Intersection double brackets

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(joint work with Vladimir Turaev)

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Contents

Quasi-Poisson algebras

$\mathbb{K}$: a commutative ring
$(G, \mathfrak{g})$: a group G acting on a Lie algebra $\mathfrak{g}$

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Definition

A $(G, \mathfrak{g})$-algebra is a commutative algebra $\mathcal{A}$ on which G and $\mathfrak{g}$ act in compatible ways: $\forall v \in \mathfrak{g}, \forall g \in G, \forall a \in \mathcal{A},\ (g \cdot v) \cdot a = g \cdot \big(v \cdot (g^{-1} \cdot a)\big)$.

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Definition (≃ Alekseev, Kosmann-Schwarzbach & Meinrenken 2002)

A quasi-Poisson bracket is a $(G, \mathfrak{g})$-invariant map $\{-,-\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that

$$\{a, b\} = -\{b, a\}, \quad \{a, b_1 b_2\} = b_1 \{a, b_2\} + \{a, b_1\} b_2$$

and

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = \phi \cdot (a \otimes b \otimes c).$$

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Remark

$\{-,-\}$ restricts to a Poisson bracket on $\mathcal{A}^{\mathfrak{g}}$.

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$$a_{ij} \quad \text{for all } a \in A \text{ and } i, j \in \{1, \ldots, N\}$$

with relations $(ka + lb)_{ij} = ka_{ij} + lb_{ij},\ 1_{ij} = \delta_{ij},\ (ab)_{ij} = \sum_{r=1}^{N} a_{ir} b_{rj}$.

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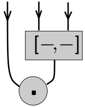

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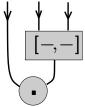

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Problem

Give to the **non-commutative** algebra A enough structure so that the **commutative** algebra A_N inherits a (quasi-)Poisson bracket.

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A double bracket in A is a linear map

$$\{\!\{-,-\}\!\} : A \otimes A \longrightarrow A \otimes A$$

such that

- $\{\!\{a, bc\}\!\} = b \cdot \{\!\{a, c\}\!\} + \{\!\{a, b\}\!\} \cdot c,$
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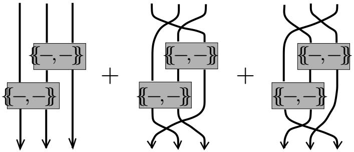

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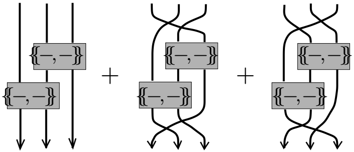
$\{\!\{-,-,-\}\!\} :=$  $: A^{\otimes 3} \longrightarrow A^{\otimes 3}$

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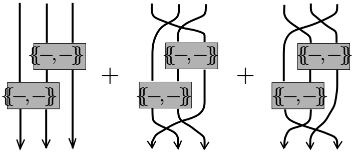
$\{\!\{-,-\}\!\}$ is Poisson if $\{\!\{a, b, c\}\!\} = 0$, and $\{\!\{-,-\}\!\}$ is quasi-Poisson if

$$\begin{aligned}\{\!\{a, b, c\}\!\} \quad = \quad & a \otimes 1 \otimes bc + 1 \otimes ab \otimes c + ca \otimes b \otimes 1 + c \otimes a \otimes b \\ & -1 \otimes a \otimes bc - a \otimes b \otimes c - ca \otimes 1 \otimes b - c \otimes ab \otimes 1.\end{aligned}$$

$\{\!\{-,-,-\}\!\} :=$  $: A^{\otimes 3} \longrightarrow A^{\otimes 3}$

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Theorem (Van den Bergh 2008)

Let $\{\!\{-,-\}\!\}$ be a Poisson (resp. quasi-Poisson) double bracket in A.

- $\{\!\{-,-\}\!\}$ *induces a Poisson (resp. quasi-Poisson) bracket $\{-,-\}$ on A_N defined by*

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Corollary (Van den Bergh 2008)

$\{-,-\}$ *restricts to a Poisson bracket on* A_N^t *and* $\mathrm{tr} : \check{A} \longrightarrow A_N$ *preserves the brackets.*

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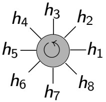

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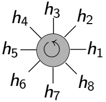
$\check{\mathsf{T}} = \mathsf{T} / [\mathsf{T}, \mathsf{T}] \quad \ni$ 

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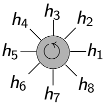
$\check{\mathsf{T}} = \mathsf{T}/[\mathsf{T}, \mathsf{T}] \quad \ni$ 

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For any quiver Q, Van den Bergh constructs a Poisson double bracket on the path algebra of the double of Q, which generalizes the necklace Lie algebra.

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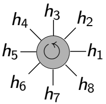
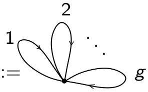
$\check{\mathrm{T}} = \mathrm{T} / [\mathrm{T}, \mathrm{T}] \quad \ni$ 

$Q :=$ 

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There is a Fox pairing $\kappa : \mathrm{T}(H) \times \mathrm{T}(H) \longrightarrow \mathrm{T}(H)$ defined by

$$\kappa(a, b) := \big(a - \varepsilon(a)1\big) \underset{\text{by } \omega}{\overset{\text{contraction}}{\leftrightsquigarrow}} \big(b - \varepsilon(b)1\big).$$

From Fox pairings to double brackets

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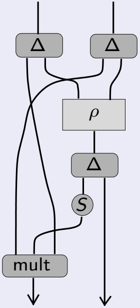

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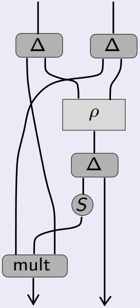

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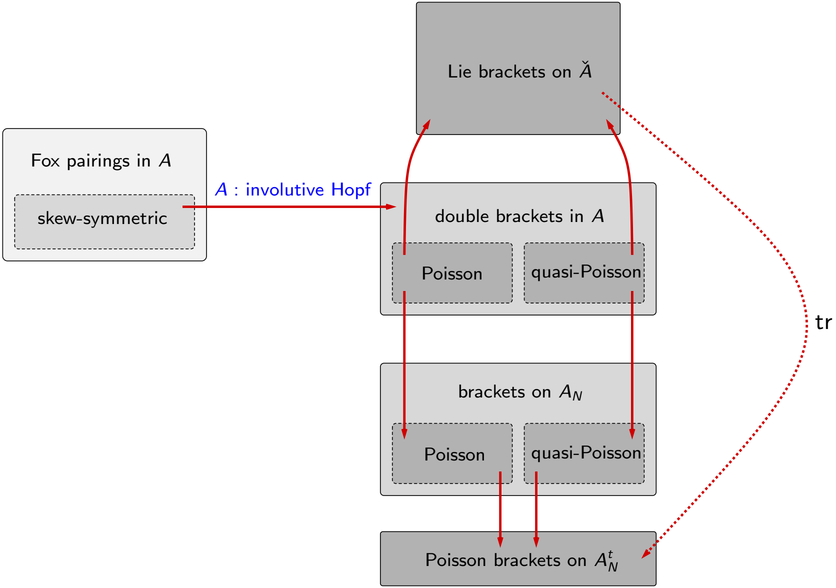
$\{\!\{-,-\}\!\}_{\mathrm{VdB}} : \mathrm{T}(H) \otimes \mathrm{T}(H) \to \mathrm{T}(H) \otimes \mathrm{T}(H)$ is induced by the Fox pairing κ.

Summary


Lie brackets on $\check{A}$
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skew-symmetric
A : involutive Hopf
double brackets in A
Poisson
quasi-Poisson
tr
brackets on A_N
Poisson
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Poisson brackets on A_N^t

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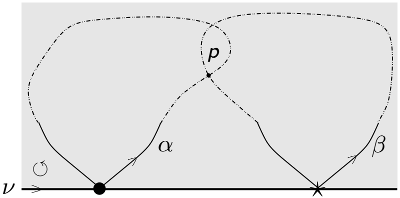

Definition (Papakyriakopoulos 1975; Turaev 1979)

The homotopy intersection form of Σ is the bilinear map $\eta : A \times A \longrightarrow A$ defined by

$$\eta(a,b) := \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha,\beta)\ [\overline{\nu}_{\star\bullet}\alpha_{\bullet p}\beta_{p\star}]$$

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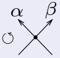

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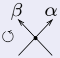

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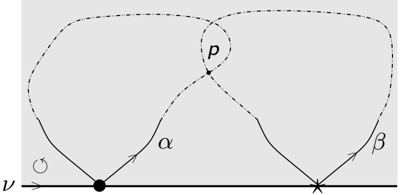

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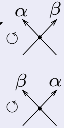

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η is a Fox pairing and $\eta^s := 2\eta + \rho_1$ is skew-symmetric.

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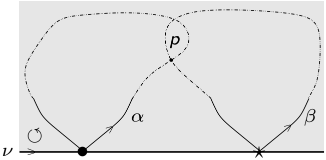

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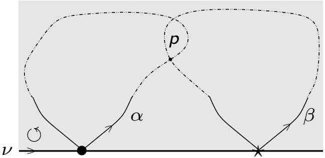

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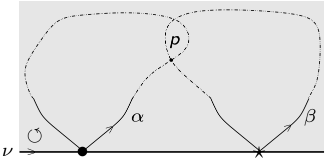

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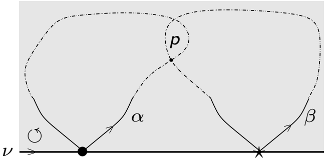

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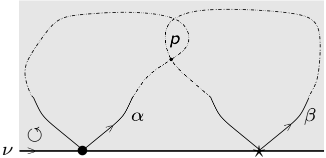

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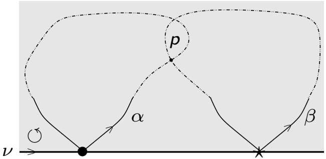

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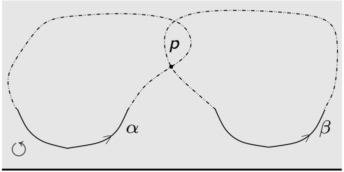

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"Push-out" $\{-,-\}$ along $\begin{cases} A_N \longrightarrow C^\infty(\mathcal{H}) \\ a_{ij} \longmapsto \big(r \mapsto (i,j)\text{-th entry of } r(a) \big). \end{cases}$ □

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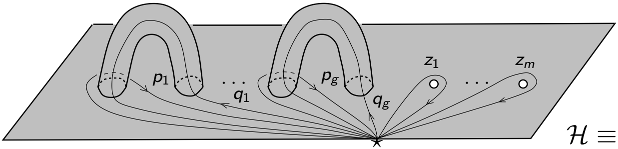

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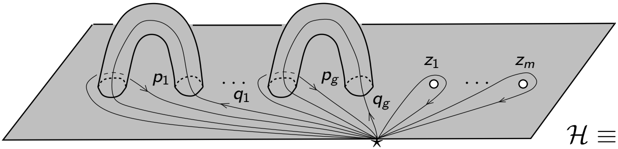

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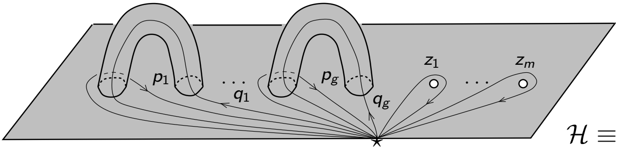

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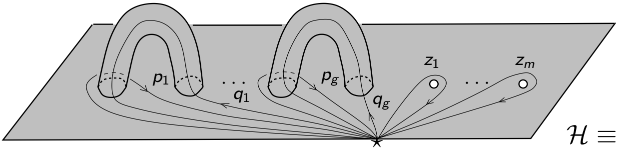

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(2) Li-Bland & Ševera (2012) and Nie (2013) have generalized this intrinsic description of $\{-,-\}_{\mathrm{AKsM}}$ to any Lie group G. $\rightsquigarrow$ Fock & Rosly (1999)

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Definition

An expansion $\theta : \pi \longrightarrow \hat{\mathrm{T}}$ is symplectic if $\theta(\nu) = e^{-\omega}$.

A tensorial description of $\{\!\{-,-\}\!\}$

Theorem (Kawazumi & Kuno 2010)

Any symplectic expansion θ of π induces (... after completions ...) an iso. from Goldman's Lie algebra $\mathbb{Q}\check{\pi}$ to Kontsevich's Lie algebra $\mathfrak{a}_g$.

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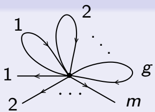
Remark

This tensorial description of $\{\!\{-,-\}\!\}$ can be generalized to any surface Σ of genus g with $m+1$ boundary comp.

$Q :=$ 

Tensorial description of $\{\!\{-,-\}\!\}$: sketch of proof

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Corollary

The graded commutative algebra representing the functor

$$B_* \longmapsto \mathrm{Hom}_{\mathcal{A}lg_*}\big(A_*, \mathrm{Mat}_N(B_*)\big)$$

has a canonical structure of Gerstenhaber algebra of degree $(2-n)$.