OIST workshop : MCM2016, October 25, 2016 at OIST. Central Building, C209

"Some tensor field on the Teichmüller space"

N. Kawazumi (Univ. Tokyo)

$g \geqq 1$

C : compact Riemann surface of genus g (without boundary)

$H^0(C;\Omega^1_C) = \{$holomorphic 1-forms on $C\} \cong \mathbb{C}^g$

$\{\psi_i\}_{i=1}^g \subset H^0(C;\Omega^1_C)$ orthonormal basis (o.n.b.)

i.e., $\frac{\sqrt{-1}}{2}\int_C \psi_i \wedge \overline{\psi_j} = \delta_{ij} \quad (1 \leq i, j \leq g)$

$B := \frac{\sqrt{-1}}{2g}\sum_{i=1}^g \psi_i \wedge \overline{\psi_i}$ volume form on C, indep of the choice of $\{\psi_i\}_{i=1}^g$

$\int_C B = 1.$

$A^p(C) := \{\mathbb{C}$-valued p-currents on $C\}$ $\quad p = 0, 1, 2$
(p-forms)

$* : A^1(C) \to A^1(C)$, Hodge $*$-operator $\left(\frac{\text{local}}{}\ z : \text{complex coordinate} \quad *dz = -\sqrt{-1}\,dz,\ *\overline{dz} = +\sqrt{-1}\,\overline{dz}\right)$

$\exists!\ \hat{\Phi} : A^2(C) \to A^0(C)$ the Green operator associated with the volume form B

s.t. $\begin{cases} \text{(i)} & d*d\,\hat{\Phi}(\Omega) = \Omega - (\int_C \Omega) B \\ \text{(ii)} & \int_C \hat{\Phi}(\Omega) B = 0 \end{cases} \quad ({}^\forall \Omega \in A^2(C))$

- $\overline{\hat{\Phi}} = \hat{\Phi}$: real operator
- $\hat{\Phi}|_{\mathrm{Ker}(\int_C)} : \mathrm{Ker}(\int_C : A^2(C) \to \mathbb{C}) \to A^0(C)/\mathbb{C}$ depends only on the cpx str. C

  const. function

- $\forall \Omega, \forall \Omega' \in A^2(C)$ $\quad \int_C \hat{\Phi}(\Omega)\Omega' = \int_C \Omega \hat{\Phi}(\Omega')$

$$\int_C \hat{\Phi}(\Omega)\Omega' = \frac{1}{4\pi}\iint_{(P_1,P_2)\in C\times C} (\Omega)_{P_1}(\log G(P_1,P_2))(\Omega')_{P_2}$$

(where $G(\cdot,\cdot)$ the Arakelov - Green function)

"some tensor field" in the title

tentative symbol

$$A_{i\bar{j}k\bar{l}} \overset{\mathrm{def}}{=} \int_C \psi_i \wedge \overline{\psi_j}\ \hat{\Phi}(\psi_k \wedge \overline{\psi_l}) \in \mathbb{C} \qquad (1 \le i,j,k,l \le g)$$

Question 1 Is it a "structure constant" of the Riemann surface ?

(more precisely) Is the map induced by $(A_{i\bar{j}k\bar{l}})$

$$(\text{the Torelli space}) \longrightarrow \mathbb{C}^{g^4}$$

injective ?

where (the Torelli space) $:= \{(C, \{A_i, B_i\}_{i=1}^g) :$ C : compact Riemann surface of genus g, $\{A_i, B_i\}_{i=1}^g < H_1(C;\mathbb{Z})$ symplectic basis $\} / \text{biholo}$

elementary properties

(1) $\overline{A_{i\bar{j}k\bar{l}}} = \int_C \overline{\psi_i} \wedge \psi_j\ \hat{\Phi}(\overline{\psi_k} \wedge \psi_l) = \int_C \psi_j \wedge \overline{\psi_i}\ \hat{\Phi}(\psi_l \wedge \overline{\psi_k}) = A_{j\bar{i}l\bar{k}}$

(2) $A_{i\bar{j}k\bar{l}} = \int_C \psi_i \wedge \overline{\psi_j}\ \hat{\Phi}(\psi_k \wedge \overline{\psi_l}) = \int_C \hat{\Phi}(\psi_i \wedge \overline{\psi_j})\ \psi_k \wedge \overline{\psi_l} = A_{k\bar{l}i\bar{j}}$

<u>Kawazumi – Zhang invariant</u> (K. 0801.4218, Zhang 0812.0371 ⇒ Invent. math.)

(cf) Pioline's talk)

$$a_g = a_g(C) := \sum_{i,j=1}^{g} A_{i\bar{j}j\bar{i}} \in \mathbb{R}$$

$$\left(\because \ \overline{a_g} = \sum_{i,j} A_{j\bar{i}i\bar{j}} = a_g \right)$$

(<u>Fact</u> $\forall g \geq 2$, $\forall C$, $a_g(C) > 0$)

<u>K.</u> $$\frac{-2\sqrt{-1}(2g-2)^2}{2g(2g+1)} \partial\bar{\partial} a_g = e_1^F - e_1^J \in A^2(\mathbb{M}_g)$$

<u>where</u> $\mathbb{M}_g := \{ C$: compact Riemann surface of genus $g \} /$ biholo

the moduli space of compact Riemann surfaces of genus g

$e_1^F, e_1^J \in A^2(\mathbb{M}_g)$ differential forms representing

will be explained later

the 1^{st} Mumford – Morita – Miller (MMM, tautological) class $e_1 = \kappa_1$

<u>Zhang</u> – Arakelov geometry, decomposition of the Faltings delta invariant δ

<u>de Jong</u> – relation with the invariant δ and the Hain – Reed function β

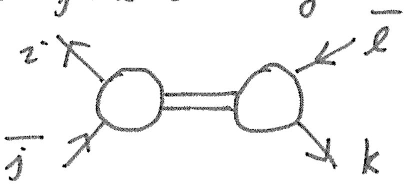
– asymptotic behavior

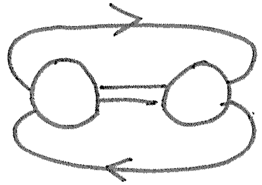
<u>d'Hoker – Green</u> – Application of a_2 $(g=2)$ to Physics

<u>Pioline</u> – Application of a_2 to Physics

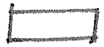
– explicit formula of a_2 in terms of the theta function

Diagrams for $A_{i\bar{j}k\bar{l}}$ and a_g

$A_{i\bar{j}k\bar{l}} =$ 

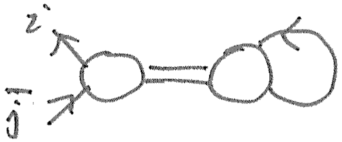
, $a_g =$ 

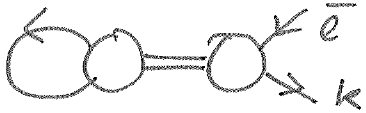
(where 

$\leftrightarrow \psi_i \wedge \overline{\psi_j}$,  $\leftrightarrow \log G(\cdot,\cdot)$)

(cf. Morita's construction : trivalent graph $\longmapsto$ tautological class $\in A^*(\mathbb{M}_g)$
(cf. Sakasai's talk))

Question 2 What do other combinations of $(A_{i\bar{j}k\bar{l}})$'s mean ?

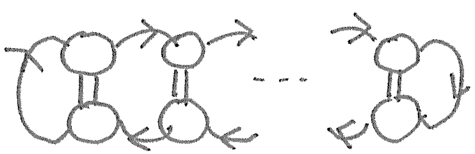
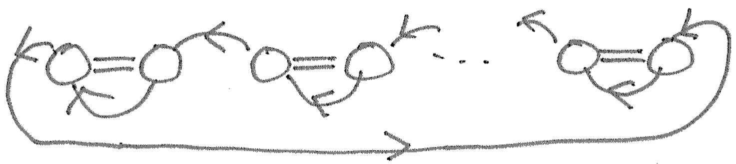
Remark 

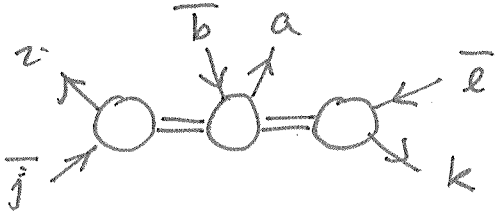
$=$ 

$= 0$

$(\because)$ $\hat{\Phi}\left(\sum_{i=1}^{g} \psi_i \wedge \overline{\psi_i}\right) = \hat{\Phi}(B) = 0$ //)

Other combinations

  and so on



$\overset{\text{def}}{=} \int_C \hat{\Phi}(\psi_i \wedge \overline{\psi_j})\, \psi_a \wedge \overline{\psi_b}\, \hat{\Phi}(\psi_k \wedge \overline{\psi_l})$

$= \int_C \psi_i \wedge \overline{\psi_j}\, \hat{\Phi}(\psi_a \wedge \overline{\psi_b}\, \hat{\Phi}(\psi_k \wedge \overline{\psi_l}))$ and so on

Differential forms on $\mathbb{M}_g$

$\mathbb{M}_g = \{C : \text{compact Riemann surface of genus } g\} / \text{biholo}$

the moduli space of compact Riemann surfaces

holomorphic cotangent space of $\mathbb{M}_g$ at $[C] \in \mathbb{M}_g$

dual

$$T^*_{[C]}\mathbb{M}_g = H^0(C ; (\Omega^1_C)^{\otimes 2}) \cong \mathrm{Coker}\left(C^\infty(TC) \xrightarrow{\bar\partial} C^\infty(TC \otimes \overline{T^*C})\right)^*$$

$$= \{\text{holomorphic quadratic differentials on } C\}$$

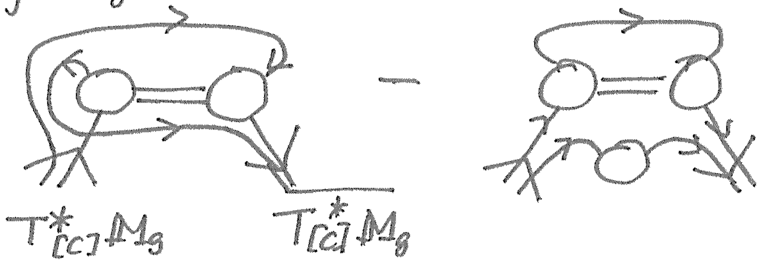
$$\left(\text{ex) } \psi_\ell \psi_j = \psi_j \psi_\ell \in T^*_{[C]}\mathbb{M}_g, \quad \bar\psi_i \bar\psi_k = \bar\psi_k \bar\psi_i \in \overline{T^*_{[C]}\mathbb{M}_g} \right)$$

Theorem (K.) The $(1,1)$-form

$$4\sqrt{-1} \sum_{i,j,k,\ell} \left(\psi_\ell \psi_j A_{i\bar j k \bar\ell} \bar\psi_i \bar\psi_k - \psi_\ell \psi_j A_{i\bar j k \bar i} \bar\psi_k \bar\psi_\ell\right) \in T^*_{[C]}\mathbb{M}_g \otimes \overline{T^*_{[C]}\mathbb{M}_g}$$

represents $\frac{1}{12} e_1 = \frac{1}{12}\kappa_1 \in H^2(\mathbb{M}_g ; \mathbb{C})$

corresponding diagram

$4\sqrt{-1}\Bigg($  $\Bigg)$

Question 3 Can we construct higher $e_n = (-1)^{n+1}\kappa_n \in H^{2n}(\mathbb{M}_g ; \mathbb{C})$ from $(A_{i\bar j k \bar\ell})$?

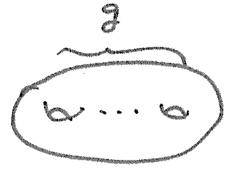
In order to explain the background of Theorem, we need

__twisted cohomology of $\mathbb{M}_g$__

$$H := \coprod_{[C] \in \mathbb{M}_g} H_1(C : \mathbb{C}) \longrightarrow \mathbb{M}_g$$ flat vector bundle $\Rightarrow$ can consider the twisted cohomology

mapping class group

$$H^*(\mathbb{M}_g : \Lambda^* H) = H^*(\pi_1^V(\mathbb{M}_g) : \Lambda^* H_1(\Sigma_g : \mathbb{C}))$$

$\Sigma_g =$  C^∞ closed surface of genus g

de Rham cohomology group cohomology ($\Leftarrow$ contractibility of the Teichmüller space)

$(A_{i\bar{j}k\bar{l}})$ can be regarded as $\mathbb{R}$-valued section of $\mathrm{Sym}^2(\Lambda^2 H) \subset H^{\otimes 4}$ $\rightarrow$ "tensor field"

$\mathbb{C}_g := \{(C, P_0) : C$: compact Riemann surface of genus g, $P_0 \in C\} /$ biholo

$$\begin{array}{ccc} \mathbb{C}_g & \ni & [C, P_0] \\ \pi \downarrow & & \downarrow \\ \mathbb{M} & \ni & [C] \end{array}$$

(__Morita__
$$H^*(\mathbb{C}_g : \Lambda^* H) \cong H^*(\mathbb{M}_g : \Lambda^* H) \oplus H^{*-1}(\mathbb{M}_g : H \otimes \Lambda^* H) \oplus H^{*-2}(\mathbb{M}_g : \Lambda^* H)$$)

__the Mumford-Morita-Miller classes__ $n \geq 0$

$$e_n = (-1)^{n+1} \kappa_n = \int_{\text{fiber}} e^{n+1} \in H^{2n}(\mathbb{M}_g : \mathbb{Q})$$

(where $e = c_1(T\mathbb{C}_g/\mathbb{M}_g) \in H^2(\mathbb{C}_g : \mathbb{Q})$) ($e_0 = 2 - 2g$)

$\rightsquigarrow$ generalization to twisted classes $\in H^*(\underset{\sim}{\mathbb{C}_g} : \Lambda^* H)$ (K.)

$$\pi : \mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \to \mathbb{C}_g, \quad [C, P_0, P_1] \mapsto [C, P_0]$$

$$k_0 \in H^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g : H) \cong H^1(\pi_1^V(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g) : H_1(\Sigma_g : \mathbb{C}))$$

$$k_0 : \pi_1^V(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g) \cong \pi_1(\Sigma_g) \rtimes \pi_1^V(\mathbb{C}_g) \xrightarrow{1^{st}\text{ proj}} \pi_1(\Sigma_g) \to H_1(\Sigma_g : \mathbb{C})$$
$$(\gamma, \varphi) \longmapsto [\gamma]$$

twisted 1-cocycle (Morita)

$i, j \geqq 0$, $2i+j \geqq 2$

$$\begin{tikzcd} e^i k_0^j = c_1(T\mathbb{C}_g/\mathbb{M}_g)^i k_0^j \arrow[d, mapsto] & \in H^{2i+j}(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g : \Lambda^j H) \arrow[d, "\pi_* = \int_{fiber}"] \\ m_{ij} \stackrel{def}{=} \int_{fiber} e^i k_0^j & \in H^{2i+j-2}(\mathbb{C}_g : \Lambda^j H) \end{tikzcd}$$

the $(i, j)^{th}$ twisted MMM class (K.)

$m_{i+1,0} = e_i$, $m_{0,2} = 2(\text{symplectic form}) \in H^0(\mathbb{C}_g : \Lambda^2 H)$

$\frac{1}{6} m_{0,3}$ = the extended 1^{st} Johnson homomorphism $\in H^1(\mathbb{C}_g : \Lambda^3 H)$

<u>canonical differential form representing</u> k_0 $\quad P_0 \neq P_1$

$[C, P_0, P_1] \in \mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \setminus$ diagonal

$\delta_{P_0} \in A^2(C)$ delta current at P_0 i.e., $\forall f \in C^\infty(C : \mathbb{C})$, $\int_C f \delta_{P_0} = f(P_0)$

$[* d \hat{\Phi}(\delta_{P_1} - \delta_{P_0})] \in H^1(C \setminus \{P_0, P_1\} : \mathbb{C}) \hookleftarrow H^1(C : \mathbb{C})$

the normalized Abelian integral of the 3^{rd} kind

$\hat{k}_0 := (\text{the } 1^{st} \text{ variation of } [* d \hat{\Phi}(\delta_{P_1} - \delta_{P_0})]) \in A^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \setminus \text{diagonal} : H)$

naturally extends to the diagonal

$$\begin{tikzcd} {[\hat{k}_0]} & \in H^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \setminus \text{diagonal} : H) \\ k_0 \arrow[u, mapsto] & \in H^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g : H) \arrow[u, "\cong"'] \end{tikzcd}$$

$\cong$ ← Gysin sequence associated with the diagonal

$\hat{k}_0 \in A^1(\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g : H)$ canonical diff. form.

canonical differential form representing $\frac{1}{6} m_{0,3}$, the extended 1^{st} Johnson homomorphism

$$\tilde{k} := \frac{1}{6} \int_{fiber} \hat{k}_0^{\,3} \in A^1(\mathbb{C}_g : \Lambda^3 H)$$

explicit description of $\tilde{k}$ in terms of the Green operator $\hat{\Phi}$

$$\varphi_a = \varphi_a' + \varphi_a'' \in H^1(C : \mathbb{C}) \overset{\text{identify}}{=} \{\text{harmonic 1-forms on } C\}, \quad a = 1, 2, 3$$

$$\varphi_a', \overline{\varphi_a''} \in H^0(C : \Omega^1_C) \text{ holomorphic}$$

$$Q(\varphi_1, \varphi_2, \varphi_3) \overset{\text{def}}{=} -\sqrt{-1}\, \varphi_1' \partial \hat{\Phi}(\varphi_2 \wedge \varphi_3 - (\textstyle\int_C \varphi_2 \wedge \varphi_3)\delta_{P_0}) - \sqrt{-1}\, \varphi_2' \partial \hat{\Phi}(\varphi_3 \wedge \varphi_1 - (\textstyle\int_C \varphi_3 \wedge \varphi_1)\delta_{P_0})$$
$$- \sqrt{-1}\, \varphi_3' \partial \hat{\Phi}(\varphi_1 \wedge \varphi_2 - (\textstyle\int_C \varphi_1 \wedge \varphi_2)\delta_{P_0})$$
$$\in T^*_{[C,P_0]} \mathbb{C}_g = \left\{ \text{mero. quad. diff. on } C;\ \begin{array}{l} \text{holo. on } C \setminus \{P_0\} \\ \text{order at } P_0 \geq -1 \end{array} \right\}$$

Theorem (K.) $\forall \varphi_a \in H^1(C : \mathbb{C})$ $(a = 1, 2, 3)$

$$\langle \tilde{k}, [\varphi_1] \otimes [\varphi_2] \otimes [\varphi_3] \rangle \text{ at } [C, P_0] \in \mathbb{C}_g$$
$$= -Q(\varphi_1, \varphi_2, \varphi_3) - \overline{Q(\varphi_1, \varphi_2, \varphi_3)} \in T^*_{[C,P_0]} \mathbb{C}_g \oplus \overline{T^*_{[C,P_0]} \mathbb{C}_g}$$

Corollary $\tilde{k}$ = the 1^{st} variation of the pointed harmonic volume —
(B. Harris) (cf. Todokoro's talk)

Morita's construction (cf) Sakasai's talk)

$[\tilde{k}] = \frac{1}{6} m_{0,3} \in H^1(\mathbb{C}_g : \Lambda^3 H)$, the extended 1^{st} Johnson homomorphism.

$N \geqq 0$, $f \in \mathrm{Hom}(\Lambda^N(\Lambda^3 H_1(\Sigma_g : \mathbb{C})), \mathbb{C})^{Sp_{2g}(\mathbb{C})}$ invariant linear form $\longleftrightarrow$ trivalent graph

$$\Longrightarrow f_* : H^N(\mathbb{C}_g : \Lambda^N(\Lambda^3 H)) \longrightarrow H^N(\mathbb{C}_g : \mathbb{C})$$
$$[\tilde{k}]^N \longmapsto f_*([\tilde{k}]^N)$$

$$\Longrightarrow \alpha : \mathrm{Hom}(\Lambda^*(\Lambda^3 H_1(\Sigma_g : \mathbb{C})), \mathbb{C})^{Sp_{2g}(\mathbb{C})} \longrightarrow H^*(\mathbb{C}_g : \mathbb{C}) \quad \text{algebra homom}$$
$$f \longmapsto f_*([\tilde{k}]^{\otimes})$$

$\mathrm{Im}\,\alpha \supset \mathbb{C}[e, e_n : n \geqq 1]$. $e = c_1(T\mathbb{C}_g/\mathbb{M}_g)$, (Morita)

$\mathrm{Im}\,\alpha = \mathbb{C}[e, e_n : n \geqq 1]$ (even in the unstable range) (Morita-K.)

$$\alpha : \mathrm{Hom}(\Lambda^2(\Lambda^3 H_1(\Sigma_g : \mathbb{C})), \mathbb{C})^{Sp_{2g}(\mathbb{C})} \xrightarrow{\cong} H^2(\mathbb{C}_g : \mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}e_1 \quad (\text{if } g \geqq 3)$$

$\exists!\, f_0$, $\exists!\, f_1$ $\qquad f_{0*}[\tilde{k}]^2 = e$, $f_{1*}[\tilde{k}]^2 = e_1$,

$e^J := f_{0*}\tilde{k}^2 \in A^2(\mathbb{C}_g)$, $\quad e_1^J := f_{1*}\tilde{k}^2 \in \mathrm{Im}\,(A^2(\mathbb{M}_g) \xhookrightarrow{\pi^*} A^2(\mathbb{C}_g))$

$e_1^F := \int_{fiber}(e^J)^2 \in A^2(\mathbb{M}_g) \quad \left(\rightsquigarrow \frac{-2\sqrt{-1}(2g-2)^2}{2g(2g+1)} \partial\bar{\partial} a_g = e_1^F - e_1^J \right)$

By chance, I found

$$\frac{g^2}{(2g-2)^2} e_1^F - \frac{2g^2+2g-1}{3(2g-2)^2} e_1^J \left(\backsim \frac{1}{12} e_1\right) \text{ is expressed in terms of } (A_{i\bar{j}k\bar{l}}).$$

$\rightsquigarrow$ Theorem

<u>Question 4</u> Is $(A_{i\bar{j}k\bar{l}})$ related to $m_{0,4} = [\int_{fiber} \hat{k}_0{}^4] \in H^2(\mathcal{C}_g ; \Lambda^4 H)$?

<u>Problem</u> $\int_{fiber} \hat{k}_0{}^4$ seems to have non-zero components in $A^{2,0} \oplus A^{0,2}$. - - - -

//