POLYNOMIAL FUNCTORS AND THE JOHNSON FILTRATION

(with Djament)

Aim: 2 generalizations of the notion of polynomial functors

- Classical polynomial functors

50': Eilenberg and MacLane $\rightsquigarrow$ polynomial functors $R\text{-Mod} \to R\text{-Mod}$.

Typical example

$T^n : R\text{-Mod} \to R\text{-Mod}$. polynomial of degree n
$M \mapsto M^{\otimes n}$

$\rightsquigarrow$ stable by quotient, subobjets: S^n, Λ^n, Γ^n poly of degree n.

This definition can be extended to functors from $(\mathcal{C}, \oplus, 0)$ monoidal where 0 is null object

ex: $\cdot (\Gamma, \vee, [0])$ finite pointed sets + pointed maps.
$\cdot (ab, \oplus, 0)$ finitely generated free abelian group.
$\cdot (gr, *, 0)$

$\mathcal{P}ol_n(\mathcal{C}, \mathbb{K}\text{-Mod})$ complicated but thick subcategory of $\mathrm{Func}(\mathcal{C}, \mathbb{K}\text{-Mod})$

$\rightsquigarrow \mathcal{P}ol_n(\mathcal{C}, \mathbb{K}\text{-Mod}) / \mathcal{P}ol_{n-1}(\mathcal{C}, \mathbb{K}\text{-Mod})$ well-understood.

- Motivation: Lot of interesting functors having polynomial properties defined only on $(\mathcal{C}, \oplus, 0)$ where 0 is initial

ex: $\cdot (FI, \sqcup, \emptyset)$ Finite sets with Injections
$\cdot (S(ab), \oplus, 0)$ Same objects as ab
$S(ab)(\mathbb{Z}^n, \mathbb{Z}^m) = \{ \mathbb{Z}^n \underset{v}{\overset{u}{\rightleftarrows}} \mathbb{Z}^m \,/\, u \circ v = Id \}$.
$\cdot (\mathcal{G}, *, 0)$ Same object as gr
$\mathcal{G}(\mathbb{Z}^{*n}, \mathbb{Z}^{*m}) = \{ (\mathbb{Z}^{*n} \overset{u}{\hookrightarrow} \mathbb{Z}^{*m}, K) \,/\, \mathbb{Z}^{*m} = K * u(\mathbb{Z}^{*n}) \}$

Aim: Extend notion of poly. functors of EML to this setting.

2 notions: $\to$ strong polynomial functors
$\to$ weak polynomial functors

<u>Examples</u>

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$$\begin{tikzcd} \circlearrowleft \mathrm{Aut}(F_n) & \circlearrowleft \mathrm{Aut}(F_n) \text{ by conjugaison} & \\ IA_n = \mathrm{Ker}(\varphi) \arrow[r] \arrow[d, dashed] & \mathrm{Aut}(F_n) \arrow[r, two heads, "\varphi"] \arrow[d] & GL_n(\mathbb{Z}) \arrow[d] \\ IA_{n+1} \arrow[r] & \mathrm{Aut}(F_{n+1}) \arrow[r, two heads] & GL_{n+1}(\mathbb{Z}) \end{tikzcd}$$

$\Rightarrow$ $IA_\bullet : \mathcal{G} \to \underbrace{Gr}$ category of all groups

$\mathbb{Z}^{*n} \mapsto IA_n$

Lower central series.

$\gamma_k(IA_\bullet)/\gamma_{k+1}(IA_\bullet) : \mathcal{G} \to \underbrace{Ab}$ category of all abelian gps

$\rightsquigarrow$ $\gamma_k(IA_\bullet)/\gamma_{k+1}(IA_\bullet) : S(ab) \to Ab.$

<u>Thm 1</u>: (Djament–V.) This functor is strong polynomial of degree $\leq 3k$.

$\Rightarrow$ weak poly of degree $\leq 3k$.

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$$\begin{tikzcd} \circlearrowleft \mathrm{Aut}(F_n) & \circlearrowleft \mathrm{Aut}\,F_n & \\ \mathcal{A}_k(F_n) := \mathrm{Ker}\,\psi \arrow[r] \arrow[d, dashed] & \mathrm{Aut}(F_n) \arrow[r, "\psi"] \arrow[d] & \mathrm{Aut}(F_n/\gamma_{k+1}(F_n)) \arrow[d] \\ \mathcal{A}_k(F_{n+1}) \arrow[r] & \mathrm{Aut}(F_{n+1}) \arrow[r] & \mathrm{Aut}(F_n/\gamma_{k+1}(F_n)) \end{tikzcd}$$

$\rightsquigarrow$ $\mathcal{A}_k : \mathcal{G} \to Gr.$

$\mathcal{A}_k/\mathcal{A}_{k+1} : S(ab) \longrightarrow Ab$

<u>Thm 2</u>: (Djament–V.) This functor is | strong polynomial
| weak poly of degree $k+2$

<u>Motivation</u>: $\gamma_k IA_n \subset \mathcal{A}_k(F_n)$

Andreadakis conjecture: is it an equality?

NO: Bartholdi (2017) $\mathcal{A}_4(F_3)/\gamma_4(F_3) \otimes \mathbb{Q} \simeq \mathbb{Q}^3$

Stable Andreadakis conjecture: is it an equality <u>for $n \gg k$</u>?

Plan: I Strong polynomial functors
II Weak ———
III Applications.

$\mathcal{M}on$: category of small sym monoidal category $(\mathcal{M}, \oplus, 0)$
$\mathcal{M}on_{ini}$: ——— where 0 initial
$\mathcal{M}on_{nul}$: ——— where 0 null

Remark: Here symmetric but true in the a more general setting of prebraided category (Randal-Williams Wahl) (See talk of Soulié).

I Strong polynomial functors

1- Definition

$\mathcal{A}$ abelian category
$\mathcal{M} \in \mathcal{M}on_{ini}$
$x \in \mathcal{M}$

- Shift functor $\tau_x : \mathrm{Func}(\mathcal{M}, \mathcal{A}) \to \mathrm{Func}(\mathcal{M}, \mathcal{A})$, $F \mapsto F(x \oplus -)$

as 0 is initial $\exists!\ 0 \to x$

$$F = \tau_0 F \xrightarrow{i_x(F)} \tau_x F$$

$$0 \to \underbrace{\kappa_x = \mathrm{Ker}(i_x)}_{\text{the evanescence functor}} \to \mathrm{Id} \xrightarrow{i_x} \tau_x \to \underbrace{\delta_x := \mathrm{coKer}(i_x)}_{\text{the difference functor}} \to 0$$

Def: $F : \mathcal{M} \to \mathcal{A}$ strong poly of $\underline{o} \leq d$ if $\forall (a_0, \dots a_d) \in \mathcal{M}^{d+1}$ $\delta_{a_0} \delta_{a_1} \dots \delta_{a_d} F = 0$

Rem: If $\mathcal{M}$ generated by t

| Ex: FI generated by 1
| $S(ab)$ ——— $\mathbb{Z}$
| $\mathcal{G}$ ——— $\mathbb{Z}$

F strong poly of $\underline{o} \leq d$ iff $\delta_t^{d+1} F = 0$

τ_x is exact.

for $0 \to F \to G \to H \to 0$ by the snake lemma

$$0 \to \kappa_x F \to \kappa_x G \to \kappa_x H \to \delta_x F \to \delta_x G \to \delta_x H \to 0$$

δ_x is <u>right</u> exact

Rem: <u>What's happen if $\mathcal{M} \in \mathcal{M}on_{nul}$</u> ?

$0 \to x$ split so i_x split.

$\Rightarrow \kappa_x = 0$

$\leadsto$ we recover the usual def of poly functors of EMcL.

2- <u>Examples</u> on FI

- $\mathbb{Z}: FI \to Ab$
  $k \to \mathbb{Z}$

  $\tau_1 \mathbb{Z} = \mathbb{Z} \Rightarrow \delta_1 \mathbb{Z} = 0$ so <u>$\mathbb{Z}$ is strong poly of degree 0</u>

- $\mathbb{Z}_i : FI \to Ab$ (atomic functor)
  $k \mapsto \begin{cases} \mathbb{Z} & \text{if } k = i \\ 0 & \text{else} \end{cases}$

  $\tau_1 \mathbb{Z}_i = \mathbb{Z}_{i-1}$ $\quad \delta_1 \mathbb{Z}_i = \mathbb{Z}_{i-1} \Rightarrow \delta_1^{i+1}(\mathbb{Z}_i) = 0$

  <u>$\mathbb{Z}_i$ is strong poly of degree i</u>

- $\mathbb{Z}_{\geq i} : FI \to Ab.$
  $k \mapsto \begin{cases} \mathbb{Z} & \text{if } k \geq i \\ 0 & \text{if } k < i \end{cases}$

  $\delta_1 \mathbb{Z}_{\geq i} = \mathbb{Z}_{i-1}$ $\quad$ <u>$\mathbb{Z}_{\geq i}$ strong poly of degree i</u>

<u>But</u>: $\mathbb{Z}_{\geq i}$ is a subfunctor of $\mathbb{Z}$
so "strong poly" is not stable by subfunctor

- **Relation with finitely generated FI-modules.**

Thm: (Church-Ellenberg-Farb)

$$n \mapsto H^i(\mathrm{Conf}_n(M), \mathbb{Q})$$

$$n \mapsto H^i(\mathcal{M}_{g,n}, \mathbb{Q})$$

...

are finitely generated FI-modules

Prop: (Djament-V.)
$F \in \mathrm{Fonc}(FI, \mathcal{A})$ is finitely generated
iff F is strong polynomial with finitely generated values.

- **Limits of the notion of strong poly functors**

1- $\mathcal{P}ol_n^{\text{strong}}$ stable by quotient, extensions, colimits **but** not by sub-objects

2- We would like a notion of poly functors adapted to stable phenomena

FI	$\mathbb{Z}$	$\mathbb{Z}_{\geq N}$
0	$\mathbb{Z}$	0
1	$\mathbb{Z}$	$\vdots$
$\vdots$	$\vdots$	
$N-1$	$\mathbb{Z}$	0
N	$\mathbb{Z}$	$\mathbb{Z}$
$N+1$	$\mathbb{Z}$	$\mathbb{Z}$
$\vdots$	$\vdots$	$\vdots$
	strong poly of d° 0	strong poly of d° N

but these 2 functors are equals for n big enough!

We would like to identify these two functors

II Weak polynomial functors

Idea: To work in a quotient category in which we kill all the unstable phenomena.

1- The stable category $St(\mathcal{M}, \mathcal{A})$

$$\mathcal{K}(F) := \sum_{x \in Ob(\mathcal{M})} \mathcal{K}_x(F)$$

Def: F is stably zero if $\mathcal{K}(F) = F$

Prop: If $\mathcal{C}$ generated by one object t
F stably zero iff $\operatorname{colim}_{n \in \mathbb{N}} F(t^{\oplus n}) = 0$

Ex: $\mathbb{Z}i : FI \to Ab$ is stably zero.

$Sn(\mathcal{M}, \mathcal{A})$ full subcategory of $Func(\mathcal{M}, \mathcal{A})$ of stably zero functors

Prop: $\mathcal{M} \in \mathcal{M}onini$ $\mathcal{A}$ Grothendieck category
$Sn(\mathcal{M}, \mathcal{A})$ is a thick subcategory of $Func(\mathcal{M}, \mathcal{A})$ stable under colimits

$$St(\mathcal{M}, \mathcal{A}) := Func(\mathcal{M}, \mathcal{A}) / Sn(\mathcal{M}, \mathcal{A})$$
$$\uparrow \pi_{\mathcal{M}}$$
$$Func(\mathcal{M}, \mathcal{A}).$$

$\mathcal{K}_x$ takes its values in $Sn(\mathcal{M}, \mathcal{A})$ so

Prop: ses of endofunctors of $St(\mathcal{M}, \mathcal{A})$

$$0 \to Id \to \tau_x \to \delta_x \to 0.$$

δ_x is exact.

2- Definition

<u>Def</u>: • $F \in St(\mathcal{M}, \mathcal{A})$ is <u>poly</u> of $\stackrel{\circ}{=} \leq d$ if
$\forall a_0, \dots, a_d \quad \delta_{a_0} \cdots \delta_{a_d}(F) = 0$

• $F \in Func(\mathcal{M}, \mathcal{A})$ is <u>weak poly</u> of $\stackrel{\circ}{=} \leq d$
if $\pi_{\mathcal{M}}(F)$ is poly of $\stackrel{\circ}{=} \leq d$

<u>Rem</u>: If F is strong poly of $\stackrel{\circ}{=} n$, F is weak poly of $\stackrel{\circ}{=} \leq n$

<u>Rem</u>: When $\mathcal{M} \in \mathcal{M}on_{nul}$ $\quad St(\mathcal{M}, \mathcal{A}) = Func(\mathcal{M}, \mathcal{A})$
So strong poly = weak poly = poly = usual poly

<u>Example illustrating the rôle of strong $\stackrel{\circ}{=}$ and weak $\stackrel{\circ}{=}$</u>

$\mathcal{M} \in \mathcal{M}on_{ini}$ generated by t.
$F: \mathcal{M} \to \mathcal{A}$
Assume that $\exists N$ s.t. $\forall n \geq N \quad F(t^n) = F(t^N) = M$.

FI	F	$\tau_t F$	$\delta_t F$	0
⋮	⋮	⋮	⋮	
$N-1$	M	M	0	
N	M	M	0	
$N+1$				

$\operatorname{colim} \delta_t F = 0 \qquad$ so $\pi_{\mathcal{M}}(\delta_1 F) = 0$

$\boxed{F \text{ weak poly of } \stackrel{\circ}{=} 0}$

$\delta_1^{N+1} F = 0$

$\boxed{F \text{ strong poly of } \stackrel{\circ}{=} N}$

III Application

Thm: (D-V) $\mathcal{A}_n/\mathcal{A}_{n+1}: S(ab) \to Ab$ is weak poly of $\overset{\circ}{=} n+2$

[$\mathcal{L}^{n+1}: Ab \to Ab$ Lie functor poly of $\overset{\circ}{=} n$.

Proof:

① Johnson homomorphism

$$t_n : (\mathcal{A}_n/\mathcal{A}_{n+1})(V) \hookrightarrow Hom_{Ab}(V, \mathcal{L}^{n+1}(V))$$

GL(V)-equivariant.

⇝ natural transformation

$$t_n : (\mathcal{A}_n/\mathcal{A}_{n+1}) \hookrightarrow Hom_{Ab}(-, \mathcal{L}^{n+1}(-)).$$

② $S(ab) \to Ab$.

$V \mapsto Hom_{Ab}(V, \mathcal{L}^{n+1}(V))$.

$$S(ab) \underset{\text{strong monoidal}}{\xrightarrow{\Delta}} ab^{op} \times ab \underset{\text{poly of } \overset{\circ}{=} n+2}{\xrightarrow{Hom(-,\mathcal{L}^{n+1}(-))}} Ab$$

} strong | weak | poly of $\overset{\circ}{=} = n+2$

③ Stability by subobjects

⇝ $\boxed{\deg \pi_{S(ab)}(\mathcal{A}_n/\mathcal{A}_{n+1}) \le n+2}$

④ $t'_n : (\gamma_n/\gamma_{n+1})(IA) \to \mathcal{A}_n/\mathcal{A}_{n+1} \xrightarrow{t_n} Hom_{Ab}(-, \mathcal{L}^{n+1}(-))$

Thm: (Satoh 2012)

For $n \geq 2$ and $k \geq n+2$

$$Coker\, t'_n \otimes \mathbb{Q}(\mathbb{Z}^{\oplus k}) \simeq C_n^{\mathbb{Q}}(\mathbb{Z}^k)$$

where $C_n^{\mathbb{Q}}(H) = H^{\otimes n} / \langle a_1 \otimes \cdots \otimes a_n - a_2 \otimes \cdots \otimes a_n \otimes a_1 \rangle$

$T^n \twoheadrightarrow C_n^{\mathbb{Q}}$ so $C_n^{\mathbb{Q}}$ is poly of $\overset{\circ}{=} n$.

$Coker\, t'_n \otimes \mathbb{Q}$ ⇝ weak poly of $\overset{\circ}{=} n$. } stability by quotient.

⇓

$Coker\, t_n \otimes \mathbb{Q}$ ⇝ — — — — —

$Hom_{Ab}(-, \mathcal{L}^{n+1}(-)) \to Coker\, t_n \to 0$

$0 \to \mathcal{A}_n/\mathcal{A}_{n+1} \xrightarrow{t_n} Hom_{Ab}(-, \mathcal{L}^{n+1}(-))$

$-\otimes \mathbb{Q}$ and $\pi_{S(ab)}$ are exact.

$$0 \to \pi_{S(ab)}(\mathcal{A}_n/\mathcal{A}_{n+1} \otimes \mathbb{Q}) \to \underbrace{\pi_{S(ab)}(Hom_{Ab}(-, \mathcal{L}^{n+1}(-)) \otimes \mathbb{Q})}_{\text{poly } \overset{\circ}{=} = n+2} \to \underbrace{\pi_{S(ab)}(Coker\, t_n \otimes \mathbb{Q})}_{\text{poly } \overset{\circ}{=} \le n} \to 0$$

$$\boxed{\deg \pi_{S(ab)}(\mathcal{A}_n/\mathcal{A}_{n+1}) \geq \deg \pi_{S(ab)}(\mathcal{A}_n/\mathcal{A}_{n+1} \otimes \mathbb{Q}) \underset{\text{stability by ses}}{=} n+2.}$$