A representation theoretic approach to the Johnson cokernels I

Joint work with

Enomoto, Naoya (Nara Women's University)

Satoh, Takao

(Tokyo University of Science)

Automorphism groups of free groups

- $F_n := \langle x_1, \ldots, x_n \rangle$: Free group of rank $n \geq 2$
- $H := H_1(F_n, \mathbb{Z}) \cong \mathbb{Z}^{\oplus n}$: Abelianization of F_n

$$\begin{array}{ccc} \rho : \mathrm{Aut}\, F_n & \xrightarrow{\text{surj.}} & \mathrm{Aut}(H) \\ & & \| \\ & & \mathrm{GL}(n, \mathbb{Z}) \end{array}$$

$$\mathrm{GL}(n, \mathbb{Z}) := \{ A \in M(n, \mathbb{Z}) \mid \det A = \pm 1 \}$$

IA-automorphism groups

$$\mathrm{IA}_n := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbb{Z}))$$

- Magnus, 1935

  IA_n is finitely generated by

  $$K_{ij} : x_i \mapsto x_j{}^{-1} x_i x_j,$$

  $$K_{ijk} : x_i \mapsto x_i\,[x_j, x_k], \quad j < k$$

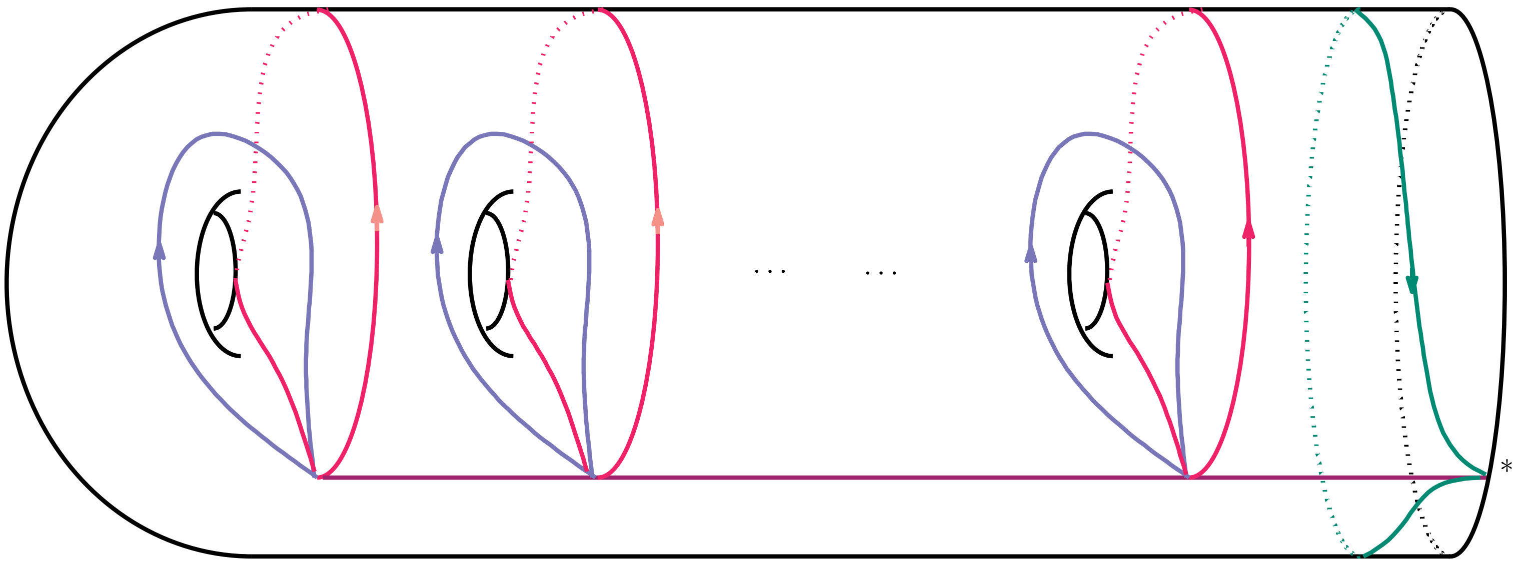
Problem For $n \geq 3$,

Find a presentation for IA_n.

Mapping class groups of surfaces

- $g \geq 1$,

  $\Sigma_{g,1} :=$



$$\pi_1(\Sigma_{g,1}, *) \cong F_{2g}$$

- $\mathcal{M}_{g,1} := \mathrm{Diff}^+(\Sigma_{g,1}, \partial)/\mathrm{isotopy}$

Theorem (Dehn, Nielsen) $g \geq 1$

$$\iota : \mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}\, F_{2g}$$

$$\mathrm{Im}(\iota) = \{\sigma \in \mathrm{Aut}\, F_{2g} \,|\, \zeta^\sigma = \zeta\}$$

- **Torelli group**

$$\mathcal{I}_{g,1} := \mathrm{IA}_{2g} \cap \mathcal{M}_{g,1}$$

$$\begin{array}{ccccccccc}
1 & \to & \mathrm{IA}_{2g} & \to & \mathrm{Aut}\, F_{2g} & \xrightarrow{\rho} & \mathrm{GL}(2g,\mathbb{Z}) & \to & 1 \\
 & & \uparrow & & \iota \uparrow & & \uparrow & & \\
1 & \to & \mathcal{I}_{g,1} & \to & \mathcal{M}_{g,1} & \to & \mathrm{Sp}(2g,\mathbb{Z}) & \to & 1
\end{array}$$

Andreadakis-Johnson filtration of $\mathrm{Aut}\, F_n$

- $\Gamma_n(k)$: lower central series of F_n

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n]$$

- $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$

Facts. (Magnus, Witt, Hall)

$\mathcal{L}_n := \bigoplus_{k \geq 1} \mathcal{L}_n(k)$ is the free Lie algebra generated by H.

- **Andreadakis-Johnson filtration** $\quad k \geq 1$

$$\mathcal{A}_n(k) := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)))$$

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

Theorem (Andreadakis, 1965)

(1) $[\mathcal{A}_n(k), \mathcal{A}_n(l)] \subset \mathcal{A}_n(k+l)$

(2) $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is **free abelian**.

- $\mathrm{gr}^k(\mathcal{A}_n)$: sequence of **approximations** of IA_n.

Remark For any $\sigma \in \mathcal{A}_n(k)$ and $x \in F_n$,

$$x^{-1}x^{\sigma} \in \Gamma_n(k+1)$$

- The k-th Johnson homomorphism of Aut F_n

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to \mathrm{Hom}_{\mathbb{Z}}(H, \mathcal{L}_n(k+1))$$

$$\overline{\sigma} \mapsto (\overline{x} \mapsto \overline{x^{-1}x^{\sigma}})$$

Facts.

(1) τ_k is injective.

(2) τ_k is a $\mathrm{GL}(n,\mathbb{Z})$-equivariant homomorphism

(3) Under the identification

$$\mathrm{Hom}_{\mathbb{Z}}(H, \mathcal{L}_n(k+1)) = \mathrm{Der}(\mathcal{L}_n)(k),$$

the graded sum

$$\tau := \bigoplus_{k\geq 1} \tau_k : \bigoplus_{k\geq 1} \mathrm{gr}^k(\mathcal{A}_n) \to \mathrm{Der}(\mathcal{L}_n)$$

is a Lie algebra homomorphism.

Problem

- **Determine** $\mathrm{Im}(\tau_k)$ **and** $\mathrm{Coker}(\tau_k)$.
- **Determine** $\mathrm{Im}(\tau_{k,\mathbb{Q}})$ **and** $\mathrm{Coker}(\tau_{k,\mathbb{Q}})$.

- A, B : $\mathbb{Z}$**-module**, $\quad f : A \to B$: $\mathbb{Z}$**-map**

$$A_{\mathbb{Q}},\ A^{\mathbb{Q}} := A \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$f_{\mathbb{Q}},\ f^{\mathbb{Q}} := f \otimes \mathrm{id}_{\mathbb{Q}}$$

Johnson cokernels for Aut F_n

- $\mathcal{A}'_n(k)$: Lower central series of IA_n

$$
\begin{array}{ccccccc}
\mathcal{A}_n(1) & \supset & \mathcal{A}_n(2) & \supset & \mathcal{A}_n(3) & \supset & \cdots \\
\| & & \|_{\text{Bachmuth}} & & \cup_{\text{Pettet}} & & \\
\mathcal{A}'_n(1) & \supset & \mathcal{A}'_n(2) & \supset & \mathcal{A}'_n(3) & \supset & \cdots
\end{array}
$$

Conjecture (Andreadakis) For any $n, k \geq 3$,

$$\mathcal{A}_n(k) = \mathcal{A}'_n(k)$$

- $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$ $\qquad H^* := \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to \mathrm{gr}^k(\mathcal{A}_n) \xrightarrow{\tau_k} H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1)$$

$$\mathrm{Coker}(\tau'_{k,\mathbb{Q}}) \xrightarrow{\text{surj.}} \mathrm{Coker}(\tau_{k,\mathbb{Q}})$$

Theorem (S., 2009) For $k \geq 2$, $n \geq k+2$,

$$\mathrm{Coker}(\tau'_{k,\mathbb{Q}}) \cong \mathcal{C}_n^{\mathbb{Q}}(k)$$

$$\mathcal{C}_n(k) := H^{\otimes k} / \langle a_1 \otimes a_2 \otimes \cdots \otimes a_k - a_2 \otimes \cdots \otimes a_k \otimes a_1 \mid a_i \in H \rangle$$

How to detect $\mathcal{C}_n^{\mathbb{Q}}(k)$

- $\mathcal{L}_n(k) \hookrightarrow H^{\otimes k}$

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

- $H^* \otimes_{\mathbb{Z}} H^{\otimes k+1} \to H^{\otimes k}$

$$x_i^* \otimes x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_1}) \cdot x_{j_2} \otimes \cdots \otimes x_{j_{k+1}}$$

Contraction map

$$\Phi_k : H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \to H^{\otimes k}$$

- $H^{\otimes k} \to \mathcal{C}_n(k)$: **projection**

$$\overline{\Phi}_k : H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \xrightarrow{\Phi_k} H^{\otimes k} \to \mathcal{C}_n(k)$$

Examples

- A Magnus generator of IA_n.

$$K_{ij} : x_i \mapsto {x_j}^{-1} x_i x_j$$

- Johnson images. For any distinct i, j, and l,

$$\tau_1(K_{ij}) = x_i^* \otimes [x_i, x_j]$$

$$\begin{aligned} \tau_2'([K_{ij}, K_{il}]) &= [x_i^* \otimes [x_i, x_j],\ x_i^* \otimes [x_i, x_l]] \\ &= x_i^* \otimes [[x_i, x_l], x_j] - x_i^* \otimes [[x_i, x_j], x_l] \\ &= x_i^* \otimes [[x_j, x_l], x_i] \end{aligned}$$

- $H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \to H^* \otimes_{\mathbb{Z}} H^{\otimes k+1}$

$$\tau_2'([K_{ij}, K_{il}]) = x_i^* \otimes [[x_j, x_l], x_i]$$
$$\mapsto x_i^* \otimes ([x_j, x_l] \otimes x_i - x_i \otimes [x_j, x_l])$$

Hence,

$$\Phi_2(\tau_2'([K_{ij}, K_{il}])) = -(x_j \otimes x_l - x_l \otimes x_j) \in H^{\otimes 2}$$

$$\mapsto 0 \in \mathcal{C}_n(2)$$

Theorem (S., 2009)

For any $n \geq k+2$,

$$\mathrm{Im}(\tau'_{k,\mathbb{Q}}) = \mathrm{Ker}(\overline{\Phi}_k^{\mathbb{Q}})$$

$$\mathrm{gr}^k_{\mathbb{Q}}(\mathcal{A}'_n) \xrightarrow{\tau'_{k,\mathbb{Q}}} H^*_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathcal{L}_n^{\mathbb{Q}}(k+1) \xrightarrow{\overline{\Phi}_k^{\mathbb{Q}}} \mathcal{C}_n^{\mathbb{Q}}(k) \to 0$$

(exact)

Generators of $\mathrm{Im}(\tau'_{k,\mathbb{Q}})$

- $x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}]$ for $i_1, \ldots, i_{k+1} \neq i$
- $x_i^* \otimes [x_{i_1}, \ldots, x_{i_k}, x_i]$ for $i_1, \ldots, i_k \neq i$
- $x_i^* \otimes [x_i, x_{i_1}, \ldots, x_{i_k}] - x_j^* \otimes [x_j, x_{i_k}, x_{i_1}, \ldots, x_{i_{k-1}}]$

  for $i, j \neq i_1, \ldots, i_k$ and $i \neq j$,
- $x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}]$

  $- \sum_{l=1}^{k} \delta_{i,i_l} x_m^* \otimes [x_{i_1}, \ldots, x_{i_{l-1}}, x_m, x_{i_{l+1}}, \ldots, x_{i_k}, x_{i_{k+1}}]$

  for $m \neq i_1, \ldots, i_{k+1}$
- $x_i^* \otimes [x_i, x_{i_1}, \ldots, x_{i_k}] - x_i^* \otimes [x_i, x_{i_k}, x_{i_1}, \ldots, x_{i_{k-1}}]$ for $i \neq i_1, \ldots, i_k$
- $x_i^* \otimes [x_i, x_{i_1}, \ldots, x_{i_k}] - x_j^* \otimes [x_j, x_{i_1}, \ldots, x_{i_k}]$

  for $i, j \neq i_1, \ldots, i_k$ and $i \neq j$

Observation

For any $1 \leq l \leq k+1$

- $H^* \otimes_{\mathbb{Z}} H^{\otimes k+1} \to H^{\otimes l-1} \otimes H^{\otimes k-l+1}$

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_l} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_l}) \cdot x_{j_1} \otimes \cdots \otimes x_{j_{k+1}}$$

$$\overline{\Phi}_k^l : H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \to \mathcal{C}_n(l-1) \otimes \mathcal{C}_n(k-l+1)$$

Proposition For any $1 \leq l \leq k+1$,

$$\mathrm{Im}(\tau'_{k,\mathbb{Q}}) = \mathrm{Ker}(\overline{\Phi}^1_{k,\mathbb{Q}}) \subset \mathrm{Ker}(\overline{\Phi}^l_{k,\mathbb{Q}})$$

- **Enoplomoto and S. (2010)**

We give a combinatorial description of the irreducible GL-decomposition of $\mathcal{C}_n^{\mathbb{Q}}(k)$ for $n \geq k+2$.

k	$\mathcal{C}_n^{\mathbb{Q}}(k)$
2	(2)
3	$(3) \oplus (1^3)$
4	$(4) \oplus (2,2) \oplus (2,1^2)$
5	$(5) \oplus (3,2) \oplus 2(3,1^2) \oplus (2^2,1) \oplus (1^5)$

- **(λ) : the irreducible polynomial representation corresponding to a Young diagram λ**

Morita obstruction

- $S^k H_{\mathbb{Q}} \cong (k)$: the Symmetric tensor product of $H_{\mathbb{Q}}$ of degree k

- Morita's trace map

$$\mathrm{Tr}_{(k)} : H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \xrightarrow{\Phi_k} H^{\otimes k} \to S^k H$$

Theorem. (Morita) For any $n, k \geq 2$,

(1) $\mathrm{Tr}_{(k)}$ is surjective.

(2) $\mathrm{Tr}_{(k)}$ vanishes on $\mathrm{Im}(\tau_k)$.

$$S^k H_{\mathbb{Q}} \subset \mathrm{Coker}(\tau_{k,\mathbb{Q}})$$

- $\Lambda^k H_{\mathbb{Q}} \cong (1^k)$: the exterior product of H of degree k
- Trace map for (1^k)

$$\mathrm{Tr}_{(1^k)} : H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \xrightarrow{\Phi_k} H^{\otimes k} \to \Lambda^k H$$

Theorem. (S.) For any $k \geq 3$ and $n \geq k+1$,

if k is odd,

(1) $\mathrm{Tr}_{(1^k)}$ is surjective.

(2) $\mathrm{Tr}_{(1^k)}$ vanishes on $\mathrm{Im}(\tau'_k)$.

$$\Lambda^k H_{\mathbb{Q}} \subset \mathrm{Coker}(\tau'_{k,\mathbb{Q}})$$

- **Enomoto-S., 2010** For any $n \geq k+2$,

(1) $[\mathcal{C}_n^{\mathbb{Q}}(k) : (k)] = 1$ for any $k \geq 2$

(2) $[\mathcal{C}_n^{\mathbb{Q}}(k) : (1^k)] = 1$ for any odd $k \geq 3$