A representation theoretic approach to the Johnson cokernels II

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Plan

§1. A new class in the Johnson cokernels of MCG of surfaces.

§2. Recall from rep. theory of $Sp(2g, \mathbb{Q})$

Interlude: A combinatorial mult. formula for $\mathfrak{h}^{\mathbb{Q}}_{g,1}(k)$

§3. Generators of the space of maximal vectors in $\mathfrak{h}^{\mathbb{Q}}_{g,1}(k)$.

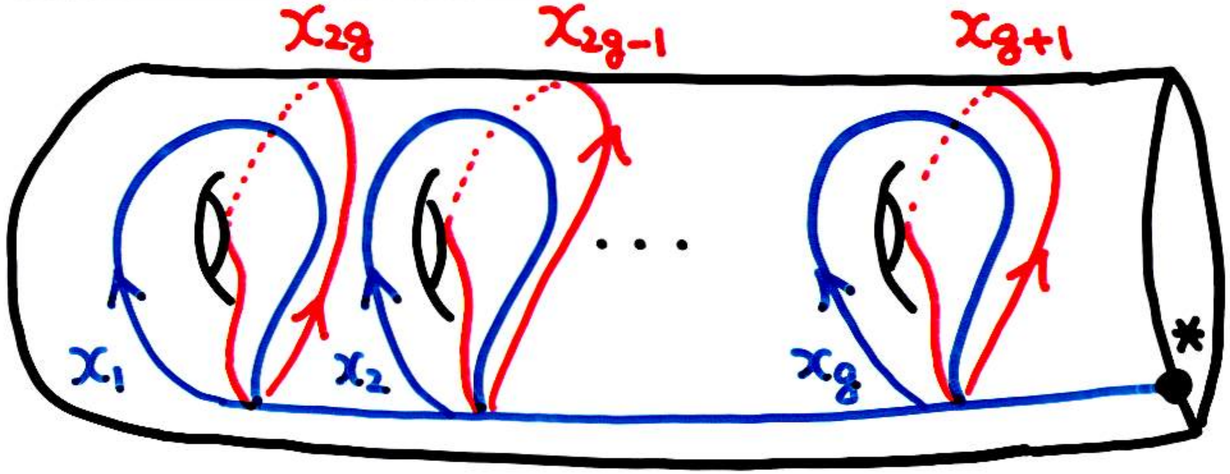
§4. Some results and conjectures.

§5. Comparison with Kawazumi-Kuno's results.

§1 A new class in the Johnson cokernels

Set up

$\Sigma_{g,1} :=$



$\mathcal{M}_{g,1} := \mathrm{Diff}^+(\Sigma_{g,1})/\text{isotopy}$

$\pi_1(\Sigma_{g,1}, *) \cong F_{2g} = \langle x_1, \cdots, x_{2g} \rangle$

$H := H_1(\Sigma_{g,1}, \mathbb{Z}) \cong F_{2g}^{ab} \quad e_i := [x_i]$

symplectic basis

$\{ e_1, \cdots, e_g, e_{g+1}, \cdots, e_{2g} \}$

$$\begin{tikzcd}
1 \arrow[r] & IA_{2g} \arrow[r] & \mathrm{Aut}\, F_{2g} \arrow[r] & GL(2g, \mathbb{Z}) \arrow[r] & 1 \\
1 \arrow[r] & \mathcal{I}_{g,1} \arrow[r] \arrow[u, hook] & \mathcal{M}_{g,1} \arrow[r] \arrow[u, hook, "\text{Dehn-Nielsen}"'] & Sp(2g, \mathbb{Z}) \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

From now, we assume $g \geq k+2$ (stable range) and over $\mathbb{Q}$.

$$\begin{tikzcd}
\operatorname{Im} \tau'_{k,\mathbb{Q}} \arrow[r, hook] & H^*_{\mathbb{Q}} \otimes \mathcal{L}^{\mathbb{Q}}_{2g}(k+1) \arrow[rr, "\overline{\Phi}^{\mathbb{Q}}_k"] \arrow[d, hook] & & C^{\mathbb{Q}}_{2g}(k) \\
& H^*_{\mathbb{Q}} \otimes H^{\otimes k+1}_{\mathbb{Q}} \arrow[r] & H^{\otimes k}_{\mathbb{Q}} \arrow[ur] &
\end{tikzcd}$$

Johnson Image for the lower central series of IA_{2g}

12-contraction

$$C^{\mathbb{Q}}_{2g}(k) := H^{\otimes k}_{\mathbb{Q}} \Big/ \left\langle \begin{array}{c} v_1 \otimes \cdots \otimes v_k \\ - v_2 \otimes \cdots \otimes v_k \otimes v_1 \\ v_i \in H_{\mathbb{Q}} \end{array} \right\rangle = H^{\otimes k}_{\mathbb{Q}} / \mathrm{Cyc}_k$$

<u>Theorem</u> (Satoh 2009)

$$\operatorname{Im} \tau'_{k,\mathbb{Q}} = \operatorname{Ker} \overline{\Phi}^{\mathbb{Q}}_k$$

Johnson image

$\Downarrow$

(cf. Kontsevich 1993, 1994)

$$\mathrm{Im}\,\tau^{M}_{k,\mathbb{Q}} \hookrightarrow \mathfrak{h}^{\mathbb{Q}}_{g,1}(k) \hookrightarrow H_{\mathbb{Q}} \otimes \mathcal{L}^{\mathbb{Q}}_{2g}(k+1) \twoheadrightarrow \mathcal{L}^{\mathbb{Q}}_{2g}(k+2)$$

$$x \otimes Y \longmapsto [x, Y]$$

S. Morita 1993

R. Hain 1997

$$\|$$

$$\mathrm{Im}\,\tau'^{M}_{k,\mathbb{Q}}$$

$\Uparrow$

Johnson image for the lower central series of $\mathrm{Torelli}_{g,1}$

<u>Theorem</u> (Morita 1993)

$$\mathrm{Tr}_k : \mathfrak{h}^{\mathbb{Q}}_{g,1}(k) \to S^k H_{\mathbb{Q}}$$

$\mathrm{Tr}_k \circ \tau^{M}_{k,\mathbb{Q}} \equiv 0$ for any odd $k \geqq 3$

<u>Theorem</u> (R. Hain 1997)

$\bigoplus_{k \geqq 1} \mathrm{Im}\,\tau^{M}_{k,\mathbb{Q}}$ is generated by $\mathrm{Im}\,\tau^{M}_{1,\mathbb{Q}} \left(= \Lambda^3 H_{\mathbb{Q}}\right)$ as a graded Lie algebra

$$
\begin{tikzcd}
\operatorname{Im}\tau'_{k,\mathbb{Q}} \arrow[rrr, hookrightarrow] & & & H^*_{\mathbb{Q}} \otimes \mathcal{L}^{\mathbb{Q}}_{2g}(k+1) \arrow[r, "\overline{\Phi}^{\mathbb{Q}}_k", "\text{Satoh}"'] \arrow[d, "\cong", "\text{as Sp-mod}"'] & C^{\mathbb{Q}}_{2g}(k) \\
\operatorname{Im}\tau'^{M}_{k,\mathbb{Q}} \arrow[u, leftrightarrow, "\text{Dehn-Nielsen}"'] \arrow[r, equal, "\text{Hain}"'] & \operatorname{Im}\tau^{M}_{k,\mathbb{Q}} \arrow[r, hookrightarrow, "\text{Morita}"'] & \mathfrak{h}^{\mathbb{Q}}_{g,1}(k) \arrow[rru, "c_k"'] \arrow[r, hookrightarrow] & H_{\mathbb{Q}} \otimes \mathcal{L}^{\mathbb{Q}}_{2g}(k+1) \arrow[r, twoheadrightarrow] & \mathcal{L}^{\mathbb{Q}}_{2g}(k+2)
\end{tikzcd}
$$

<u>An new class</u> (Satoh-E 2010)

$$\operatorname{Im}\tau^{M}_{k,\mathbb{Q}} \subset \operatorname{Ker} c_k \subset \mathfrak{h}^{\mathbb{Q}}_{g,1}(k).$$

Problem: Study Sp-mod structures of them.

§2. Recall from rep. theory of $Sp(2g,\mathbb{Q})$

$$GL(2g,\mathbb{Q}) \supset Sp(2g,\mathbb{Q}) := \{ g \in GL(2g,\mathbb{Q}) \mid {}^t g J g = J \}$$

$$\cup \qquad\qquad \cup \qquad J = \left(\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array} \right)$$

$$T_{2g} = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{2g} \end{pmatrix} \middle| \begin{array}{l} \forall i \\ t_i \neq 0 \end{array} \right\} \supset T_g^{Sp} := \left\{ \begin{pmatrix} t_1 & & & & & 0 \\ & \ddots & & & & \\ & & t_g & & & \\ & & & t_g^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & t_1^{-1} \end{pmatrix} \middle| \begin{array}{l} \forall i \\ t_i \neq 0 \end{array} \right\}$$

$$U_{2g} = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \supset U_g^{Sp} := U_{2g} \cap Sp(2g,\mathbb{Q})$$

unipotent subgroup

weight decomposition

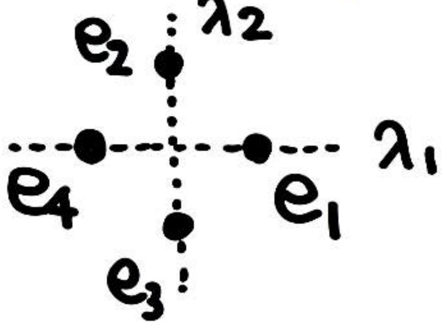
V: fin. dim. (rational) Sp-mod $\qquad V = \bigoplus_{(\lambda_1,\cdots,\lambda_g)\in\mathbb{Z}^g} V_{\lambda_1,\cdots,\lambda_g}$

$$V_{\lambda_1,\cdots,\lambda_g} := \left\{ v \in V \;\middle|\; \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_g & & & \\ & & & t_g^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \cdot v = t_1^{\lambda_1} \cdots t_g^{\lambda_g} v \right\}$$

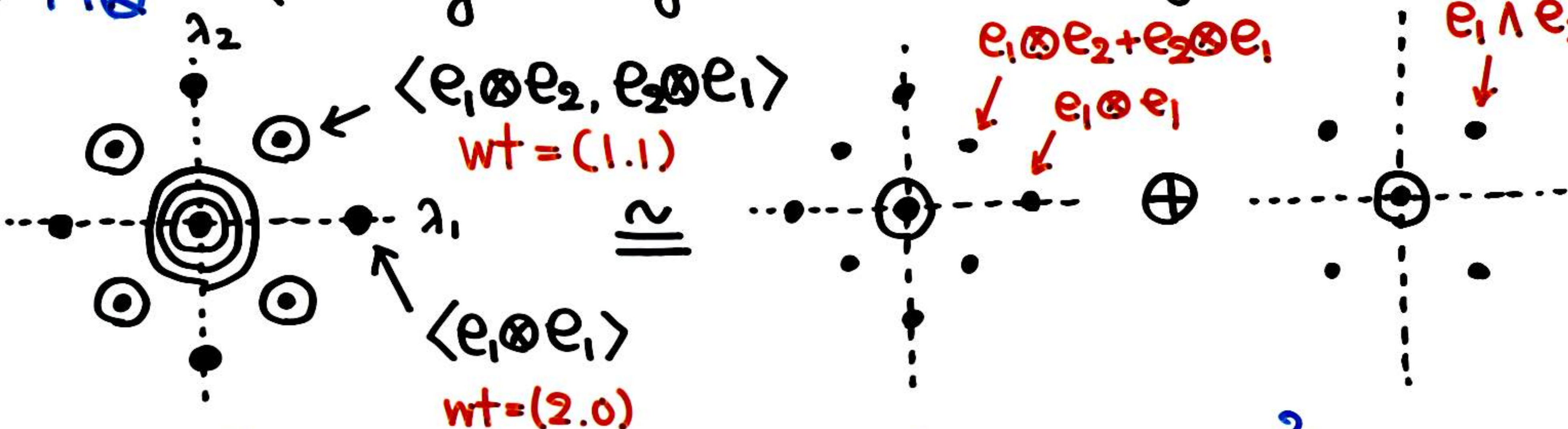
Example ($g=2$)

① natural rep. $Sp(2g,\mathbb{Q}) \curvearrowright \mathbb{Q}^4 = H_{\mathbb{Q}} = \langle e_1, e_2, e_3, e_4 \rangle$

weight $(1.0)\ (0.1)\ (0.-1).(-1.0)$



② $H_{\mathbb{Q}}^{\otimes 2} = \langle e_i \otimes e_j \mid 1 \leqq i.j \leqq 4 \rangle$ $\quad wt(e_i \otimes e_j) = wt(e_i) + wt(e_j)$



$$H_{\mathbb{Q}}^{\otimes 2} \cong S^2 H_{\mathbb{Q}} \oplus \Lambda^2 H_{\mathbb{Q}}$$

maximal vector

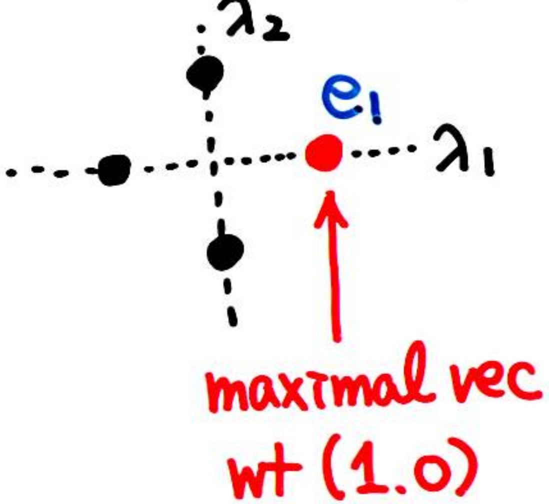
$$V^{U_g^{Sp}} := \{ v \in V \mid u \cdot v = v \text{ for } {}^\forall u \in U_g^{Sp} \}$$ space of maximal vectors of V

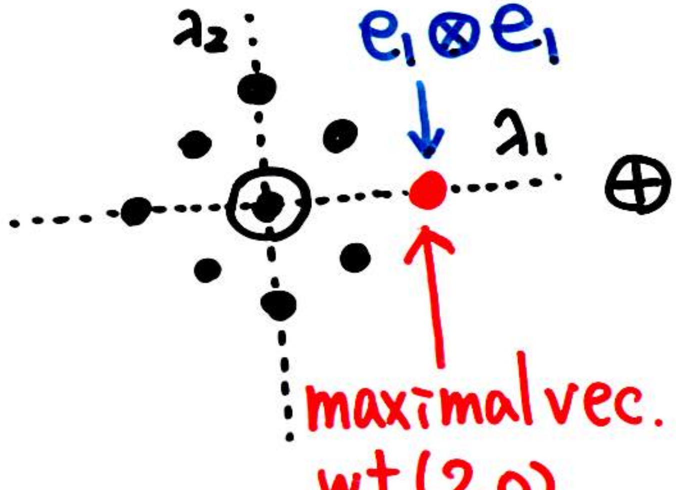
$$= \bigoplus_{\lambda_1, \cdots, \lambda_g} \left(V^{U_g^{Sp}} \right)_{\lambda_1, \cdots, \lambda_g}$$

$\rightsquigarrow (\lambda_1, \cdots, \lambda_g) \in \mathbb{Z}^g$ satisfies $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_g \geqq 0.$

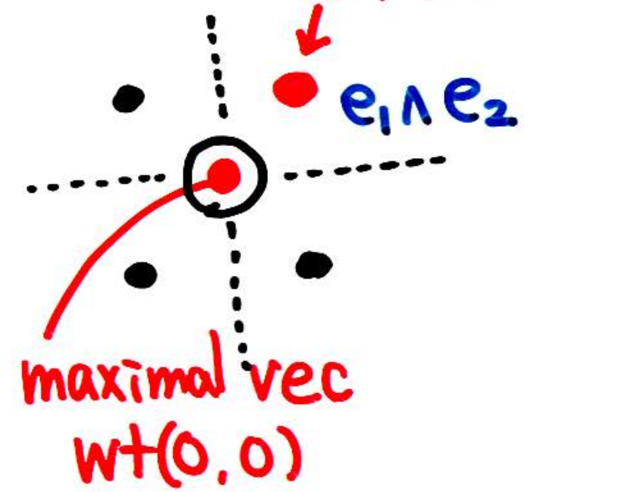
Example $(g=2)$

① natural rep. $H_{\mathbb{Q}}$ ② $H_{\mathbb{Q}}^{\otimes 2} \cong S^2 H_{\mathbb{Q}} \oplus \bigwedge^2 H_{\mathbb{Q}}$





$\oplus$



$\omega = e_1 \wedge e_4 + e_2 \wedge e_3$

Classification of Sp-irr. mod

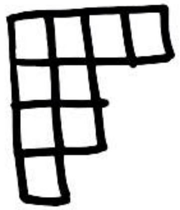
$\{$ isom. class of fin. dim. (rational) Sp-irr. mod $\} \ni L =: [\lambda]$

$\updownarrow$ 1:1

$P_{2g}^{+} = \{ (\lambda_1, \cdots, \lambda_g) \in \mathbb{Z}^g \mid \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_g \geqq 0 \} \ni {}^{\exists 1}\lambda$: "highest weight"

such that $L^{U_g^{Sp}} = (L^{U_g^{Sp}})_\lambda$ is 1-dim. and generates L as Sp-module.

$\updownarrow$

$\{$ Young diagram of length $\leqq g \}$ (4.2.2.1) $\leftrightarrow$ 

Theorem Any fin. dim. (rational) Sp-mod V are completely reducible.

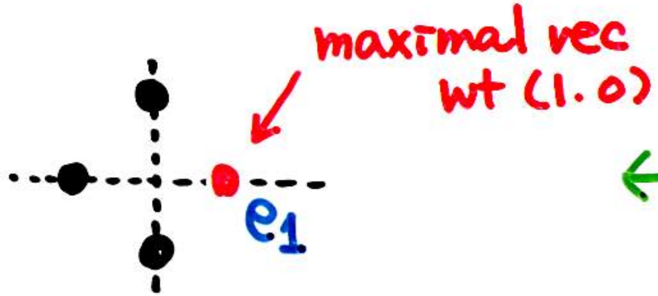
(= decompose a direct sum of Sp-irr.)

How to compute multiplicities

$$[V : [\lambda]] = \dim (V^{U_g^{Sp}})_\lambda .$$

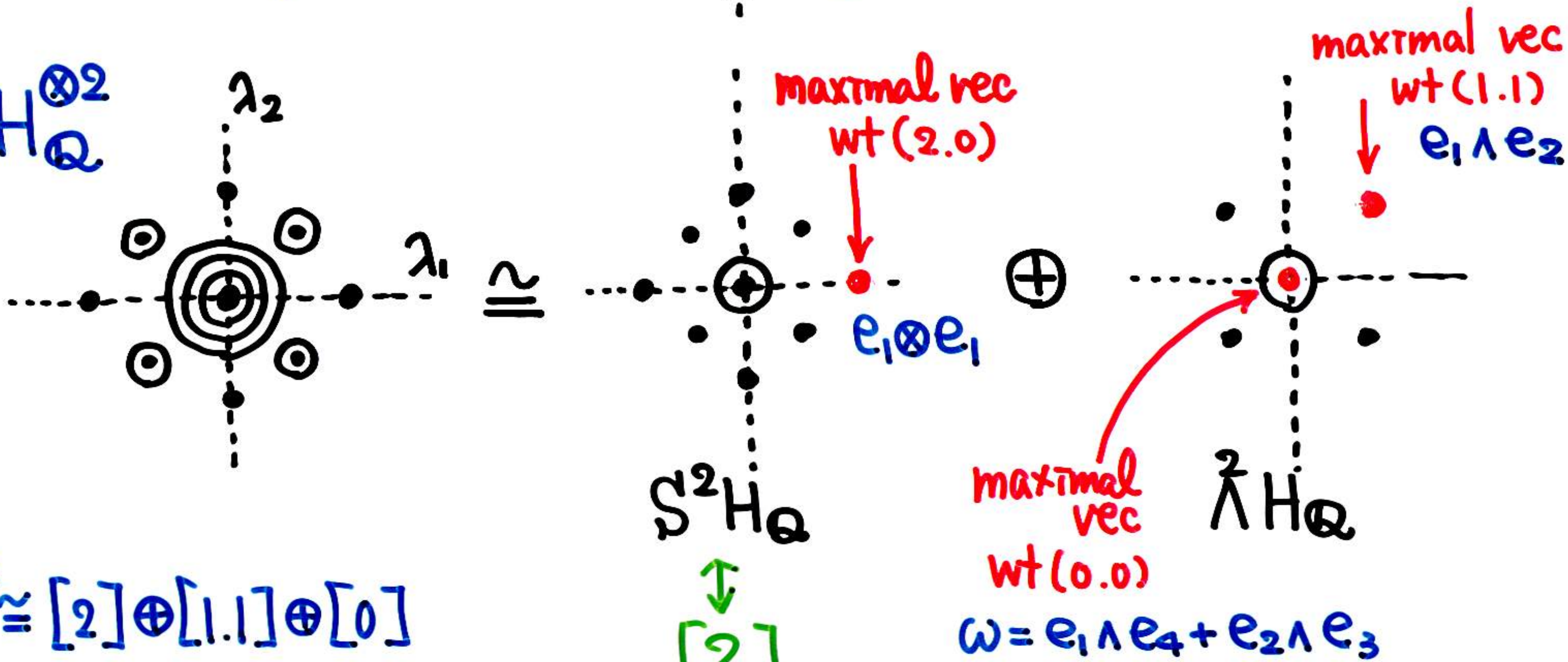
Example ($g=2$)

① natural rep $H_{\mathbb{Q}}$



$\longleftrightarrow$ [1]

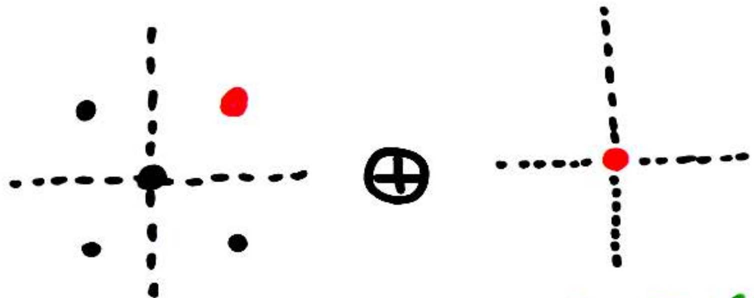
② $H_{\mathbb{Q}}^{\otimes 2}$



$$H_{\mathbb{Q}}^{\otimes 2} \cong [2] \oplus [1.1] \oplus [0]$$

$\updownarrow$

[2]



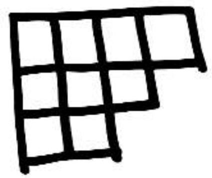
[1.1] [0] (triv. rep)

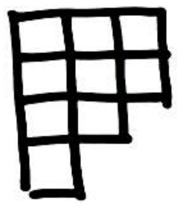
Interlude : A combinatorial multiplicity formula for $\mathfrak{h}^{\mathbb{Q}}_{g,1}(k)$

$$\mathfrak{h}^{\mathbb{Q}}_{g,1}(k) := \mathrm{Ker}\left(H_{\mathbb{Q}} \otimes \mathcal{L}^{\mathbb{Q}}_{2g}(k+1) \twoheadrightarrow \mathcal{L}^{\mathbb{Q}}_{2g}(k+2)\right)$$

- Zhuravlev's formula 1996.
  - A combinatorial multiplicity formula of irreducible GL-mod in $\mathcal{L}^{\mathbb{Q}}_{2g}(m)$
  - description of $[\mathcal{L}^{\mathbb{Q}}_{2g}(m) : (\lambda)] - [\mathcal{L}^{\mathbb{Q}}_{2g}(m) : \lambda^T]$
- Nakayama-Murnaghan's formula
- Pieri's rule
- Branching rules for GL to Sp

λ^T = transpose of λ

$\lambda =$  (4.3.2)

$\longleftrightarrow \lambda^T =$ 

(3.3.2.1)

Example

$[\lambda]$	mult. in $\mathfrak{h}^{\mathbb{Q}}_{g,1}(k)$	mult in $C^{\mathbb{Q}}_{2g}(k)$
$[k]$	$\delta_{k:\mathrm{odd}}$	$\delta_{k:\mathrm{odd}}$
$[1^k]$	$\delta_{k\equiv 1,2 \bmod 4}$	$\delta_{k:\mathrm{odd}}$
$[k-2]$	$\dfrac{k^2+2k}{8} - \delta_{k:\mathrm{odd}}\dfrac{6k-3}{8}$	$\dfrac{k-2+\delta_{k:\mathrm{odd}}}{2}$
$[1^{k-2}]$	$[\mathfrak{h}^{\mathbb{Q}}_{g,1}(k):[k-2]] - \dfrac{3k}{4}\delta_{k\equiv 0 \bmod 4} + \dfrac{3k-3}{4}\delta_{k\equiv 1 \bmod 4}$	$\dfrac{k-2+\delta_{k:\mathrm{odd}}}{2}$

But in general,

$$c_k : \mathfrak{h}^{\mathbb{Q}}_{g,1}(k) \longrightarrow C^{\mathbb{Q}}_{2g}(k)$$ is NOT surjective and NOT injective

(even if on λ-isotypic components)

$\rightsquigarrow$ We consider the image of the space of maximal vectors in $\mathfrak{h}^{\mathbb{Q}}_{g,1}(k)$ by c_k.

§3. Generators of the space of maximal vec. in $\mathfrak{h}^{\mathbb{Q}}_{g,1}(k)$

$$\mathfrak{h}^{\mathbb{Q}}_{g,1}(k) \subset H_{\mathbb{Q}} \otimes \mathcal{L}^{\mathbb{Q}}_{2g}(k+1) \subset H_{\mathbb{Q}}^{\otimes k+2}$$

$\lambda = (\lambda_1, \cdots, \lambda_g) \in P^+_{2g}$ $\quad \lambda_1 + \cdots + \lambda_g = k+2-2j \quad \left(0 \leq j \leq \left\lfloor \frac{k+2}{2} \right\rfloor\right)$

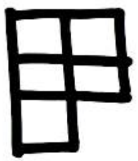
Brauer-Schur-Weyl duality

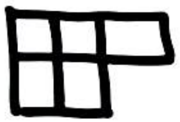
$$\left(H_{\mathbb{Q}}^{\otimes k+2}\right)^{U_g^{Sp}}_{\lambda} = v_\lambda \cdot \mathbb{Q}\mathfrak{S}_{k+2}$$

$$v_\lambda := \omega^{\otimes j} \otimes (e_1 \wedge \cdots \wedge e_{\lambda^T_1}) \otimes (e_1 \wedge \cdots \wedge e_{\lambda^T_2}) \otimes \cdots$$

Example $\quad \omega$: symplectic form, λ^T : transpose of λ

$\lambda = (2.2.1)$ $\quad \lambda^T = (3.2)$ $\quad k = 7.$





$$v_\lambda = \omega \otimes \omega \otimes (e_1 \wedge e_2 \wedge e_3) \otimes (e_1 \wedge e_2)$$

Dynkin-Specht-Wever's idempotent

$$\left(H_{\mathbb{Q}} \otimes \mathcal{L}^{\mathbb{Q}}_{2g}(k+1) \right)^{\mathrm{Sp}}_{\lambda} = v_\lambda \cdot \mathbb{Q}\mathfrak{S}_{k+2} \cdot \theta$$

$$\theta := (1-s_2)(1-s_3 s_2) \cdots (1-s_{k+1} \cdots s_3 s_2) \in \mathbb{Q}\mathfrak{S}_{k+2}$$

$$s_i = (i, i+1) \in \mathfrak{S}_{k+2}$$

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_{k+2}) \cdot \theta = v_1 \otimes [\cdots [[v_2, v_3], v_4] \cdots] \text{ in } H_{\mathbb{Q}}^{\otimes k+2}$$

Morita's characterization

For $v \in H_{\mathbb{Q}}^{\otimes k+2}$

$$v \in \mathfrak{h}^{\mathbb{Q}}_{g,1}(k) \iff v \cdot \theta = (k+1) v \text{ and } v \cdot \sigma_{k+2} = v$$

$$\sigma_{k+2} := (1\, 2 \cdots k+2) \in \mathfrak{S}_{k+2}$$

$$\zeta_{k+2} = 1 + \sigma_{k+2} + \sigma_{k+2}^2 + \cdots + \sigma_{k+2}^{k+1} \in \mathbb{Q}\mathfrak{S}_{k+2}.$$

<u>Theorem</u>

$$\left(\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)\right)^{U_g^{Sp}}_{\lambda} = v_\lambda \cdot \mathbb{Q}\mathfrak{S}_{k+2} \cdot \theta \cdot \zeta_{k+2}$$

Recall $c_k : \mathfrak{h}_{g,1}^{\mathbb{Q}}(k) \longrightarrow C_{2g}^{\mathbb{Q}}(k)$

$$\operatorname{Im} \tau_{k,\mathbb{Q}}^{M} \subset \operatorname{Ker} c_k \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$$

<u>Corollary</u>

$$\left[\mathfrak{h}_{g,1}^{\mathbb{Q}}(k) / \operatorname{Ker} c_k : [\lambda]\right] = \dim\, c_k\left(v_\lambda \cdot \mathbb{Q}\mathfrak{S}_{k+2} \cdot \theta \cdot \zeta_{k+2}\right)$$

§4. Some results and conjectures.

Theorem 1 (Revisit to Morita (-Nākamura) 1993)

$$\exists^1 [k] \subset \mathfrak{h}^{\mathbb{Q}}_{g,1}(k)/\mathrm{Ker}\, c_k \quad \text{for any odd } k \geqq 3.$$

$[k] \updownarrow S^k H_{\mathbb{Q}}$

"Morita obstruction"

Remark

- $[\mathfrak{h}^{\mathbb{Q}}_{g,1}(k) : [k]] = \delta_{k:\mathrm{odd}}$.
- $v_\lambda = \omega \otimes \underbrace{e_1 \otimes \cdots \otimes e_1}_{k}$ and $v_k \,\theta\, \zeta_{k+2} \neq 0$ for any odd k.
- $c_k(v_\lambda \,\theta\, \zeta_{k+2}) \neq 0$. for any odd $k \geqq 3$.

Theorem. 2 (Satoh - E 2010)

"anti-Morita obstruction"

$\exists^1 [1^k] \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)/\mathrm{Ker}\, c_k$ for any $k \equiv 1 \bmod 4$ and $k \geq 5$

$\updownarrow$

largest irr. component of $\Lambda^k H_{\mathbb{Q}}$

Remark

- $\left[\mathfrak{h}_{g,1}^{\mathbb{Q}}(k) : [1^k] \right] = \delta_{k \equiv 1,2 \bmod 4}$
- $v_\lambda = \omega \otimes (e_1 \wedge \cdots \wedge e_k)$ and $v_\lambda \cdot \theta\, S_{k+2} \neq 0$ for $k \equiv 1,2 \bmod 4$.
- $c_k(v_\lambda \cdot \theta\, S_{k+2}) \begin{cases} \neq 0 & \text{if } k \equiv 1 \bmod 4 \text{ and } k \geq 5 \\ = 0 & \text{if } k \equiv 2 \bmod 4 \end{cases}$

Remark H. Nakamura conjectured Thm. 2 in 1996.

Theorem. 3 (H. Enomoto – N. E.)

$$\exists\, [r+1, 1^{k-r-1}] \subset \mathfrak{h}^{\otimes}_{g,1}(k) / \operatorname{Ker} c_k$$

for $r \geqq 1$ and $k-r-1 \geqq 2$

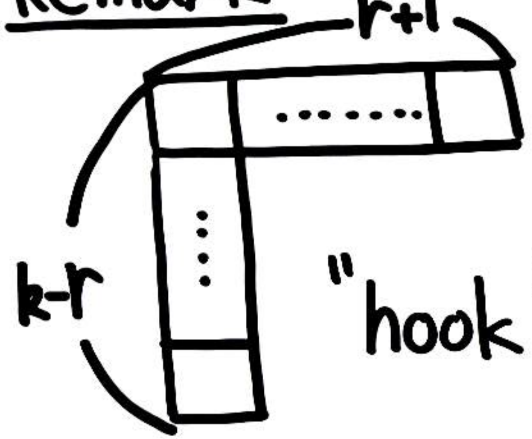
except for the following 3 cases :

(i) $r = 1$ and k is odd

(ii) $r = 1$ and k is even s.t. $k - r \equiv 1 \mod 4$

(iii) r is even and k is odd s.t. $k - r \equiv 3 \mod 4$

Remark



"hook"

- $v_\lambda = \omega \otimes (e_1 \wedge \cdots \wedge e_{k-r}) \otimes \overbrace{e_1 \otimes \cdots \otimes e_1}^{r}$
- $c_k(v_\lambda \,\theta\, \xi_{k+2}) \neq 0$

  except for (i), (ii), (iii) (and $(r, k) = (3, 8)$)

Example in $\mathfrak{h}^{\mathbb{Q}}_{g,1}(k)/\mathrm{Ker}\, c_k$

$\exists\, [2.1^{k-2}]$ if $k \equiv 0 \mod 4$, $k \geqq 4$

$\exists\, [3.1^{k-3}]$ if $k \not\equiv 1 \mod 4$, $k \geqq 5$

$\exists\, [4.1^{k-4}]$ if $k \geqq 6$

$\exists\, [5.1^{k-5}]$ if $k \not\equiv 3 \mod 4$, $k \geqq 7$ etc....

Remark For (i), (ii), (iii), we conjecture that

$\nexists\, [r+1, 1^{k-r-1}]$ in $\mathfrak{h}^{\mathbb{Q}}_{g,1}(k)/\mathrm{Ker}\, c_k$.

Conjecture 4 (H. Enomoto and N.E)

$$\left[\mathfrak{h}^{\mathbb{Q}}_{g,1}(k)/\operatorname{Ker} c_k : [k-2] \right] = \delta_{k:\text{even}} \left\lfloor \frac{k}{3} \right\rfloor$$

$$\left[\mathfrak{h}^{\mathbb{Q}}_{g,1}(k)/\operatorname{Ker} c_k : [1^{k-2}] \right]$$

$$= \delta_{\substack{k\equiv 0 \\ \text{mod} 4}} \left\lfloor \frac{k}{6} \right\rfloor + \delta_{\substack{k\equiv 1 \\ \text{mod} 4}} \left(\left\lfloor \frac{k}{3} \right\rfloor + 1 \right) + \delta_{\substack{k\equiv 2 \\ \text{mod} 4}} \left\lfloor \frac{k}{4} \right\rfloor$$

Summary and Discussions

$[1^4], [1^2], [0]$ for $k=6$

[k] for odd $k \geqq 3$
$[1^k]$ for $k \equiv 1 \bmod 4$ and $k \geqq 5$
$[2.1^{k-2}], [3.1^{k-3}], \ldots$ etc
for infinitely many k

$$\begin{array}{ccccc} & & \mathrm{Ker}(c_k) & & \\ & \subset & & \subset & \\ \mathrm{Im}\,\tau^M_{k,\mathbb{Q}} & & & & \mathfrak{h}^{\mathbb{Q}}_{g,1}(k) \\ & \subset & & \subset & \\ & & \mathrm{Ker}^{ab}_k & & \end{array}$$

conjectually
$[k-2], [1^{k-2}] \ldots$

[k] for odd $k \geqq 3$.

Conant-Kassabov-Voghtmann

Ker^{ab}_k $\parallel$ degree k-part of
$$\mathrm{Ker}(\mathfrak{h}^{\mathbb{Q}}_{g,1} \to (\mathfrak{h}^{\mathbb{Q}}_{g,1})^{ab}) = [\mathfrak{h}^{\mathbb{Q}}_{g,1}, \mathfrak{h}^{\mathbb{Q}}_{g,1}](k)$$

- low degree case ($k \leqq 6$) Morita-Sakasai-Suzuki
- Galois obstructions
- Recently, Kawazumi-Kuno introduced another new class in the Johnson cokernels.

§5 Comparison with Kawazumi-Kuno's obstructions

the leading term of Kawazumi-Kuno's obstruction

$$\delta_k^{alg} : H_{\mathbb{Q}}^{\otimes k+2} \longrightarrow \bigoplus_{p+q=k} C_{2g}^{\mathbb{Q}}(p) \otimes C_{2g}^{\mathbb{Q}}(q)$$

<u>Theorem</u> (Kawazumi-Kuno)

$$\delta_k^{alg} \circ \tau_{k,\mathbb{Q}}^M = 0$$

$$\rightsquigarrow \operatorname{Im} \tau_{k,\mathbb{Q}}^M \subset \operatorname{Ker} \delta_k^{alg} \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$$

↑ Morita obstruction [k] for any odd $k \geq 3$.

$$\begin{tikzcd}
H^*_{\mathbb{Q}} \otimes \mathcal{L}^{\otimes}_{2g}(k+1) \arrow[r, hook] \arrow[d, "\wr"] & H^*_{\mathbb{Q}} \otimes H_{\mathbb{Q}}^{\otimes k+1} \arrow[r, "\overline{\Phi}_k^j"'] \arrow[dr, dashed, "\overline{\Phi}_k^j"'] & H_{\mathbb{Q}}^{\otimes j-1} \otimes H_{\mathbb{Q}}^{\otimes k-j+1} \arrow[d] \\
H_{\mathbb{Q}} \otimes \mathcal{L}^{\otimes}_{2g}(k+1) \arrow[rr, "c_{1.j}"'] & & C^{\otimes}_{2g}(j-1) \otimes C^{\otimes}_{2g}(k-j+1)
\end{tikzcd}$$

$c_{1.j}$ $(1 \leqq j \leqq k+1)$

$c_{1.1} = c_k$.

Lemma

$$\operatorname{Ker} \delta_k^{alg} = \bigcap_{j \geqq 2} \operatorname{Ker} c_{1.j} \quad \text{in } \mathfrak{h}^{\otimes}_{g.1}(k)$$

Proposition

$$\operatorname{Ker} c_k \subset \operatorname{Ker}(c_{1j}) \text{ for any } j \geqq 2. \text{ in } \mathfrak{h}^{\otimes}_{g.1}(k).$$

Theorem. 5 (Kuno - Satoh - E.)

$$\mathrm{Im}\,\tau_{k,\mathbb{Q}}^{M} \subset \mathrm{Ker}\, c_k \subset \mathrm{Ker}\, \delta_k^{alg} \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$$

Theorem. 6 (E.)

$$[1^k] \subsetneq \mathrm{Ker}\,\delta_k^{alg} / \mathrm{Ker}\, c_k$$

$(k \equiv 1 \bmod 4,\ k \geq 5)$

(!!! $\nexists\, [1^k]$ in $C_{2g}^{\mathbb{Q}}(p) \otimes C_{2g}^{\mathbb{Q}}(k-p)$ for any $p \geq 1$.)

Remark δ_k^{alg} is only the <u>leading term</u> of KK's obs.

They conjectured that the <u>total</u> of KK's obs. captures the Johnson cokernels.

Thank you very much
for your attention!