

Poisson Geometry Program at the Simons Center for Geometry and Physics

June 19, 2018, 10am - 12am

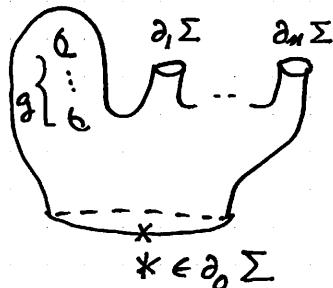
"Formal description of the Goldman bracket, revisited"

Nariya Kawazumi (U. Tokyo)

joint work with A. Alekseev (U. Geneva), Y. Kuno (Tsuda U.) and F. Naef (MIT).

$$g, n \geq 0$$

$$\Sigma = \Sigma_{g, n+1} :=$$



$$\partial \Sigma = \bigcup_{j=0}^m \partial_j \Sigma \neq \emptyset$$

$$\pi := \pi_1(\Sigma, *) \cong F_{2g+n}. \text{ free group of rank } 2g+n$$

group-like expansions: "coordinates" of the free group π

\mathbb{K} : field of char. 0

$\mathbb{K}\pi$: the group ring of π

$\Delta : \mathbb{K}\pi \rightarrow \mathbb{K}\pi \otimes \mathbb{K}\pi$ coproduct, algebra homom.

$\forall x \in \pi \quad \Delta x = x \otimes x \quad \rightarrow \mathbb{K}\pi : \text{Hopf algebra}$

$I\pi := \text{Ker } \text{aug} : \mathbb{K}\pi \rightarrow \mathbb{K}, \sum_{x \in \pi} a_x x \mapsto \sum_{x \in \pi} a_x$ augmentation ideal

$\widehat{\mathbb{K}\pi} := \varprojlim_{P \rightarrow \infty} \mathbb{K}\pi / (I\pi)^P$

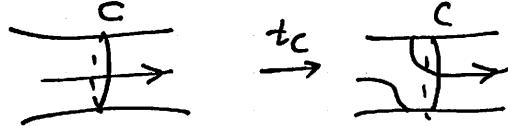
complete group ring

\longrightarrow complete Hopf algebra

Personal Remarks: Why I began to consider the formality of the Goldman bracket.

$$\pi = \pi(\Sigma, *), * \in \partial \Sigma$$

Dehn twist $C \subset \Sigma$ Simple closed curve. $t_C \in \text{MCG}(\Sigma, \partial \Sigma)$



$$\begin{array}{c} \downarrow \\ \text{Aut}(\pi) \\ \downarrow \\ \text{Aut}(\widehat{\mathbb{K}\pi}) \end{array}$$

Y. Kuno (Nov. 2009): explicit description of the action of t_C on $\pi/\langle [\pi, [\pi, \pi]] \rangle$
using a symplectic expansion, for $\Sigma_{g,1}$ (the extended 1st Johnson homom)

(Rem) At that time, the notion of a symplectic expansion was already introduced
by G. Massuyeau

(cf). Picard-Lefschetz formula $\forall X \in H_1(\Sigma)$

$$t_{C*}(X) = X - (X \cdot [C])[C] \in H_1(\Sigma), \quad [C]^{\otimes 2} \in H_1(\Sigma)^{\otimes 2}$$

t_C seemed to be described by $\frac{1}{2} (\log |C|)^2$

{ some explicit computations

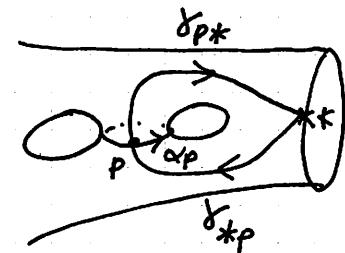
$\log t_C$ seemed to equal $\frac{1}{2} (\log |C|)^2$

This can be interpreted as an element of $|\widehat{\mathbb{K}\pi}|$: the completed Goldman Lie algebra

$\sigma: (\widehat{K\pi}) \rightarrow \text{Der}(\widehat{K\pi})$: Lie algebra homom.

$\alpha \in \widehat{\pi} = \pi/\text{conj.}, \gamma \in \pi \quad (\text{in general position})$

$$\sigma(\alpha)(\gamma) \stackrel{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \epsilon_p(\alpha, \gamma) \gamma_{*p} \alpha_p \gamma_{p*} \in \mathbb{Z}\pi$$



Theorem (Σg.11 Kuno-K., general K-K, Massuyeau-Turaev)

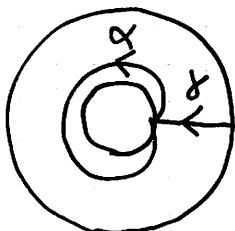
$$t_c = \exp(\sigma(\frac{1}{2}(\log c)^2)) \in \text{Aut}(\widehat{K\pi})$$

or equivalently

$$\log(t_c) = \sigma(\frac{1}{2}(\log c)^2) \in \text{Der}(\widehat{K\pi})$$

1st proof \Leftarrow formal description of the Goldman bracket and the homom. σ

2nd proof without using the formal description (K-K.)



$$t_{|\alpha|} \gamma = \gamma \alpha$$

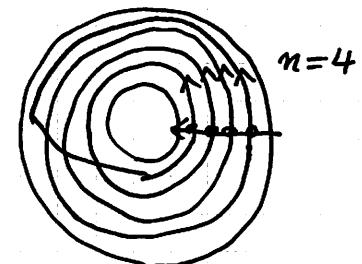
$$(\log t_{|\alpha|})(\gamma) = \gamma \log \alpha$$

$$\forall n \in \mathbb{Z} \quad \sigma(|\alpha^n|)(\gamma) = n \gamma \alpha^n$$

$$\forall f \in K[[z^{-1}]] \quad \sigma(|f(\alpha)|)(\gamma) = \gamma \alpha f'(\alpha)$$

Solve the equation: $\log z = z f'(z)$

$$f(z) = \int_1^z \frac{1}{z} \log z \, dz = \frac{1}{2} (\log z)^2 //$$



"Skein" " Skein-ization of the Dehn twist formula

HOMFLY-PT skein algebra ($\Sigma \times [0,1]$)

$|K\pi|$ Turaev

Goldman Lie alg

Kauffman skein algebra ($\Sigma \times [0,1]$)

Theorem (S. Tsujii)

(1) an explicit formula for

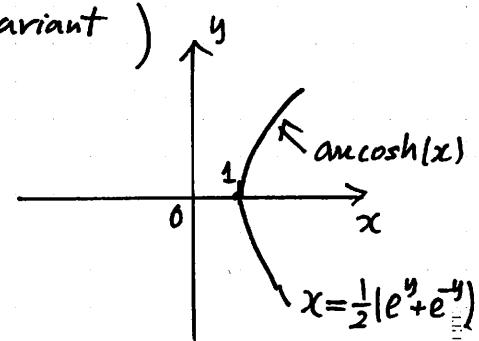
$\log |t_C|$ on HOMFLY-PT : very complicated \rightsquigarrow recovers the $\frac{1}{2}(\log C)^2$ formula
 in particular

$$\log |t_C| \text{ on Kauffman} = \frac{-A + A^{-1}}{4 \log(-A)} (\operatorname{arccosh}(-\frac{C}{2}))^2$$

(2) Heegaard splittings of 3-mfd's

new construction of series of invariants for \mathbb{Z} -homology S^3 's

(The 1st terms of both series are the Casson invariant)



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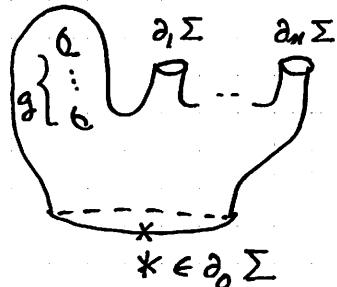
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$\forall x \in \pi \quad \Delta x = x \otimes x \quad \rightarrow \mathbb{K}\pi : \text{Hopf algebra}$

$I\pi := \text{Ker } \text{aug} : \mathbb{K}\pi \rightarrow \mathbb{K}, \sum_{x \in \pi} a_x x \mapsto \sum_{x \in \pi} a_x$ augmentation ideal

$\widehat{\mathbb{K}\pi} := \varprojlim_{P \rightarrow \infty} \mathbb{K}\pi / (I\pi)^P$ complete group ring
 \longrightarrow complete Hopf algebra

$\widehat{K\pi} \cong (\text{the completed tensor algebra over } H := H, [\pi : K] = H, [\Sigma : K])$ Hopf algebra isomorphism
 a group-like expansion (\exists many choices)

$$H := H, [\pi : K] = H, [\Sigma : K] = [\pi / [\pi, \pi]] \otimes_{\mathbb{Z}} K$$

$$\gamma \in \pi \mapsto [\gamma] := [\gamma \bmod [\pi, \pi]] \otimes 1 \in H$$

$$\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m} \quad \text{completed tensor algebra over } H$$

$$T = T(H) := \bigoplus_{m=0}^{\infty} H^{\otimes m} \quad \text{tensor algebra over } H \quad \text{complete tensor product}$$

$$\Delta : \widehat{T}(H) \rightarrow \widehat{T}(H) \otimes \widehat{T}(H) \quad \text{coproduct, algebra homom}$$

$$x \in H \mapsto \Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\widehat{L} = \widehat{L}(H) := \{a \in \widehat{T}(H), \Delta a = a \otimes 1 + 1 \otimes a\} \quad \text{Lie-like elements}$$

completed free Lie algebra over H

$$\widehat{L} = \prod_{m=1}^{\infty} (\widehat{L} \cap H^{\otimes m}) \quad \text{homogeneous}$$

$$L = L(H) := \bigoplus_{m=1}^{\infty} (L \cap H^{\otimes m}) \quad \text{free Lie algebra over } H$$

$$T(H) = U(L(H)) \quad \text{universal enveloping algebra of } L(H)$$

(\because universal mapping properties for T, L and U)

$\exp(\hat{L}) = \{a \in \hat{T}(H) \setminus \{0\}; \Delta a = a \otimes a\}$ group-like elements
 < mult. group ($\hat{T}(H)$) subgroup

$$\exp(\hat{L}) \xrightleftharpoons[\text{exp}]{\log} \hat{L}$$

Definition $\theta: \pi \rightarrow \exp(\hat{L})$ group-like expansion.

- $\overset{\text{def}}{\iff}$
- 1) θ : group homomorphism
 - 2) $\forall \gamma \in \pi \quad \log \theta(\gamma) \equiv [\gamma] \pmod{\prod_{m \geq 2} H^{\otimes m}}$

$\Rightarrow \hat{K}\pi \xrightleftharpoons[\cong]{\theta} \hat{T}(H)$ isom. of complete Hopf algebras

$$\sum a_\gamma \gamma \mapsto \sum a_\gamma \theta(\gamma)$$

$\hat{K}\pi$: filtration coming from $(I\pi)^P$, $p \geq 0$

$\hat{T}(H)$: filtration given by $\prod_{m \geq p} H^{\otimes m}$

θ : filtration preserving

$$\text{gr } \theta: \text{gr}(\hat{K}\pi) = \prod_{m=0}^{\infty} (I\pi)^m / (I\pi)^{m+1} \xrightarrow{\cong} \text{gr}(\hat{T}(H)) = \hat{T}(H)$$

canonical isom., independent of the choice of θ

example $\pi = \langle \delta_1, \delta_2, \dots, \delta_m \rangle$, $m = 2g+n$, free generators

$\theta_{\text{exp}} : \pi \rightarrow \exp(\widehat{L})$, $\theta_{\text{exp}}(\delta_j) := \exp[\delta_j]$, $1 \leq j \leq m$, exponential expansion

$\forall \theta$: group-like expansion $\exists! F_i \in \text{Aut}_{\text{Hopf}}^+(\widehat{T}(H)) = \text{Aut}^+(\widehat{L}(H))$ w.r.t. $\{\delta_1, \dots, \delta_m\}$

s.t. $\theta = F^{-1} \circ \theta_{\text{exp}} : \pi \xrightarrow{\theta_{\text{exp}}} \exp(\widehat{L}) \xrightarrow{F^{-1}} \exp(\widehat{L})$ ("") $\widehat{K}\pi \xrightarrow[\theta_{\text{exp}}]{\cong} \widehat{T}(H)$

Lie algebra abelianization of an associative algebra

A : (topological) associative \mathbb{K} -algebra

$[a, b] := ab - ba$ ($a, b \in A$) $\longrightarrow A$: Lie algebra

$[A, A] := \overline{(\mathbb{K}\text{-linear subspace of } A \text{ spanned by } \{[a, b]; a, b \in A\})}$ closure

$|A| := A/[A, A]$ Lie algebra abelianization of A

$|\cdot| : A \rightarrow |A|$, quotient map.

$|\widehat{T}(H)| = \prod_{m=0}^{\infty} (H^{\otimes m})_{\mathbb{Z}/m}$ cyclic coinvariants ($\not\cong \widehat{\text{Sym}}(H)$)

$|\widehat{K}\pi| = \mathbb{K}(\pi/\text{conj})$

$|\sum a_\gamma \gamma| \mapsto |\sum a_\gamma| \gamma$, where $|\gamma| \in \pi/\text{conj} = [S^1, \Sigma]$ (free loops) : conj. class of γ

$\theta : |\widehat{K}\pi| \xrightarrow{\cong} |\widehat{T}(H)|$ isom. of \mathbb{K} -vector spaces

$\text{gr } \theta : \text{gr} |\widehat{K}\pi| \xrightarrow{\cong} \text{gr} |\widehat{T}(H)| = |\widehat{T}(H)|$, canonical, indep. of the choice of θ

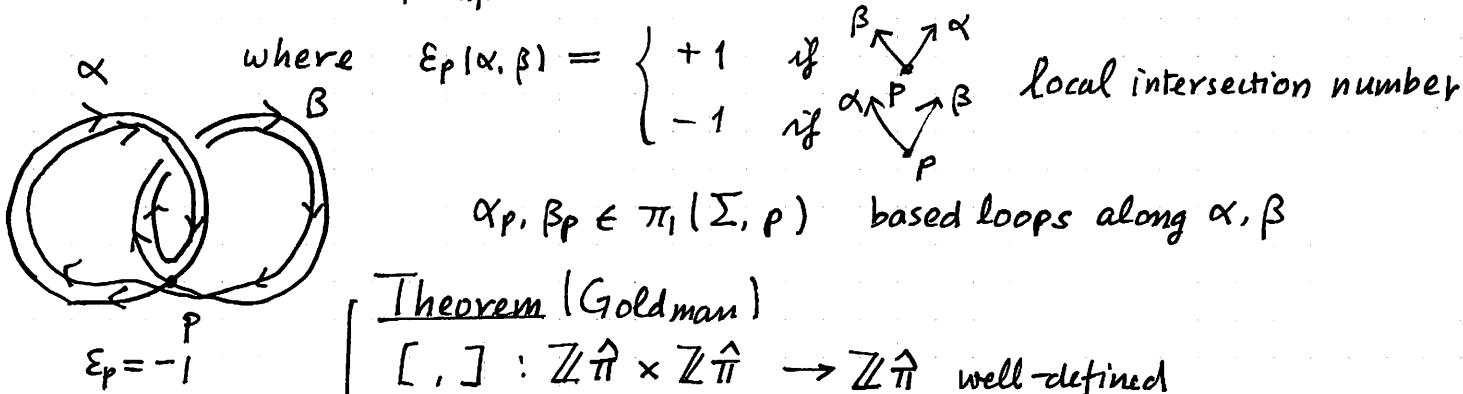
Goldman bracket

$\widehat{\pi} := [S^1, \Sigma] = \pi/\text{conj}$, the free homotopy set of free loops on Σ

$K\widehat{\pi} = |K\pi|$, Lie algebra abelianization

$\alpha, \beta \in \widehat{\pi}$ represented by generic immersions $\rightsquigarrow \alpha \cap \beta$: finite and transverse

$$[\alpha, \beta] \stackrel{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \epsilon_p(\alpha, \beta) [\alpha_p \beta_p] \in |K\pi| = K\widehat{\pi} \quad \text{Goldman bracket}$$



Theorem (Goldman)

$$[,] : \mathbb{Z}\widehat{\pi} \times \mathbb{Z}\widehat{\pi} \rightarrow \mathbb{Z}\widehat{\pi} \text{ well-defined}$$

$(\mathbb{Z}\widehat{\pi}, [,]) \text{ Lie algebra}$

$$|(I\pi)^p|, |(I\pi)^q| \subset |(I\pi)^{p+q-2}| \quad (\forall p, q \geq 1)$$

$\Rightarrow [,]$ descends to

$$[,] : |K\widehat{\pi}| \times |K\widehat{\pi}| \rightarrow |K\widehat{\pi}|$$

$$\text{gr}[,] : \text{gr}|K\widehat{\pi}| \times \text{gr}|K\widehat{\pi}| \rightarrow \text{gr}|K\widehat{\pi}|$$

Recall $\text{gr}|K\widehat{\pi}| = \prod_{m=0}^{\infty} |(I\pi)^m| / |(I\pi)^{m+1}| = \prod_{m=0}^{\infty} (H^{\otimes m})_{\mathbb{Z}/m} = |\widehat{T}(H)|$

Formality problem: $(|\widehat{K}\pi|, [,]) \stackrel{?}{=} (\text{gr}|\widehat{K}\pi|, [,]) \text{ as Lie algebras}$

\uparrow
 $\exists? \theta: \pi \rightarrow \exp(\widehat{L})$ group-like expansion
 s.t. $\theta: |\widehat{K}\pi| \xrightarrow{\cong} |\widehat{T}(H)| (= \text{gr}|\widehat{K}\pi|)$ Lie algebra isom

Theorem 1 (Kuno-K., Massuyeau-Turaev)

$\forall n \geq 0, \exists \theta: \text{group-like expansion}$ special / symplectic expansion
 s.t. $\theta: |\widehat{K}\pi| \xrightarrow{\cong} |\widehat{T}(H)|$ Lie algebra isom.

Poisson geometry proof
by Naef.

Main result in this talk is

Theorem 2

$\theta: \text{group-like expansion}$ (If $g \neq 0$ and $n \neq 0$, we need some modification
of the definition of group-like expansions)

$\theta: |\widehat{K}\pi| \xrightarrow{\cong} |\widehat{T}(H)|$ Lie algebra isom

$\Rightarrow \theta$ is conjugate to a special / symplectic expansion

2 proofs

by { - non-comm. geom
- elem. algebra

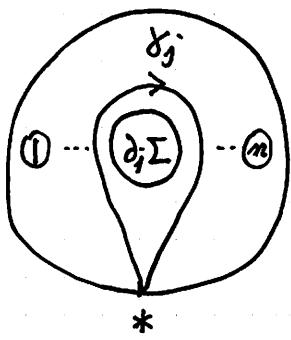
this talk

For simplicity, we confine ourselves to the two extreme cases

$$(g=0) \quad \Sigma = \Sigma_{0,n+1} \quad \text{special expansions}$$

$$(n=0) \quad \Sigma = \Sigma_{g,1} \quad \text{symplectic expansions}$$

$(g=0) \quad \Sigma = \Sigma_{0,n+1} \quad (\neq \Sigma_{0,2})$ (For $\Sigma_{0,2}$, the theorem is trivial
In fact, a group-like expansion is unique for $\Sigma_{0,2}$)



$$1 \leq j \leq n$$

$\pi = \langle \delta_1, \delta_2, \dots, \delta_n \rangle$ free group of rank n

$\delta_0 := \delta_1 \delta_2 \cdots \delta_n \in \pi$ negative boundary loop along $\partial_0 \Sigma$

$$z_j = [\delta_j] \in H = H_1(\Sigma; K), \quad 0 \leq j \leq n$$

$$z_0 = \sum_{j=1}^n z_j$$

Definition

$\theta: \pi \rightarrow \exp(\hat{L})$ special expansion

def

1) θ : group-like expansion

tangential \rightarrow 2) $1 \leq j \leq n, \exists g_j \in \exp(\hat{L}), \theta(\delta_j) = g_j \exp(z_j) g_j^{-1}$

special \rightarrow 3) $\theta(\delta_0) = \exp\left(\sum_{j=1}^n z_j\right)$

(Rmk) - θ_{\exp} w.r.t. $\{\delta_1, \delta_2, \dots, \delta_n\}$ ($\theta_{\exp}(\delta_j) = e^{z_j} (1 \leq j \leq n)$) is not special:

$$\log \theta_{\exp}(\delta_0) = \log(e^{z_1} e^{z_2} \cdots e^{z_n}) \neq \sum_{j=1}^n z_j = z_0$$

- $\theta = F^{-1} \circ \theta_{\exp}$: special expansion $F \in \text{Aut}^+(\hat{L})$

\iff 1) (tangential) $1 \leq j \leq n, \exists g_j \in \exp(\hat{L}), F(z_j) = g_j^{-1} z_j g_j$
 2) (KVI) $F(z_1 + z_2 + \cdots + z_n) = \log(e^{z_1} e^{z_2} \cdots e^{z_n})$

The Goldman bracket on $\text{gr}[\widehat{\mathbb{K}\pi}]$ for $\Sigma_{0,n+1}$

$$u = u_1 u_2 \dots u_\ell, v = v_1 v_2 \dots v_m \in \widehat{T}(H), \quad u_j, v_k \in \{z_1, \dots, z_m\}$$

$$\{u, v\}_{\text{gr}} := \sum_{j,k} \delta_{u_j, v_k} (v_1 \dots \overset{\leftarrow}{v_{k-1}} u_{j+1} \dots u_\ell \overset{\leftarrow}{u_1} \dots u_{j-1} \overset{\leftarrow}{u_j} v_{k+1} \dots v_m - v_1 \dots \overset{\leftarrow}{v_{k-1}} u_j \overset{\leftarrow}{u_{j+1}} \dots u_\ell u_1 \dots u_{j-1} v_{k+1} \dots v_m) \in \widehat{T}(H)$$

$$\{|u|, |v|\}_{\text{gr}} := |\{u, v\}_{\text{gr}}| \in |\widehat{T}(H)|$$

- $(|\widehat{T}(H)|, \{ \cdot, \cdot \}_{\text{gr}}) = (\text{gr}[\widehat{\mathbb{K}\pi}], \text{gr}[\text{Goldman bracket}])$

- $|\widehat{T}(H)| \rightarrow \text{Der}(\widehat{T}(H)), |u| \mapsto \{u, -\}_{\text{gr}}$, Lie algebra homom.

Theorem (K.-K., M.-T.)

$\theta : \pi \rightarrow \exp(\widehat{L})$ (conjugate of) a special expansion

$$\Rightarrow \theta : (|\widehat{\mathbb{K}\pi}|, \text{Goldman bracket}) \xrightarrow{\cong} (|\widehat{T}(H)|, \{ \cdot, \cdot \}_{\text{gr}})$$

Theorem 2 for $\Sigma_{0,n+1}$,

$\theta : \pi \rightarrow \exp(\widehat{L})$ group-like expansion

$$\theta : (|\widehat{\mathbb{K}\pi}|, \text{Goldman bracket}) \xrightarrow{\cong} (|\widehat{T}(H)|, \{ \cdot, \cdot \}_{\text{gr}}) \text{ Lie algebra isom}$$

$\Rightarrow \theta$ is conjugate to a special expansion.

i.e., $0 \leq b_j \leq n \quad \exists g_j \in \exp(\widehat{L}) \text{ s.t. } \log \theta(x_j) = g_j z_j g_j^{-1}$

combining our previous result (AKKN)

Corollary 3 $\theta = F \circ \theta_{\text{exp}} : \pi \rightarrow \exp(\hat{\mathbb{L}})$ group-like expansion

Then,

F is conjugate to a solution to the Kashiwara-Vergne problem
 $\Leftrightarrow \theta : (\widehat{K\pi}, \text{Goldman bracket}, \text{Turaev cobracket } \delta^f \text{ w.r.t. framing of } T\Sigma) \xrightarrow{\cong} (\widehat{|T(H)|}, \{-,-\}_{\text{gr}}, \delta_{\text{gr}}^f)$ Lie bialgebra isomorphism

(Rmk) $l \in \hat{\mathbb{L}}$, $l = l_{(1)} + \text{higher degree terms}$, $l_{(1)} \in H$
 $\Rightarrow j(\text{Ad}_{\exp(l)}) = [l_{(1)}] \in Z(\widehat{|T(H)|}, \{-,-\}_{\text{gr}})$

$$(m=0) \quad \Sigma = \Sigma_{g,1}$$

$H = H_1(\Sigma_{g,1}; \mathbb{K})$ symplectic vector space

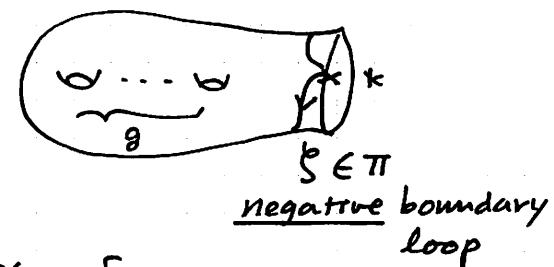
$\cdot : H \otimes H \rightarrow \mathbb{K}$ intersection number.

$\{x_i, y_i\}_{i=1}^g \subset H$ symplectic basis

$$\begin{aligned} x_i \cdot y_j &= -y_j \cdot x_i = \delta_{ij} \\ x_i \cdot x_j &= y_i \cdot y_j = 0 \end{aligned}$$

$$\omega := \sum_{i,j} x_i \cdot y_i - y_i \cdot x_i \in H^{\otimes 2} \cap \hat{\mathbb{L}}$$

symplectic form, independent of the choice of $\{x_i, y_i\}$



negative boundary loop

Definition (Massuyeau)

$\theta : \pi \rightarrow \exp(\widehat{L})$ symplectic expansion

def

\iff 1) $\theta : \pi \rightarrow \exp(\widehat{L})$ group-like expansion

$$2) \log \theta(\zeta) = w \in H^{\otimes 2} \cap \widehat{L}$$

the Goldman bracket on $gr| \widehat{K\pi} |$ for $\Sigma_{g,1}$

$$u = u_1 u_2 \cdots u_e, v = v_1 v_2 \cdots v_m \in \widehat{T}(H), u_j, v_k \in H$$

$$\{u, v\}_{gr} := \sum_{j, k} (u_j \cdot v_k) v_1 \cdots v_{k-1} u_{j+1} \cdots u_e u_1 \cdots u_{j-1} v_{k+1} \cdots v_m$$

$$\{u, v\}_{gr} := |\{u, v\}_{gr}|$$

- $(|\widehat{T}(H)|, \{.,.\}_{gr}) = (gr| \widehat{K\pi} |, gr(\text{Goldman bracket}))$

- $|\widehat{T}(H)| \rightarrow \text{Der}(|\widehat{T}(H)|), u \mapsto \{u, -\}_{gr}$ Lie algebra homom.

Theorem (K.-K.)

$\theta : \pi \rightarrow \exp(\widehat{L})$ (conjugate of) a symplectic expansion

$$\Rightarrow \theta : (| \widehat{K\pi} |, \text{Goldman bracket}) \xrightarrow{\cong} (|\widehat{T}(H)|, \{.,.\}_{gr}) \text{ Lie algebra isomorphism}$$

Theorem 2 for $\Sigma_{g,1}$, $\theta : \pi \rightarrow \exp(\widehat{L})$ group-like expansion

$$\theta : (| \widehat{K\pi} |, \text{Goldman bracket}) \xrightarrow{\cong} (|\widehat{T}(H)|, \{.,.\}_{gr}) \text{ Lie algebra isomorphism}$$

$$\Rightarrow \theta \text{ is conjugate to a symplectic expansion i.e., } \exists g_0 \in \exp(\widehat{L}) \text{ s.t. } \log \theta(\zeta) = g_0 w g_0^{-1}$$

Keys to the proof of Theorem 2.

- (i) PBW type decomposition: $|T(H)| = \bigoplus_{m=0}^{\infty} |\text{Sym}^m L|$
- (ii) Look at the center $Z(|\hat{T}(H)|, \{-, -\}_{\text{gr}})$

Theorem (Etingof ($n+1=0$), Gadgil ($n+1 \geq 1$)) $\forall g. \forall n$

$$Z(|K\pi|, \text{Goldman bracket}) = \begin{cases} K\mathbb{1} & \text{if } n+1 = 0 \\ \sum_{j=0}^n \sum_{i=0}^{\infty} K[\partial_j]^{(i)} & \text{if } n+1 \geq 1 \end{cases}$$

where $\mathbb{1} = |1| \in \hat{\pi}$ const. loop

∂_j : positive boundary loop along $\partial_j \Sigma$, $0 \leq j \leq n$.

Theorem (general) Crawley-Boevey - Etingof - Ginzburg

$$Z(|\hat{T}(H)|, \{-, -\}_{\text{gr}}) = \left| \sum_{j=0}^n K[[z_j]] \right|$$

where $z_j = \begin{cases} [\partial_j] & \in H = H_1(\Sigma : K) \\ -[\partial_0] & \end{cases} \quad (1 \leq j \leq n) \quad (j=0)$

Technical Part

- (1) PBW type decomposition for $U(\mathfrak{g})$
- (2) Inner derivation lemma for $\text{Der}(\hat{T}(H))$
- (3) Elementary proof of Thm ($C=B-E-G$) for $\Sigma_{0,n+1}$ and $\Sigma_{g,1}$
- (4) Proof of Thm 2 for $\Sigma_{0,n+1}$ and $\Sigma_{g,1}$

(1) H : fin.dim. \mathbb{K} -vect. space

$L = L(H)$: free Lie algebra over H

$$T(H) = U(L)$$

in general,

\mathfrak{g} : Lie algebra / \mathbb{K}

$U(\mathfrak{g}) = T(\mathfrak{g}) / \text{ideal gen. by } \{xy - yx - [x,y] : x, y \in \mathfrak{g}\}$
universal enveloping algebra

Poincaré-Birkhoff-Witt Theorem

$$\bigoplus_{m=0}^{\infty} \text{Sym}^m \mathfrak{g} \xrightarrow{\cong} U(\mathfrak{g}) \quad \dots : (*)$$

$$x_1, x_2, \dots, x_m \mapsto \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)}$$

$x_i \in \mathfrak{g}$

$$[\mathbb{U}(\mathfrak{g}), \mathbb{U}(\mathfrak{g})] = (\text{K-vec. subsp. spanned by } \{ab - ba : a, b \in \mathbb{U}(\mathfrak{g})\})$$

$$\stackrel{\text{PBW}}{=} \sum_{p=1}^{\infty} [\text{Sym}^p \mathfrak{g}, \mathbb{U}(\mathfrak{g})]$$

$$|\mathbb{U}(\mathfrak{g})| = \mathbb{U}(\mathfrak{g}) / [\mathbb{U}(\mathfrak{g}), \mathbb{U}(\mathfrak{g})], \quad \text{I/I: } \mathbb{U}(\mathfrak{g}) \rightarrow |\mathbb{U}(\mathfrak{g})|, \text{ quotient map}$$

(Rmk) $\forall V: \text{K-vec.sp. } \text{Sym}^m V \text{ is gen. by } \{v^m : v \in V\}$

$$\text{Theorem 4} \quad |\mathbb{U}(\mathfrak{g})| = \bigoplus_{m=0}^{\infty} |\text{Sym}^m \mathfrak{g}| \quad (\text{direct sum decomposition})$$

proof It suffices to show that $[\mathbb{U}(\mathfrak{g}), \mathbb{U}(\mathfrak{g})]$ is homogeneous w.r.t. $(*)$: $\mathbb{U}(\mathfrak{g})$

i.e., $(\forall V := \bigoplus_{m=0}^{\infty} (\text{Sym}^m \mathfrak{g}) \cap [\mathbb{U}(\mathfrak{g}), \mathbb{U}(\mathfrak{g})]) \supset [\mathbb{U}(\mathfrak{g}), \mathbb{U}(\mathfrak{g})]$

By induction on $p \geq 1$, we prove

$$[\text{Sym}^p \mathfrak{g}, \mathbb{U}(\mathfrak{g})] \subset V$$

$p=1$ $x, y \in \mathfrak{g}, m \geq 1$

$$[x, y^m] = \sum_{i=1}^m y^{i-1} [x, y] y^{m-i} \in \text{Sym}^m \mathfrak{g}$$

$$[\mathfrak{g}, \text{Sym}^m \mathfrak{g}] \subset (\text{Sym}^m \mathfrak{g}) \cap [\mathbb{U}(\mathfrak{g}), \mathbb{U}(\mathfrak{g})] \subset V$$

$p \geq 1$ $[x^{p+1}, y^m] = [x^p, y^m] x + x^p [x, y^m] = [x^p, y^m x] + [x, x^p y^m]$

\uparrow ind. assumption
V

$\uparrow p=1$ // Thm 4
V

$$T(H) = \bigoplus_{p=0}^{\infty} H^{\otimes p}$$

$$\hat{T}(H) = \prod_{p=0}^{\infty} H^{\otimes p}$$

$$L = \bigoplus_{p=0}^{\infty} (H^{\otimes p} \wedge L) \text{ homogeneous}$$

$$\downarrow \quad \text{Sym}^m L = \bigoplus_{p=0}^{\infty} (H^{\otimes p} \wedge \text{Sym}^m L) \text{ homogeneous } \forall m \geq 1$$

$$(\text{Sym}^m L)^{\wedge} := \prod_{p=0}^{\infty} (H^{\otimes p} \wedge \text{Sym}^m L) \hookrightarrow \text{Sym}^m (\hat{L})$$

$$|T(H)| = \bigoplus_{m=0}^{\infty} |\text{Sym}^m L| \text{ from Thm 4.}$$

$$\boxed{\text{Corollary 5} \quad |\hat{T}(H)| = \prod_{m=0}^{\infty} |(\text{Sym}^m L)^{\wedge}|}$$

in particular,

$$u, v \in \hat{L}, |\exp u| = |\exp v| \Rightarrow \forall m \geq 0 \quad |u^m| = |v^m| \in |\hat{T}(H)|.$$

$$(2) \quad H: \text{fin. dim. } K\text{-vect. space}, \quad \hat{T}(H) = \prod_{m=0}^{\infty} H^{\otimes m}$$

$\text{Der}(\hat{T}(H))$: continuous derivations

Lemma 6 $x \in H \setminus \{0\}, a \in \hat{T}(H)$

$$\forall m \geq 1 \quad |x^m a| = 0$$

$$\Rightarrow \exists a' \in \hat{T}(H), a = [x, a'] \in \hat{T}(H)$$

(proof: elementary but complicated //)

$$T(H) \xrightarrow{\cong} L(L)$$

$$\begin{array}{c} \uparrow G \uparrow G \uparrow \\ H \rightarrow L \end{array}$$

$$\left. \begin{aligned} & (\text{Rmk}) \\ & |x_1 [x_2 [[x_1, x_2], a]]| \\ & = |[x_1, x_2] [[x_1, x_2], a]| \\ & = |[[x_1, x_2], [x_1, x_2]] a| = 0 \end{aligned} \right\}$$

Lemma 7 (Inner derivation lemma)

$D \in \text{Der}(\hat{T}(H))$, $\forall a \in \hat{T}(H) \quad |Da| = 0$

$\Rightarrow \exists w \in \hat{T}(H)_{\geq 1}, D = \text{ad}_w (= [w, -])$

proof $H = \bigoplus_{j=1}^n \mathbb{K} x_j$

$$\forall m \geq 1, 0 = |D(x_j^{m+1})| = \sum_{k=0}^m |x_j^k (D x_j) x_j^{m-k}| = (m+1) |x_j^m (D x_j)|$$

$\exists a_j \in \hat{T}(H)_{\geq 1}, D(x_j) = [a_j, x_j] \quad (\because \text{Lem 6})$

$i \neq j, \forall p, \forall q \geq 1$

$$\begin{aligned} 0 &= |D(x_i^p x_j^q)| = |[a_i, x_i^p] x_j^q| + |x_i^p [a_j, x_j^q]| = |[a_i, x_i^p] x_j^q| - |[a_j, x_i^p] x_j^q| \\ &= |[a_i - a_j, x_i^p] x_j^q| \end{aligned}$$

$[a_i - a_j, x_i^p] \in [x_i, \hat{T}(H)] \quad \forall p \geq 1 \quad (\because \text{Lem 6})$

similarly $[a_i - a_j, x_j^q] \in [x_i, \hat{T}(H)] \quad \forall q \geq 1 \quad (\because \text{Lem 6})$

↓ some elementary consideration

$a_i - a_j \in \mathbb{K}[[x_j]] + \mathbb{K}[[x_i]] \quad \text{i.e., } \exists f_{ij}(x_j) \in \mathbb{K}[[x_j]]_{\geq 1}, \exists f_{ji}(x_i) \in \mathbb{K}[[x_i]]_{\geq 1}$

$$a_i - a_j = f_{ij}(x_j) - f_{ji}(x_i)$$

$w := a_j + f_{ij}(x_j)$ is indep from i and j

$$D(x_i) = [w, x_i] \quad 1 \leq i \leq n$$

$$D(a) = [w, a] \quad \forall a \in \hat{T}(H) \quad (\because D: \text{cont.}) //$$

(3-1) Elementary proof of

$$\text{Thm } (C=B-E-G) \quad \sum = \sum_{0,n+1} \left[\mathcal{Z}(|\hat{T}(H)|, \{-\}_{gr}) = \left| \sum_{j=0}^n K[[z_j]] \right| \right]$$

where $z_j = [x_j] \in H = H_1(\sum_{0,n+1}; K)$, $0 \leq j \leq n$

Facts (1) $1 \leq v_j \leq n$. $\forall m \geq 0$ $\{ |z_j|^m |, - \}_{gr} = 0$

$$(2) \{ -, z_0 \}_{gr} = 0$$

$$(3) \{ |z_0|^m |, - \}_{gr} = -m[z_0^{m-1}, -]$$

$$(4) \text{Ker}(|\hat{T}(H)| \rightarrow \text{Der}(\hat{T}(H))) = \left| \sum_{j=1}^n K[[z_j]] \right|$$

(\Rightarrow) Facts (1)(3)

(\subset) $a \in \hat{T}(H)$, Assume $|a| \in \mathcal{Z}(|\hat{T}(H)|, \{-\}_{gr})$

$$\forall b \in \hat{T}(H), 0 = \{ |a|, |b| \}_{gr} = \{ |a|, b \}_{gr}$$

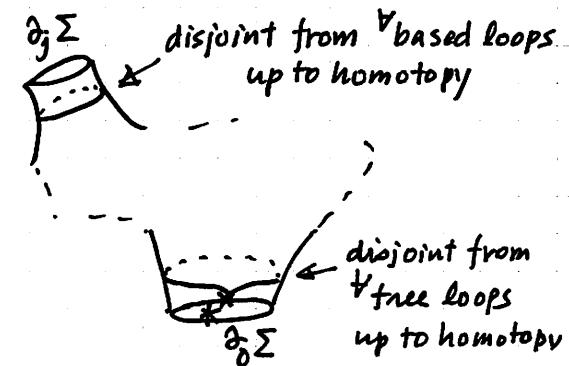
$$\exists w \in \hat{T}(H)_{\geq 1}, \{ |a|, - \}_{gr} = [w, -] \quad (\because \text{Lem 7})$$

$$0 \stackrel{\text{Fact (2)}}{=} \{ |a|, z_0 \} = [w, z_0]$$

$$\exists f(z_0) \in K[[z_0]]_{\geq 1}, w = f(z_0), \hat{f}(t) := \int_0^t f(1-t) dt \quad \text{indefinite integral}$$

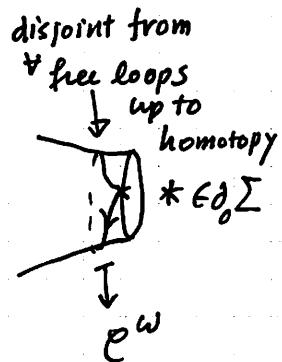
$$\{ |a|, - \}_{gr} = [f(z_0), -] = \{ |\hat{f}(z_0)|, - \}_{gr} \quad (\because \text{Fact (3)})$$

$$\{ |a| - |\hat{f}(z_0)|, - \}_{gr} = 0, \quad |a| - |\hat{f}(z_0)| \in \left| \sum_{j=1}^n K[[z_j]] \right| \quad (\because \text{Fact (4)}) //$$



(3-2) Elementary Proof of

$$\boxed{\begin{aligned} \text{Thm } (C = B - E - G) \quad \Sigma &= \Sigma_{g,1} \\ Z(|\hat{T}(H)|, \{-, -\}_{gr}) &= |\mathbb{K}[[\omega]]| \\ \text{where } \omega \in H^{\otimes 2} \cap \wedge^2 \text{ symplectic form} \end{aligned}}$$



- Facts (1) $\{-, \omega\}_{gr} = 0$
- (2) $\{|w^m|, -\}_{gr} = -m[w^{m-1}, -]$
- (3) $\text{Ker}(|\hat{T}(H)| \rightarrow \text{Der}(\hat{T}(H))) = \mathbb{K}\mathbf{1}.$ $\mathbf{1} = 11 \text{ const. loop}$

(D) Fact (2)

(C) $a \in \hat{T}(H).$ Assume $|a| \in Z(|\hat{T}(H)|, \{-, -\}_{gr})$

$$\forall b \in \hat{T}(H), \quad 0 = \{ |a|, |b| \}_{gr} = | \{ |a|, b \}_{gr} |$$

$$\exists v \in \hat{T}(H), \quad \{ |a|, - \}_{gr} = [v, -] \quad (\because \text{Lem 7})$$

$$0 \xrightarrow{\text{Fact (1)}} \{ |a|, \omega \} = [v, \omega]$$

$$\exists f(\omega) \in \mathbb{K}[[\omega]]. \quad v = f(\omega). \quad \hat{f}(t) := \int_0^t f(-t) dt$$

$$\{ |a|, - \}_{gr} = [f(\omega), -] \xrightarrow{\text{Fact (2)}} \{ |\hat{f}(\omega)|, - \}_{gr}$$

$$|a| - |\hat{f}(\omega)| \in \mathbb{K}\mathbf{1} \quad (\because \text{Fact (3)})$$

$$|a| \in |\mathbb{K}[[\omega]]| \quad //$$

(4-1) Proof of Thm 2 for $\Sigma_{0,n+1}$

θ as in the theorem, $\ell_j := \log \theta(z_j) \in \hat{L} \quad (0 \leq j \leq n)$

Claim $0 \leq j \leq n, \forall m \geq 0 \quad |\ell_j^m| = |z_j^m| \in |\hat{T}(H)| \quad \dots \quad (b)$

$$(\text{pf}) \quad |\exp(\ell_j)| = |\theta(z_j)| \in \mathbb{Z}(|\hat{T}(H)|, \{-\rangle_{\text{gr}}) = \sum_{j=0}^n |\mathbb{K}[[z_j]]|$$

Take the components in $|\text{Sym}^m L^\wedge|$ in Cor. 5

$$\forall m \geq 2 \quad |\ell_j^m| \in \sum_{j=0}^n \mathbb{K}|z_j^m|$$

$$\ell_j = z_j + \text{higher degree terms} \implies \ell_j^m = z_j^m + \text{higher degree terms}$$

$\{ |(z_1 + \dots + z_n)^m|, |z_1^m|, \dots, |z_n^m| \} : \text{linearly independent } (\because m \geq 2, \underline{m \geq 2})$

Hence $|\ell_j^m| = |z_j^m| \text{ for } m \geq 2$

For $m=1$, use the fact $|\hat{L}_{\leq 2}| = 0 //$

By induction on $k \geq 1$, we prove

$$\left[\begin{array}{l} \exists u_k \in L \cap H^{\otimes k} \\ \exp(\text{ad}_{u_k}) \exp(\text{ad}_{u_{k-1}}) \dots \exp(\text{ad}_{u_1})(\ell_j) = z_j \pmod{\hat{T}(H)_{\geq k+2}} \\ \parallel \exp(bch(u_k, u_{k-1}, \dots, u_1)) \ell_j \exp(bch(u_k, u_{k-1}, \dots, u_1))^{-1} \end{array} \right]$$

may assume

$$\Sigma \neq \Sigma_{0,2}$$

" For $\Sigma_{0,2}$
a group-like
expansion
is unique

$$\underline{k=1} \quad l_j = z_j + b_2 + (\text{degree } \geq 3 \text{ terms}), \quad b_2 \in L \cap H^{\otimes 2}$$

Taking the degree- $(m+1)$ -terms in (b): $0 \leq k_j \leq n, \quad k_m \geq 1. \quad |l_j| = |z_j|$

$$m |z_j^{m-1} b_2| = 0 \quad k_m \geq 2$$

$$\exists u_1 \in H, \quad b_2 = [z_j, u_1] \quad (\because \text{Lem 6})$$

$$\exp(\text{ad}_{u_1})(l_j) = z_j + b_2 + [u_1, z_j] + (\text{degree } \geq 3 \text{ terms}) = z_j + (\text{degree } \geq 3 \text{ terms})$$

$k \geq 2$ By the inductive assumption $\exists u_1, \dots, \exists u_{k-1}$

$$\exp(\text{ad}_{u_{k-1}}) \cdots \exp(\text{ad}_{u_1})(l_j) = z_j + b_{k+1} + \text{higher degree terms}. \quad b_{k+1} \in L \cap H^{\otimes (k+1)}$$

Taking the degree- $(m+k)$ terms in (b)

$$m |z_j^{m-1} b_{k+1}| = 0 \quad k_m \geq 2$$

$$\exists u_k \in H^{\otimes k} \quad b_{k+1} = [z_j, u_k] \quad (\because \text{Lem 6})$$

Fact $\forall z \in H \setminus \{0\}, \{a \in \hat{T}(H); [z, a] \in \hat{L}\} = \hat{L} + K[[z]]$

may assume $u_k \in H^{\otimes k} \cap L$

$$\exp(\text{ad}_{u_k}) \exp(\text{ad}_{u_{k-1}}) \cdots \exp(\text{ad}_{u_1})(l_j) = z_j + (\text{degree } \geq k+3 \text{ terms}) \quad // \text{induction}$$

$$v_\infty := \lim_{k \rightarrow \infty} b_{k+1}(u_k, u_{k-1}, \dots, u_1) \in \hat{L}$$

$$e^{v_\infty} l_j e^{-v_\infty} = z_j$$

$$g_j := e^{-v_\infty} \in \exp(\hat{L}). \quad \theta(x_j) = \exp l_j = g_j e^{z_j} g_j^{-1} \quad // \text{Thm 2 for } \Sigma_{0, n+1}$$

(4-2) Proof of Thm 2 for $\Sigma_{g,1}$

ℓ as in the theorem, $\ell := \log(\theta/\zeta)$

$$\forall m \geq 0 \quad |\ell^m| = |\omega^m| \in |\hat{T}(H)| \quad \text{--- (bb)} \quad (\because \text{Cor 5})$$

Here we need an analogue of Lemma 6 for the symplectic form ω

Lemma 8 $a \in \hat{T}(H), \forall m \geq 1, |\omega^m a| = 0 \in |\hat{T}(H)|$
 $\Rightarrow \exists a' \in \hat{T}(H), a = [\omega, a']$

Using this lemma, we prove the following by induction on $k \geq 1$.

$\exists u_k \in L \cap H^{\otimes k}$ s.t. $\exp(\text{ad}_{u_k}) \exp(\text{ad}_{u_{k-1}}) \cdots \exp(\text{ad}_{u_1})(\ell) \equiv \omega \pmod{\hat{T}(H)_{\geq k+3}}$

$k=1$ Take the degree $(2m+1)$ -terms in (bb) and use Lemma 8. (similar to 14-1)

$k \geq 2$ By inductive assumption. $\exists u_1, \dots, \exists u_{k-1}$

$$\exp(\text{ad}_{u_{k-1}}) \cdots \exp(\text{ad}_{u_1})(\ell) = \omega + b_{kr2} + \text{higher degree terms.} \quad b_{kr2} \in L \cap H^{\otimes(kr2)}$$

Taking the degree $(2m+k)$ -terms in (bb)

$$m|\omega^{m-1} b_{kr2}| = 0 \quad \forall m \geq 2$$

$$\exists u_k \in H^{\otimes k} \quad b_{kr2} = [\omega, u_k] \quad (\because \text{Lem 8})$$

$$\text{Faut } \{a \in \hat{T}(H) : [\omega, a] \in \hat{L}\} = \hat{L} + \mathbb{K}[[\omega]]$$

may assume $u_k \in H^{\otimes k} \cap L$ // induction // Thm 2 for $\Sigma_{g,1}$

///