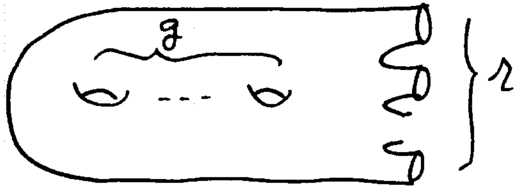
"A geometric approach to the higher Johnson homomorphisms"

Nariya Kawazumi (Univ. of Tokyo)
joint work with Yusuke Kuno (Hiroshima U./JSPS)

§0. Introduction

$S = \Sigma_{g,r} :=$



$, \quad g \geqq 1, r \geqq 1.$

$E \subset \partial S$ a finite subset

such that $^{\forall}$component of ∂S has a <u>unique</u> point of E

$\mathcal{M}(S) := \pi_0 \mathrm{Diff}_+(S, \mathrm{id} \text{ on } \partial S)$ mapping class group

$\mathcal{I}(S, E) := \mathrm{Ker}(\mathcal{M}(S) \longrightarrow \mathrm{Aut}(H_1(S, E; \mathbb{Z})))$

the "smallest" Torelli group in the sense of Putman.

Our results

(I) $^{\exists}\tau : \mathcal{I}(S,E) \hookrightarrow {}^{\exists}L^{+}(S,E)$ injective group homomorphism

a pro-nilpotent Lie algebra

⇓ Hausdorff series

pro-nilpotent Lie group

(II) $^{\exists}\delta : L^{+}(S,E) \longrightarrow {}^{\exists}$ ▨ a linear map

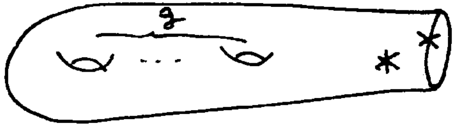
$\neq 0$ restriction of "cobracket"

$\delta \circ \tau = 0$ on $\mathcal{I}(S,E)$

$S = \Sigma_{g,1}$, $\mathcal{I}(S,E) = \mathcal{I}_{g,1}$ classical Torelli group.

§ 1. Classical results

classical ; $r = 1$, i.e., $S = \Sigma_{g,1}$ $(g \geq 1)$, $E = \{*\} \subset \partial S$



$\pi := \pi_1(\Sigma_{g,1}, *)$ free group of rank $2g$

$\mathcal{M}_{g,1} := \mathcal{M}(S)$ (after Johnson's terminology)

(lower central series

$\Gamma_1 := \pi, \quad \Gamma_{k+1} := [\Gamma_k, \Gamma_1] \triangleleft \pi, \quad k \geqq 1$

$\mathcal{M}_{g,1} \curvearrowright N_k := \pi/\Gamma_{k+1}$,

$\mathcal{M}_{g,1}(k) := \mathrm{Ker}(\mathcal{M}_{g,1} \to \mathrm{Aut}(N_k)), k \geqq 1$, Johnson filtration

$k=1, \quad N_1 = H_{\mathbb{Z}} = H_1(\Sigma_{g,1}; \mathbb{Z}), \quad \mathcal{M}_{g,1}(1) = \mathcal{I}_{g,1}$ Torelli group

$H := H_1(\Sigma_{g,1}; \mathbb{Q}) = H_{\mathbb{Z}} \otimes \mathbb{Q}$

Johnson $\forall k \geqq 1$

$$\tau_k : \mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1) \hookrightarrow H^* \otimes \mathcal{L}_{k+1}(H)$$

the k^{th} Johnson homomorphism

(where $\mathcal{L}(H)$: free Lie algebra generated by $H \subset \bigoplus_{k=0}^{\infty} H^{\otimes k}$

$\mathcal{L}_k(H) := \mathcal{L}(H) \cap H^{\otimes k}$ the k^{th} homogeneous component)

$$\tau : \mathrm{gr}(\mathcal{I}_{g,1}) \hookrightarrow H^* \otimes \mathcal{L}(H) = \mathrm{Der}(\mathcal{L}(H))$$

Lie algebra homomorphism

Morita refinement of the target of τ

$\omega \in \Lambda^2 H = \mathcal{L}_2(H)$ symplectic form ($\Leftarrow$ $H \otimes H \to \mathbb{Q}$ intersection form on $\Sigma_{g,1}$)

$\mathfrak{h}_{g,1}^{+} := \underset{\nwarrow \text{killing } \omega}{\mathrm{Der}_\omega(\mathcal{L}(H))} \cap \left(\bigoplus_{\underset{\sim}{k=1}}^{\infty} H^* \otimes \mathcal{L}_{k+1}(H) \right)$ Lie subalgebra

(We will show $L^+(\Sigma_{g,1}, \{*\}) \cong (\mathfrak{h}_{g,1}^{+})^{\wedge}$ $\leftarrow$ completion)

Morita (i)

$$\begin{tikzcd}
\mathrm{gr}(\mathcal{G}_{g,1}) \arrow[rr, hook, "\tau"] \arrow[dr, dashed, "\exists ! \, \tau"'] & \circlearrowleft & \mathrm{Der}_\omega(\mathcal{L}(H)) \\
& \mathfrak{h}_{g,1}^{+} \arrow[ur, hook] &
\end{tikzcd}$$

(ii) $\tau : \mathrm{gr}(\mathcal{G}_{g,1}) \to \mathfrak{h}_{g,1}^{+}$ is not surjective $\Leftarrow$ Morita trace.

obstructions of the surjectivity

Morita trace

Galois obstruction (Oda, Nakamura, M. Matsumoto)

Enomoto-Satoh trace

⋱ various algebraic obstructions

Summary

The action of any mapping class $\in \mathcal{M}_{g,1}$ on the group ring $\mathbb{Q}\pi$ preserves

ⓐ the subset $\pi \subset \mathbb{Q}\pi \Rightarrow \mathrm{Der}(\mathcal{L}(H))$ (Johnson)

ⓑ the intersections of any two loops
$\Rightarrow \mathfrak{h}_{g,1}^{+} (\subset \mathrm{Der}_{\omega}(\mathcal{L}(H)))$ (Morita)

ⓒ the self-intersections of any single loop
$\Rightarrow \mathrm{Ker}(\delta|_{L^{+}}) \subsetneqq (\mathfrak{h}_{g,1}^{+})^{\wedge}$ (our result (II))

↓

geometric obstruction of the surjectivity of τ

Sources of τ : $\mathrm{gr}(\mathcal{I}_{g,1}) \neq \mathcal{I}_{g,1}, \neq \mathcal{M}_{g,1}$.

Extension of τ to the whole $\mathcal{M}_{g,1}$ or $\mathcal{I}_{g,1}$

- τ_1 and τ_2 to $\mathcal{M}_{g,1}$ (Morita)
- $^{\exists}$extension of τ to the whole $\mathcal{M}_{g,1}$ (Hain)
- explicit extensions (K., Day, Massuyeau, Day, ⋯)

Magnus expansion and total Johnson map

$$\hat{T} := \prod_{m=0}^{\infty} H^{\otimes m} \quad \text{completed tensor algebra}$$

$$\hat{T}_n := \prod_{m \geqq n} H^{\otimes m} \subset \hat{T} \quad \text{two-sided ideal} \quad (n \geqq 1)$$

Definition

$\theta : \pi \to \hat{T}$ (generalized) Magnus expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\forall x \in \pi$, $\theta(x) \equiv 1 + [x] \mod \hat{T}_2$

2) $\forall x, \forall y \in \pi$, $\theta(xy) = \theta(x)\theta(y)$

$$\Longrightarrow \theta : \widehat{\mathbb{Q}\pi} := \varprojlim_{n \to \infty} \mathbb{Q}\pi / I\pi^n \xrightarrow{\cong} \hat{T} \quad \text{algebra isom.}$$

where $I\pi := \operatorname{Ker}(\varepsilon : \mathbb{Q}\pi \to \mathbb{Q})$ augmentation ideal

$\sum a_i x_i \mapsto \sum a_i$

$\varphi \in \mathcal{M}_{g,1}$.

$$\begin{tikzcd}
\widehat{\mathbb{Q}\pi} \arrow[r, "\theta", "\cong"'] \arrow[d, "\varphi"'] & \hat{T} \arrow[d, dashed, "\exists!\, T^{\theta}(\varphi)"] \\
\widehat{\mathbb{Q}\pi} \arrow[r, "\theta", "\cong"'] & \hat{T}
\end{tikzcd}$$

$T^{\theta} : \mathcal{M}_{g,1} \hookrightarrow \operatorname{Aut}(\hat{T})$ total Johnson map (K.)

$\forall k \geqq 1$, $T^{\theta}|_{\mathcal{M}_{g,1}(k)} = 1 + \tau_k + \text{higher terms.}$

Massuyeau's improvement

Definition

$\theta : \pi \to \hat{T}$ symplectic expansion

$\overset{\text{def}}{\Longleftrightarrow}$

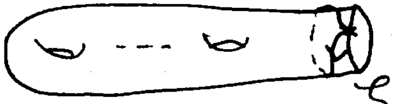
0) $\theta : \pi \to \hat{T}$ Magnus expansion

1) (group-like) $\forall x \in \pi \quad \Delta\theta(x) = \theta(x) \hat{\otimes} \theta(x)$

(where $\Delta : \hat{T} \to \hat{T} \hat{\otimes} \hat{T}$ coproduct)

2) $\theta(\zeta) = \exp(\omega) \left(= \sum_{k=0}^{\infty} \frac{1}{k!} \omega^k \right)$

(where



$\zeta \in \pi$ boundary loop)

(explicit examples K., Massuyeau, Kuno, Bene-K.-Kuno-Penner.)

Massuyeau θ : symplectic expansion

$$\begin{tikzcd}
\mathcal{I}_{g,1} \arrow[r, hook, "T^\theta"] \arrow[dr, dashed, "\exists 1"', "\tau^\theta"] & \mathrm{Aut}(\hat{T}) \arrow[d, phantom, "\circlearrowleft"', near start] \\
& (\mathfrak{h}_{g,1}^{+})^{\wedge} \arrow[u, "\exp"']
\end{tikzcd}$$

(<u>Our result</u> (I) : generalize $\tau^\theta : \mathcal{I}_{g,1} \hookrightarrow (\mathfrak{h}_{g,1}^+)^\wedge$ to the case $r \geqq 2$
without using expansions

$\Uparrow$

a completion of the Goldman Lie algebra (Kuno-K.)

§ 3, The completed Goldman Lie algebra

$I = [0,1] \subset \mathbb{R}$, $S^1 = I/_{0\sim 1} = \mathbb{R}/\mathbb{Z}$.

$S = \Sigma_{g,r}$, $g, r \geqq 1$. $E \subset \partial S$ as in § 0.

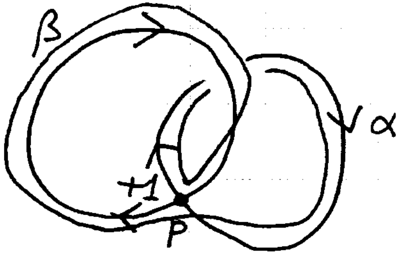
$\hat{\pi} = \hat{\pi}(S) = [S^1, S] = \pi_1(S)/\mathrm{conj}$ free loops on S

$|\ | : \pi_1(S) \to \hat{\pi}(S)$ quotient map

$\Pi S(*_1, *_2) := [(I, 0, 1), (S, *_1, *_2)]$ paths from $*_1$ to $*_2$ on S
$*_1, *_2 \in E$

$\mathbb{Q}\mathcal{C}(S, E) := \{ \mathbb{Q}\Pi S(*_1, *_2) \}_{*_1, *_2 \in E}$ small additive category

Goldman bracket



$\alpha, \beta \in \hat{\pi}$ in general position

$$[\alpha, \beta] := \sum_{p\in\alpha\cap\beta} \varepsilon(p;\alpha,\beta)\, |\alpha_p \beta_p| \in \mathbb{Q}\hat{\pi}$$

$\varepsilon(p;\alpha,\beta) \in \{\pm 1\}$ local intersection number

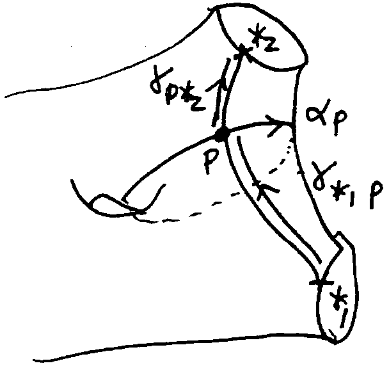
α_p (resp. β_p) $\in \pi_1(S,p)$ based loop along α (resp. β)

Goldman

[,] : well-defined

$(\mathbb{Q}\hat{\pi}, [\,,\,])$: Lie algebra

Action on $\mathbb{Q}\mathcal{C}(S,E)$



$\alpha \in \hat{\pi}$, $\gamma \in \Pi S(*_1, *_2)$ in general position

$$\sigma(\alpha)(\gamma) := \sum_{p\in\alpha\cap\gamma} \varepsilon(p;\alpha,\gamma)\, \gamma_{*_1 p}\, \alpha_p\, \gamma_{p *_2} \in \mathbb{Q}\Pi S(*_1, *_2)$$

Kuno-K.

σ : well-defined

$\sigma : \mathbb{Q}\hat{\pi} \to \mathrm{Der}\, \mathbb{Q}\mathcal{C}(S,E)$

Lie algebra homomorphism (injective)

Completion $n \geqq 0$

$\mathbb{Q}\hat{\pi}(n) := |I\pi_1(S,g)^n + \mathbb{Q}1| \subset \mathbb{Q}\hat{\pi}(S)$, indep. of the choice of $g \in S$

$1 \in \hat{\pi}$ constant loop.

$F_n \mathbb{Q}\Pi S(*_1, *_2) := \gamma_1 \, I\pi_1(S,g)^n \, \gamma_2 \subset \mathbb{Q}\Pi S(*_1, *_2)$

indep. of the choice of g, $\gamma_1 \in \Pi S(*_1, g)$ and $\gamma_2 \in \Pi S(g, *_2)$

Lemma $\forall n_1, \forall n_2 \geqq 0$

$[\mathbb{Q}\hat{\pi}(n_1), \mathbb{Q}\hat{\pi}(n_2)] \subset \mathbb{Q}\hat{\pi}(n_1+n_2-2)$

$\sigma(\mathbb{Q}\hat{\pi}(n_1))(F_{n_2}\mathbb{Q}\Pi S(*_1, *_2)) \subset F_{n_1+n_2-2}\mathbb{Q}\Pi S(*_1, *_2)$

$$\widehat{\mathbb{Q}\hat{\pi}} = \widehat{\mathbb{Q}\hat{\pi}}(S) := \varprojlim_{n\to\infty} \mathbb{Q}\hat{\pi}/\mathbb{Q}\hat{\pi}(n)$$

completed Goldman Lie algebra

$$\widehat{\mathbb{Q}\Pi S}(*_1, *_2) := \varprojlim_{n\to\infty} \mathbb{Q}\Pi S(*_1,*_2)/F_n\mathbb{Q}\Pi S(*_1,*_2) \quad (*_1, *_2 \in E)$$

$\rightsquigarrow \widehat{\mathbb{Q}\mathcal{C}}(S, E)$ small additive category

$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \to \mathrm{Der}\,\widehat{\mathbb{Q}\mathcal{C}}(S,E)$ Lie algebra homomorphism

<u>injective</u>

<u>coproduct</u> on $\widehat{\mathbb{Q}\mathcal{C}}(S,E)$

$$\Delta : \gamma \in \Pi S(*_1,*_2) \longmapsto \Delta(\gamma) := \gamma \hat{\otimes} \gamma \in \widehat{\mathbb{Q}\Pi S}(*_1,*_2) \hat{\otimes} \widehat{\mathbb{Q}\Pi S}(*_1,*_2)$$

$$L^+(S,E) := \{ u \in \widehat{\mathbb{Q}\hat{\pi}}(3) ; (\sigma(u) \hat{\otimes} \sigma(u))\Delta = \Delta\sigma(u) \text{ on } \widehat{\mathbb{Q}\mathcal{C}}(S,E) \}$$

$\subset \widehat{\mathbb{Q}\hat{\pi}}$ (closed) Lie subalgebra

$$\begin{tikzcd}
L^+(S,E) \arrow[r, hook, "\exp\circ\sigma"] \arrow[r, phantom, shift right=4, "\circlearrowleft"] & \operatorname{Aut}\widehat{\mathbb{Q}\mathcal{C}}(S,E) \quad \underline{\text{injective}} \\
& \mathcal{I}(S,E) \arrow[ul, dashed, "\exists !"] \arrow[u, "\text{injective (essentially due to Dehn-Nielsen)}"']
\end{tikzcd}$$

<u>Putman</u> $\mathcal{I}(S,E)$ is generated by some Dehn twists
if $g \geq 1$ and $r \geq 1$

<u>Kuno-K.</u> "explicit" formula for Dehn twists

$\tau : \mathcal{I}(S,E) \to L^+(S,E)$ <u>injective</u> group homomorphism

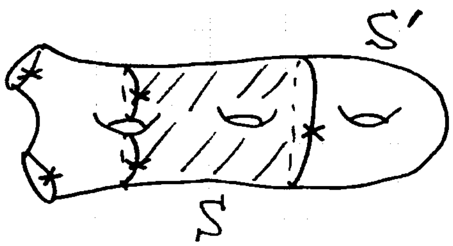
In the classical case $r = 1$

$$\begin{tikzcd}
\mathcal{I}_{g,1} \arrow[r, hook, "\tau"] \arrow[dr, "\text{Massuyean's } \tau^\theta"'] \arrow[r, phantom, shift right=4, "\circlearrowleft"] & L^+(S,E) \arrow[d, "\cong\ \lambda_\theta"] \\
& (\mathfrak{h}_{g,1}^+)^\wedge
\end{tikzcd}$$

$\lambda_\theta \Leftarrow \theta$: symplectic expansion
Kuno-K.

$\rightsquigarrow$ $L^+(S,E) = $ "$(\mathfrak{h}_{g,r}^+)^\wedge$"
for $\forall r \geq 2$

Naturality of $\tau : \mathfrak{g}(S,E) \hookrightarrow L^+(S,E)$



(S', E') : similar to (S,E)

$i : S \hookrightarrow S'$ embedding of surface

$i : \widehat{\mathbb{Q}\hat{\pi}}(S) \to \widehat{\mathbb{Q}\hat{\pi}}(S')$ inclusion homomorphism

$\iota : \mathcal{M}(S) \to \mathcal{M}(S')$ extending diffeo's by $1_{S' \setminus S}$

$\Rightarrow$ the diagram

$$\begin{tikzcd}
\mathfrak{g}(S,E) \arrow[r, "\iota"] \arrow[d, hook, "\tau"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{g}(S',E') \arrow[d, hook, "\tau"] \\
L^+(S,E) \arrow[r, "i"'] & L^+(S',E')
\end{tikzcd}$$

commutes

§4. cobracket

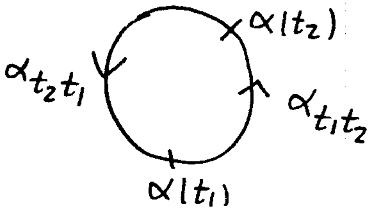
$\mathbb{Q}\hat{\pi}' := \mathbb{Q}\hat{\pi}/\mathbb{Q}1$. Lie algebra $(\because)$ $\mathbb{Q}1 \subset \mathrm{Center}(\mathbb{Q}\hat{\pi})$)

$|\ |' : \pi_1(S) \to \mathbb{Q}\hat{\pi}/\mathbb{Q}1$ quotient map.

Turaev cobracket

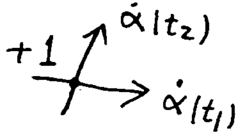
$\alpha \in \hat{\pi}$ in general position

$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 ;\ t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$ double points



$$\delta(\alpha) := \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1 t_2}|' \otimes |\alpha_{t_2 t_1}|' \in \mathbb{Q}\hat{\pi}' \otimes \mathbb{Q}\hat{\pi}'$$

$\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \in \{\pm 1\}$ local intersection number



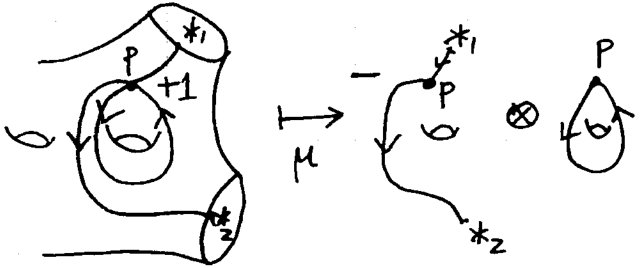
Turaev

$\delta : \mathbb{Q}\hat{\pi}' \to \mathbb{Q}\hat{\pi}' \otimes \mathbb{Q}\hat{\pi}'$ well-defined

$(\mathbb{Q}\hat{\pi}', [\ ,\ ], \delta)$: Lie bialgebra. --- Chas involutive

comodule structure on $\mathbb{Q}\widehat{\Pi S}(*_1,*_2)$

$*_1 \neq *_2$, $\gamma \in \Pi S(*_1,*_2)$ in general position



$\Gamma_\gamma := \{\text{double pts of } \gamma\} \subset S$

$p \in \Gamma_\gamma$, $\gamma^{-1}(p) = \{t_1^p, t_2^p\} \subset I$, $t_1^p < t_2^p$

$$\mu(\gamma) := -\sum_{p\in\Gamma_\gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))\,(\gamma_{0t_1^p}\gamma_{t_2^p1}) \otimes |\gamma_{t_1^pt_2^p}|'$$
$$\in \mathbb{Q}\Pi S(*_1,*_2) \otimes \mathbb{Q}\hat{\pi}'(S)$$

(In the case $*_1 = *_2$, we consider a degeneration of μ
$\Longrightarrow$ $\exists$ 1-parameter family of comodule structure maps
μ_t on $\mathbb{Q}\pi_1(S,*_1)$, $t \in \mathbb{Q}$)

Kuno-K.

$\underset{(\mu_t)}{\mu} : \mathbb{Q}\Pi S(*_1,*_2) \longrightarrow \mathbb{Q}\Pi S(*_1,*_2) \otimes \mathbb{Q}\hat{\pi}'(S)$ well-defined

$(\mathbb{Q}\Pi S(*_1,*_2), \sigma, \underset{(\mu_t)}{\mu})$: involutive $\mathbb{Q}\hat{\pi}'(S)$-bimodule

δ and μ (μ_t) are compatible with the filtrations $\{\mathbb{Q}\hat{\pi}(n)\}$ and $\{F_n \mathbb{Q}\Pi S(*_1,*_2)\}$

$\Longrightarrow$ $\widehat{\mathbb{Q}\hat{\pi}}(S)$: (complete) involutive Lie bialgebra

$\widehat{\mathbb{Q}\Pi S}(*_1,*_2)$: (complete) involutive $\widehat{\mathbb{Q}\hat{\pi}}(S)$-bimodule $(*_1,*_2 \in E)$

Consider $\delta|_{L^+} : L^+(S,E) \hookrightarrow \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\delta} \widehat{\mathbb{Q}\hat{\pi}} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}$

- $^\forall$mapping class $\in \mathcal{M}(S)$ preserves δ and μ (μ_t)
- compatibility of δ and μ (μ_t)

$\Downarrow$

Theorem (Kuno-K.)

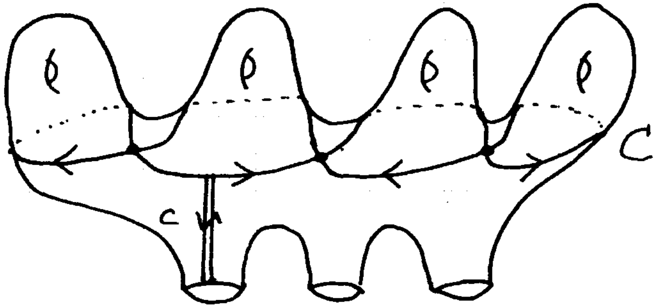
$$(\delta|_{L^+}) \circ \tau = 0 \ : \ \mathfrak{g}(S,E) \xrightarrow{\tau} L^+(S,E) \xrightarrow{\delta|_{L^+}} \widehat{\mathbb{Q}\hat{\pi}} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}$$

i.e., $\overline{\tau(\mathfrak{g}(S,E))}^{\text{Zariski closure}} \subset \operatorname{Ker}(\delta|_{L^+})$

geometric obstruction of the surjectivity of $\tau : \mathfrak{g}(S,E) \to L^+(S,E)$

Proposition $\delta|_{L^+(S,E)} \neq 0$ if genus$(S) \geq 2$

(pf)



$L(C) := |\frac{1}{2}(\log c)^2|$

$\in L^+(S,E)$

∴ C has a self intersection if $g \geq 2$

$\delta L(C) \neq 0 \in \widehat{\mathbb{Q}\hat{\pi}}(S) \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}(S)$ //

($\exp \sigma(L(C))$: generalized Dehn twist in the sense of Kuno)

Conjecture $\overline{\tau(\mathcal{I}(S,E))}^{\text{Zariski closure}} \underset{?}{⊜} \mathrm{Ker}(\delta|_{L^+(S,E)})$ —

too optimistic ?

We don't know how $\delta|_{L^+(S,E)}$ relates to known <u>algebraic</u> obstuctions.