Hibi ringとchain polytopeのEhrhart ringのnon-Gorenstein lociについて

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§1 Trace of a module

R : ring (comm. Noeth.)

M : R-module

<u>Def.</u>

$$\mathrm{tr}_R(M) = \mathrm{tr}(M) = \sum_{\varphi \in \mathrm{Hom}_R(M,R)} \varphi(M)$$

<u>Rem.</u> M : f.g. $\mathfrak{p} \in \mathrm{Spec}\, R$

$$\Rightarrow \mathrm{tr}_R(M)_{\mathfrak{p}} = \mathrm{tr}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

<u>Lemma</u> (Herzog-Hibi-Stamate)

$I \subset R$: ideal grade $I > 0$

$$\Rightarrow \mathrm{tr}(I) = I^{-1} I$$

$$I^{-1} = R :_{Q(R)} I = \{ x (\in Q(R)) \mid xI \subset R \}$$

<u>Lemma</u> (Herzog-Hibi-Stamate)

R : CM local or graded

ω_R : Rのcanonical module

$\mathfrak{p} \in \mathrm{Spec}(R)$

Then

$\mathfrak{p} \supset tr(\omega_R) \iff R_{\mathfrak{p}}$ is not Gor.

In particular

R : Gor. $\iff tr(\omega_R) \ni 1$

(proof)

$tr(\omega_R)_{\mathfrak{p}} = tr((\omega_R)_{\mathfrak{p}}) = tr(\omega_{R_{\mathfrak{p}}})$

It is enough to prove the last statement

($\Rightarrow$) $R \cong \omega_R \quad \therefore tr(\omega_R) = tr(R) \ni 1$

($\Leftarrow$) $\exists \varphi : \omega_R \longrightarrow R, \quad \varphi(\omega_R) \ni 1$

$\varphi : \omega_R \twoheadrightarrow R$ split epi

$\omega_R = R \oplus M$

$type(\omega_R) = 1$

$\therefore M = 0$ //

§2 Hibi ring と order, chain polytopes

X : set とする. $\#X$: cardinality of X

X, Y : sets

$Y^X = \{ f \mid f : X \to Y \}$

If $\#X < \infty$ then we identify $\mathbb{R}^X$ with $\mathbb{R}^{\#X}$

Def. $f \in \mathbb{R}^X$, $A \subset X$

$$f^+(A) := \sum_{a \in A} f(a)$$

$$f^+(\phi) := 0$$

Def. $f, f_1, f_2 \in \mathbb{R}^X$, $a \in \mathbb{R}$

We define af, $f_1 \pm f_2 \in \mathbb{R}^X$ by

$$(af)(x) = a(f(x))$$

$$(f_1 \pm f_2)(x) = f_1(x) \pm f_2(x)$$

$\mathcal{P}$: rational convex polytope in $\mathbb{R}^X$

($\mathcal{P}$の頂点 $\in \mathbb{Q}^X$)

$-\infty$: a new element with $-\infty \notin X$

$X^- = X \cup \{-\infty\}$

$\{T_x\}_{x \in X^-}$: a family of indet. indexed by X^-

$\mathbb{K}$: a field

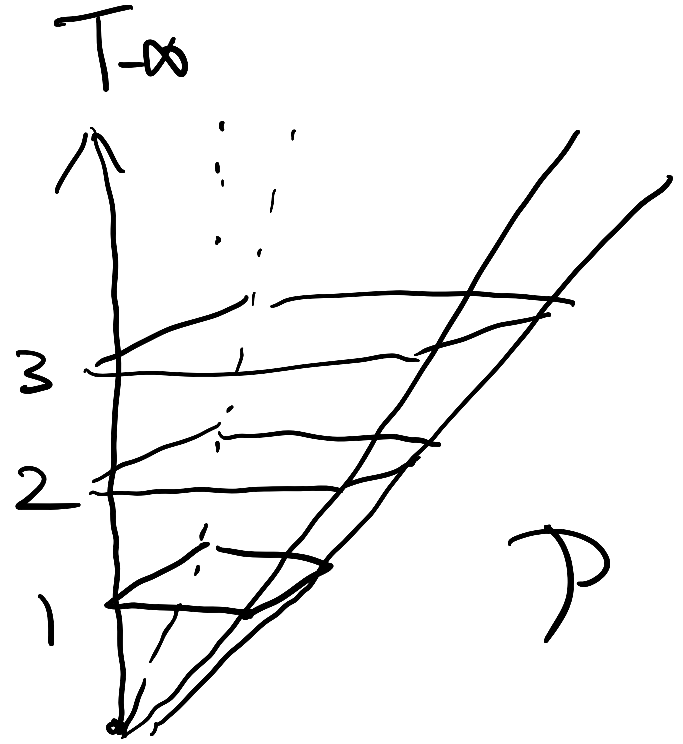
For $f \in \mathbb{Z}^{X^-}$, $T^f = \prod_{x \in X^-} T_x^{f(x)}$

Def. $E_{\mathbb{K}}[\mathcal{P}] = \mathbb{K}[T^f \mid f \in \mathbb{Z}^{X^-},$
$f(-\infty) > 0,\ \frac{1}{f(-\infty)} f \in \mathcal{P}]$

$\mathcal{P}$ の $\mathbb{K}$ 上の Ehrhart ring という.

Rem. $f(-\infty) = n > 0$

$\frac{1}{f(-\infty)} f \in \mathcal{P} \Leftrightarrow f \in n\mathcal{P}$



Fact. (Hochster)
$E_{\mathbb{K}}[\mathcal{P}]$ is noeth. normal

$\deg T_{-\infty} = 1,\ \deg T_x = 0\ (x \in X)$

$E_{\mathbb{K}}[\mathcal{P}]$ is an $\mathbb{N}$-graded ring.

If $E_{\mathbb{K}}[P]$ is a standard graded ring, then we denote $E_{\mathbb{K}}[P]$ by $\mathbb{K}[P]$.

<u>Fact</u> (Stanley)

$$\omega_{E_{\mathbb{K}}[P]} = \bigoplus_{\substack{f \in \mathbb{Z}^{X^-},\ f(-\infty) > 0, \\ \frac{1}{f(-\infty)} f \in \mathrm{rel.int.}\, P}} \mathbb{K} T^f$$

is $E_{\mathbb{K}}[P]$ の canonical module

We call $\omega_{E_{\mathbb{K}}[P]}$ the canonical ideal of $E_{\mathbb{K}}[P]$.

Q: finite poset

If $x < y$ and $x < \nexists z < y$ then y covers x and denoted by $x \lessdot y$ or $y \gtrdot x$

For $x, y \in Q$ with $x \leq y$

$[x,y]_{\mathbb{Q}} = [x,y]$
$:= \{ z \in \mathbb{Q} \mid x \leq z \leq y \}$
$[x,y)_{\mathbb{Q}} := \{ z \in \mathbb{Q} \mid x \leq z < y \}$
$(x,y]_{\mathbb{Q}} := \{ z \in \mathbb{Q} \mid x < z \leq y \}$
$x < y$ のとき
$(x,y)_{\mathbb{Q}} := \{ z \in \mathbb{Q} \mid x < z < y \}$

$\infty = +\infty$: a new element with
$\infty \notin \mathbb{Q}$

$\mathbb{Q}^+ = \mathbb{Q} \cup \{\infty\}$
For $x, y \in \mathbb{Q}^+$
$x < y \overset{}{\underset{\text{def}}{\iff}} x, y \in \mathbb{Q},\ x < y$ in $\mathbb{Q}$
or
$x \in \mathbb{Q},\ y = \infty$

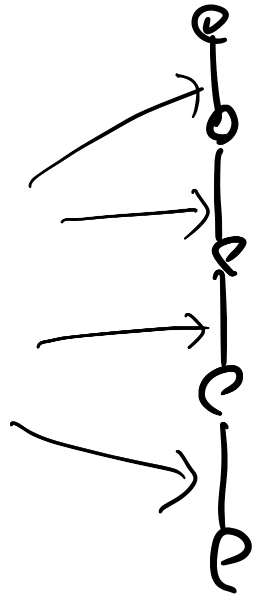
$\mathbb{Q}^- = \mathbb{Q} \cup \{-\infty\}$ defined similarly
$\mathbb{Q}^{\pm} = (\mathbb{Q}^+)^-$

$X \subset Q$

X is a chain $\underset{\text{def}}{\iff}$ X is a totally ordered subset or $X = \emptyset$

length of $X := \# X - 1$

$\operatorname{rank} Q = \max\{$ length of chains in Q $\}$



Q is pure

$\iff \forall C$ (max. chain) length of C is $\operatorname{rank} Q$

$x, y \in Q$ $x \leq y$ とするとき

$x = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_t = y$

であるとき、$z_0, z_1, \dots, z_t$ is called a saturated chain from x to y

Def. $x, y \in Q$, $x \leq y$

$\mathrm{dist}_Q(x, y) = \mathrm{dist}(x, y)$

$:= \min\{t \mid$ there is a saturated chain from x to y with length $t\}$

Rem $\mathrm{rank}([x, y])$

$= \max\{t \mid$ there is a saturated chain from x to y with length $t\}$

P: finite poset

$\{T_x\}_{x \in P^-}$: a family of indet indexed by P^-

Def. (Hibi)

$$\mathcal{R}_{\mathbb{K}}[\mathcal{J}(P)] = \mathbb{K}[T^{\nu} \mid \nu \in \mathbb{Z}^{P^-}, \quad x, y \in P^-, x < y \Rightarrow \nu(x) \geq \nu(y)]$$

($\mathcal{J}(P)$: the set of poset ideals of P)

(D : dist. lattice
$\mathcal{R}_{\mathbb{K}}[D]$)

Def. (Stanley)

$$\mathcal{O}(P) = \{ f \in \mathbb{R}^P \mid 0 \leq f(x) \leq 1 \quad \forall x \in P, \quad x < y \Rightarrow f(x) \geq f(y) \}$$

$$\mathcal{C}(P) = \{ f \in \mathbb{R}^P \mid 0 \leq f(x) \quad \forall x \in P, \quad f^+(C) \leq 1 \quad \forall C \text{ (chain)} \}$$

それぞれ, order, chain polytope と呼ぶ。

Rem. $\mathcal{R}_{\mathbb{K}}[\mathcal{J}(P)] = E_{\mathbb{K}}[\mathcal{O}(P)]$

Fact. $E_{\mathbb{K}}[\mathcal{O}(P)]$, $E_{\mathbb{K}}[\mathcal{C}(P)]$ are standard graded, i.e., $\mathcal{O}(P)$ and $\mathcal{C}(P)$ have IDP

§3 Symbolic powers of the canonical ideals

R: noeth. normal domain

$\mathrm{Div}(R)$: the set of divisorial ideals of R

$\mathrm{Div}(R)$ is a group by

$$I \cdot J = R :_{Q(R)} (R :_{Q(R)} IJ)$$

For $n \in \mathbb{Z}$ and $I \in \mathrm{Div}(R)$

$I^{(n)} :=$ the n-th power of I in $\mathrm{Div}(R)$

<u>Rem.</u> If $I \subsetneq R$, and $n > 0$, then $I^{(n)}$ is the n-th symbolic power of I.

R: CM local or graded

ω_R: canonical module of R

$\Rightarrow \omega_R$: is reflexive

In particular

$$\omega_{E_{\mathbb{K}}[\mathcal{P}]} \in \mathrm{Div}(E_{\mathbb{K}}[\mathcal{P}])$$

<u>Rem.</u> $\mathrm{tr}(\omega_{E_{\mathbb{K}}[\mathcal{P}]}) = \omega_{E_{\mathbb{K}}[\mathcal{P}]}^{(-1)} \, \omega_{E_{\mathbb{K}}[\mathcal{P}]}$

P: poset $n \in \mathbb{Z}$

<u>Def</u> $\mathcal{J}^{(n)}(P) = \mathcal{J}^{(n)}$

$$= \{ \nu \in \mathbb{Z}^{P^-} \mid \nu(x) \geq n \quad \forall x \in \max P$$
$$x \lessdot y \Rightarrow \nu(x) \geq \nu(y) + n \}$$

$\mathcal{S}^{(n)}(P) = \mathcal{S}^{(n)}$

$$= \{ \xi \in \mathbb{Z}^{P^-} \mid \xi(x) \geq n \quad \forall x \in P$$
$$\xi^+(C) + n \geq \xi(-\infty) \quad \forall C \text{ (max. chain)} \}$$

<u>Rem.</u> $$\mathbb{K}[\mathcal{O}(P)] = \bigoplus_{\nu \in \mathcal{J}^{(0)}} \mathbb{K} T^\nu$$

$$\mathbb{K}[\mathcal{C}(P)] = \bigoplus_{\xi \in \mathcal{S}^{(0)}} \mathbb{K} T^\xi$$

<u>Thm</u> (M) $n \in \mathbb{Z}$

$$\omega^{(n)}_{\mathbb{K}[\mathcal{O}(P)]} = \bigoplus_{\nu \in \mathcal{J}^{(n)}} \mathbb{K} T^\nu$$

$$\omega^{(n)}_{\mathbb{K}[\mathcal{C}(P)]} = \bigoplus_{\xi \in \mathcal{S}^{(n)}} \mathbb{K} T^\xi$$

Cor. $\mathrm{tr}(\omega_{\Bbbk[\mathcal{O}(P)]}) = \sum_{\nu_1 \in \mathcal{J}^{(1)},\, \nu_2 \in \mathcal{J}^{(-1)}} \Bbbk T^{\nu_1+\nu_2}$

$\mathrm{tr}(\omega_{\Bbbk[\mathcal{C}(P)]}) = \sum_{\xi_1 \in \mathcal{S}^{(1)},\, \xi_2 \in \mathcal{S}^{(-1)}} \Bbbk T^{\xi_1+\xi_2}$

§4 Non-Gor. loci

Thm 4.1 $\nu \in \mathcal{J}^{(0)} \setminus \{0\}$.

set $\nu(+\infty) = 0$. Then

$T^{\nu} \in \sqrt{\mathrm{tr}(\omega_{\Bbbk[\mathcal{O}(P)]})}$

$\iff$ For any $a_1, \cdots, a_n, b_1, \cdots, b_n \in P^{\pm}$ with

$$(\ast)\begin{cases} a_1 < b_1 > a_2 < b_2 > \cdots > a_n < b_n > a_1 \\ a_i \neq a_j \quad b_i \neq b_j \quad (i \neq j) \\ \sum_{i=1}^{n} \mathrm{rank}([a_i, b_i]) \\ > \sum_{i=1}^{n-1} \mathrm{dist}(a_{i+1}, b_i) + \mathrm{dist}(a_1, b_n) \end{cases}$$

it holds that $\sum_{i=1}^{n} \nu(a_i) > \sum_{i=1}^{n} \nu(b_i)$

$\Sigma f(a_i) = \Sigma f(b_i)$

Thm 4.2. $\xi \in \mathcal{S}^{(0)}$. Assume $\xi(-\infty) > 0$

Then

$$T\xi \in \sqrt{\operatorname{tr}(W_{\mathbb{K}[e(P)]})}$$

$\Longleftrightarrow$ For any $a_1, \cdots, a_u, b_1, \cdots, b_u \in P^I$ with $(\ast)$

and max. chains C_i (resp. C_i')

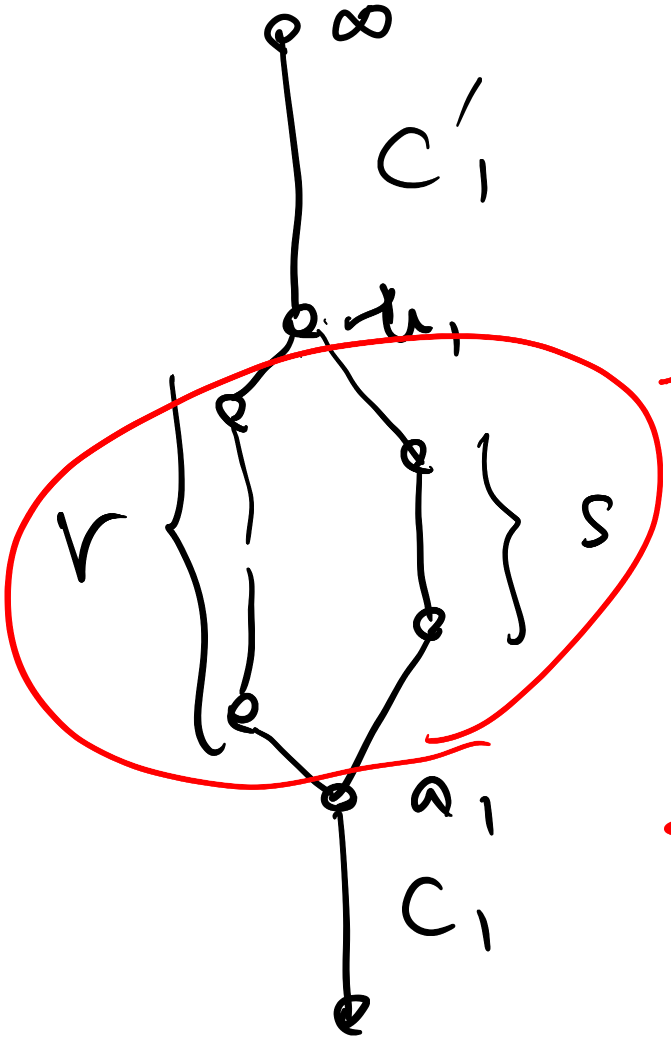
in $(-\infty, a_i]$, (resp. $[b_i, \infty)$)

it holds that

$$\sum_{i=1}^{u} \left(\xi^+(C_i) + \xi^+(C_i') \right) < u\,\xi(-\infty)$$

(points of proof $\Rightarrow$)

ex. $u = 1$ assume the contrary



$\xi(z) = 0$

$r > s$.

dim mon form $\leq \dim K[\mathcal{O}(P)] - 4$

$$\xi^+(C_1) + \xi^+(C_1') = \xi(-\infty)$$

$$\xi = \xi_1 + \xi_2, \quad \xi_1 \in S^{(1)}, \; \xi_2 \in S^{(A)}$$

$$\xi(z) \geq 0.$$

$$\xi_1(z) \geq 1, \quad \xi_2(z) \geq -1$$

$$\therefore \; \xi_1(z) = 1, \quad \xi_2(z) = -1$$

$$\xi^+(C_1) + \xi^+(C_1') = \xi(-\infty)$$

$$\xi_1^+(C_1) + r + \xi_1^+(C_1') + 1 \leq \xi_1(-\infty)$$

$$\xi_2^+(C_1) - s + \xi_2^+(C_2') - 1 \leq \xi_2(-\infty)$$

$$\xi_1^+(C_1) + \xi_2^+(C_1) + \xi_1^+(C_1') + \xi_2^+(C_1') + r - s \leq \underline{\xi_1(-\infty) + \xi_2(-\infty)}$$

$$\therefore \xi^+(C_1) + \xi^+(C_1') + r - s \leq \xi(-\infty)$$

矛盾.