $(N, \partial) \oplus (N(-|X|), -\partial)$ is liftable to A.*
3. $[j_X(\partial)] = 0 \in \mathrm{Ext}_B^{|X|+1}(N, N)$

Naïve lifting for extensions with finite variables

Let $B = A\langle X_1, \ldots, X_n\rangle$ be a free extension,
or $B = A[X_1, \ldots, X_n]$ be a polynomial extension with finite number of variables.
$(N, \partial) =$ a dg B-module that is free and bounded below as an underlying graded B-module.

Theorem (Nasseh-Ono-Yoshino)

If $\mathrm{Ext}_B^i(N, N) = 0$ *for all* $i > 0$*, then* N *is naïvely liftable to* A*.*

Diagonal ideal

Let $A \to B$ be a dg R-algebra homomorphism.
Let $\pi_B : B^e = B^o \otimes_A B \to B$ be multiplication map and define the diagonal ideal J of B^e by the exact sequence

$$0 \longrightarrow J \longrightarrow B^e \xrightarrow{\pi_B} B \longrightarrow 0$$

Take the tensor product $N \otimes_B -$ and we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes_B J & \longrightarrow & N \otimes_B B^e & \xrightarrow[N \otimes_B \pi_B]{} & N \otimes_B B & \longrightarrow & 0 \\ & & & & \| & & \| & & \\ & & & & N|_A \otimes_A B & \xrightarrow{\pi_N} & N & & \end{array}$$

Proposition

$\mathrm{Ext}^1_B(N, N \otimes_B J) = 0 \;\Rightarrow N$ is naïvely liftable to A.

Proof of Theorem

Enough to prove;

Proposition

$\mathrm{Ext}^i_B(N,N) = 0\ (i > 0)\ \Rightarrow \mathrm{Ext}^1_B(N, N \otimes_B J) = 0$

We show how to prove this in the case $B = A[X_1, \ldots, X_n]$ is a polynomial extension of A.
Then J is an ideal of B^e generated by
$\{\xi_i = X_i \otimes 1 - 1 \otimes X_i \mid i = 1, 2, \ldots, n\}$ which is something like a regular sequence, i.e.
J^m/J^{m+1} is a finitely generated free B-module for any $m > 0$.
$J^m/J^{m+1} \cong \bigoplus_{\text{finite}} B(-a)\ (a > 0)$

e.g. In the case that $|X|$ is odd, $A[X] \otimes_A A[X] \cong A[X', X'']/(X'^2, X''^2) \supset J = (X' - X'')$,

where $J^2 = ((X' - X'')^2) = (X'^2 - X'X'' - X''X' + X''^2) = (0)$, $J/J^2 = J \cong A[X](-|X|)$

Proof of Theorem

Proposition

$\mathrm{Ext}^i_B(N,N)=0\ (i>0)\ \Rightarrow \mathrm{Ext}^1_B(N,N\otimes_B J)=0$

$$\begin{aligned}
\mathrm{Ext}^i_B(N,N\otimes_B J^m/J^{m+1}) &= \mathrm{Ext}^i_B(N,N\otimes_B \bigoplus_{\text{finite}} B(-a)) \\
&= \mathrm{Ext}^i_B(N,\bigoplus_{\text{finite}} N(-a)) \\
&= \bigoplus_{\text{finite}} \mathrm{Ext}^i_B(N,N(-a)) \\
&= \bigoplus_{\text{finite}} \mathrm{Ext}^{i+a}_B(N,N)=0 \text{ for } m>0,\ i\geqq 0
\end{aligned}$$

$$\Rightarrow \mathrm{Ext}^i_B(N,N\otimes_B J/J^{m+1})=0\ (m>0,\ i\geqq 0)$$

Proof of Theorem

Proposition

$\mathrm{Ext}^i_B(N,N)=0\ (i>0)\ \Rightarrow \mathrm{Ext}^1_B(N,N\otimes_B J)=0$

$$\begin{aligned}
& \mathrm{Ext}^i_B(N,N\otimes_B J^m/J^{m+1})\ (m>0,\ i\geqq 0)\\
\Rightarrow\ & \mathrm{Ext}^i_B(N,N\otimes_B J/J^{m+1})=0\ (m>0,\ i\geqq 0)\\
\Rightarrow\ & \mathrm{Hom}_{K(B)}(N,N\otimes_B J/J^{m+1}(-i))=0\ (m>0,\ i\geqq 0)\\
\Rightarrow\ & \mathrm{Hom}_{K(B)}(N,\mathrm{holim}(N\otimes_B J/J^{m+1})(-i))=0\ (i\geqq 0)\\
\Rightarrow\ & \mathrm{Hom}_{K(B)}(N,N\otimes_B J)(-i))=0\ (i>0)\\
\Rightarrow\ & \mathrm{Ext}^i_B(N,N\otimes_B J)=0\ (i>0)\qquad \square
\end{aligned}$$

*We can prove $\mathrm{holim}(N\otimes_B J^{m+1})=0$ in $D(B)$.

The Conjecture

Let $A \to B$ be a dg R-algebra homomorphism that is a free extension or a polynomial extension, and
N be a dg B-module that is bounded below.
We conjecture the following:

Conjecture

If $\mathrm{Ext}^i_B(N, N) = \mathrm{Ext}^i_B(N, B) = 0$ for all $i > 0$, then N is naïvely liftable to A.

If this conjecture is true, then Auslander-Reiten conjecture is true for any finitely generated modules over any commutative Noetherian rings.

Note