"Mapping class groups and the Goldman-Turaev Lie bialgebra"

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S : connected compact oriented surface with $\partial S \neq \phi$ ($\Leftarrow$ technical reason)

$$\mathcal{M}(S) := \pi_0 \mathrm{Diff}_+(S \text{ rel } \partial S)$$
$$= \{\varphi : S \to S \,;\, \text{ori.pres. diffeo}, \ \varphi|_{\partial S} = \mathrm{id}_{\partial S}\} \Big/ \begin{array}{l}\text{isotopy fixing}\\ \partial S \ \underline{\text{pointwise}}\end{array}$$

the mapping class group of S

$E \subset \partial S$ finite subset with $\pi_0(E) \xrightarrow{\cong} \pi_0(\partial S)$

$$\mathcal{I}(S) := \mathrm{Ker}(\mathcal{M}(S) \longrightarrow \mathrm{Aut}(H_1(S, E; \mathbb{Z})))$$

The "smallest" Torelli group in the sense of Putman.

<u>Our motivation of research</u>:

A <u>linear</u> <u>approximation</u> of the mapping class group $\mathcal{M}(S)$ or $\mathcal{I}(S)$

(cf) G : a Lie group.
The Lie algebra $\mathrm{Lie}\, G = T_1 G$ is a linear approximation of G)

Our candidate for "the Lie algebra" of $\mathcal{M}(S)$ or $\mathcal{I}(S)$

<u>a Lie subalgebra</u> of <u>a completion</u> of the Goldman Lie algebra of S

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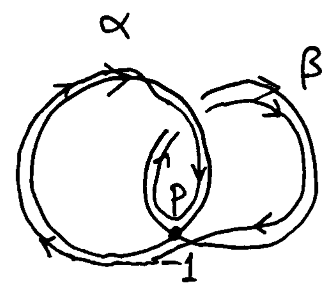
{ Turaev cobracket
"coproduct" }

Dehn twist formula
(Kuno-K.)

The Goldman Lie algebra

$\hat{\pi} = \hat{\pi}(S) \stackrel{\mathrm{def}}{=} [S^1, S] = \pi_1(S)/\mathrm{conj}$ the homotopy set of free loops on S

$|\ |: \pi_1(S) \to \hat{\pi}(S)$ forgetful map of the basepoint



$\alpha, \beta \in \hat{\pi}$ in general position

$$[\alpha, \beta] \stackrel{\mathrm{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta)\, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}$$

$\varepsilon(p; \alpha, \beta) \in \{\pm 1\}$ local intersection number

α_p (resp. β_p) $\in \pi_1(S, p)$ based loop at p along α (resp. β)

Theorem (Goldman)
- $[\ ,\ ]$: well-defined.
- $(\mathbb{Z}\hat{\pi}, [\ ,\ ])$: Lie algebra $/\mathbb{Z}$ ----- the Goldman Lie algebra of S

Action of $\mathbb{Z}\hat{\pi}$ on the free $\mathbb{Z}$-module over the fundamental groupoid ΠS

$*_0, *_1 \in E$

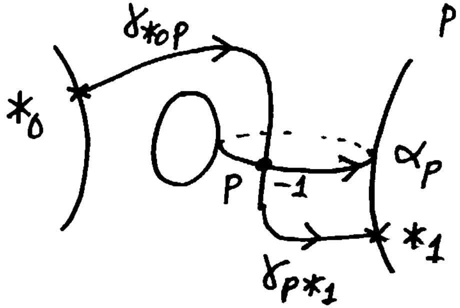
$\Pi S(*_0, *_1) := [([0,1], 0, 1), (S, *_0, *_1)]$

the homotopy set of paths on S from $*_0$ to $*_1$

$\rightsquigarrow$ the fundamental groupoid ΠS (restricted to E)

$\alpha \in \hat{\pi}(S)$, $\gamma \in \Pi S(*_0,*_1)$ in general position

$$\sigma(\alpha)(\gamma) \overset{\text{def}}{=} \sum_{p\in\alpha\cap\gamma} \varepsilon(p;\alpha,\gamma)\,\gamma_{*_0 p}\,\alpha_p\,\gamma_{p*_1} \in \mathbb{Z}\Pi S(*_0,*_1)$$

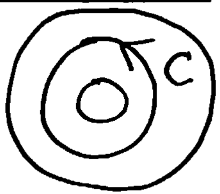


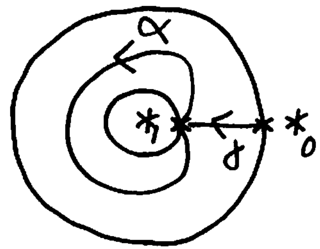
Theorem (Kuno-K.)

σ: well-defined

$\mathbb{Z}\Pi S(*_0,*_1)$: $\mathbb{Z}\hat{\pi}(S)$-module via σ

Dehn twist formula

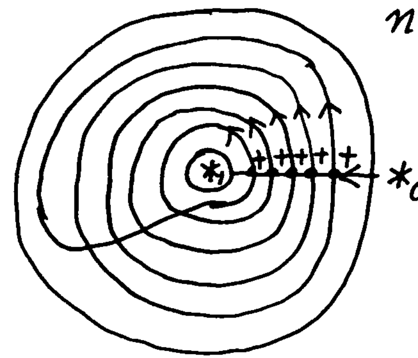
$S =$  annulus



$\alpha \in \pi_1(S,*_1)$

$\gamma \in \Pi S(*_0,*_1)$

$n=5$



$$\begin{cases} \sigma(C^n)(\gamma) = n\gamma\alpha^n & (\forall n \geq 1) \\ \sigma(C^n)(\alpha) = 0 \end{cases}$$

$f(x)$: polynomial in x

$$\begin{cases} \sigma(f(C))(\gamma) = \gamma\alpha f'(\alpha) \\ \sigma(f(C))(\alpha) = 0 \end{cases}$$

- n intersection points
- contribution of each point $= +\gamma\alpha^n$

$t_C \in \mathcal{M}(S)$ right-handed Dehn twist along C

$$\begin{cases} t_C(\gamma) = \gamma\alpha \\ t_C(\alpha) = \alpha \end{cases} \qquad \begin{cases} \log t_C(\gamma) = \gamma \log\alpha \\ \log t_C(\alpha) = 0 \end{cases} \qquad \log t_C = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(t_C - 1)^n$$

$\alpha f'(\alpha) = \log\alpha$

$$\Rightarrow f(x) = \int_1 \frac{1}{x} \log x \, dx = \frac{1}{2}(\log x)^2.$$

$$\frac{1}{2}(\log C)^2 \notin \mathbb{Q}\hat{\pi}$$

$$\in \widehat{\mathbb{Q}\hat{\pi}} := \varprojlim_{n\to\infty} \mathbb{Q}\hat{\pi} / |\mathbb{Q}1 + I\pi_1(S)^n| \qquad \left(\begin{matrix} 1 \in \pi_1(S) \\ \text{constant loop} \end{matrix} \right)$$

completion w.r. to the augmentation ideal $I\pi_1(S)$

<u>Theorem</u> (Kuno–K., arXiv: 1008.5017, 1109.6479)

S : connected compact oriented surface with $\partial S \neq \phi$

$C \subset S \setminus \partial S$ simple closed curve

$L(C) \overset{\text{def}}{=} \frac{1}{2}(\log C)^2 \in \widehat{\mathbb{Q}\hat{\pi}}(S)$ completion with respect to $I\pi_1(S)$

$\Longrightarrow \quad t_C = \exp(L(C)) \in \mathrm{Aut}(\widehat{\mathbb{Q}\pi S})$

<u>Remarks</u> (i) a highly non-trivial generalization is given by Massuyeau and Turaev (arXiv: 1109.5248)

(ii) $\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \to \mathrm{Der}(\widehat{\mathbb{Q}\Pi S})$ is injective. (an infinitesimal Dehn-Nielsen theorem)

(iii) $m \geqq 1$

$$\widehat{\mathbb{Q}\hat{\pi}}(m) := \varprojlim_{n\to\infty} \left(|\mathbb{Q}1 + I\pi_1(S)^m| \big/ |\mathbb{Q}1 + I\pi_1(S)^n| \right) \subset \widehat{\mathbb{Q}\hat{\pi}}$$

filtration on $\widehat{\mathbb{Q}\hat{\pi}}$, $\quad \widehat{\mathbb{Q}\hat{\pi}}(1) = \widehat{\mathbb{Q}\hat{\pi}}$

$$[\widehat{\mathbb{Q}\hat{\pi}}(m), \widehat{\mathbb{Q}\hat{\pi}}(\ell)] \subset \widehat{\mathbb{Q}\hat{\pi}}(m+\ell-2) \qquad \forall m, \forall \ell \geqq 1$$

$S = \Sigma_{g,1} =$ 

$$\Rightarrow \widehat{\mathbb{Q}\hat{\pi}}(m) \big/ \widehat{\mathbb{Q}\hat{\pi}}(m+1) \cong \left(H_1(\Sigma_{g,1}; \mathbb{Q})^{\otimes m} \right)^{\mathbb{Z}/m} \quad (\forall m \geqq 1)$$ cyclic invariants

$\widehat{\mathbb{Q}\hat{\pi}}(2) \big/ \widehat{\mathbb{Q}\hat{\pi}}(3) \cong \mathfrak{sp}_{2g}(\mathbb{Q})$ as Lie algebras.

Embedding of the Torelli group $\mathcal{I}(S)$ into the completed Goldman Lie algebra

exponential

$\exp \circ \sigma : \widehat{\mathbb{Q}\hat{\pi}}(S)(3) \to \mathrm{Aut}(\widehat{\mathbb{Q}\pi S})$ well-defined and injective

$u \longmapsto \exp \circ \sigma(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma(u)^n$

$\mathrm{Image}(\exp \circ \sigma) \subset \mathrm{Aut}(\widehat{\mathbb{Q}\pi S})$ subgroup ($\because$ Baker-Campbell-Hausdorff formula)

$$\begin{tikzcd}
\widehat{\mathbb{Q}\hat{\pi}}(S)(3) \arrow[r, hook, "\exp\circ\sigma"] & \mathrm{Aut}(\widehat{\mathbb{Q}\pi S}) \\
& \mathcal{I}(S) \arrow[ul, dashed, "\exists!\, \tau"] \arrow[u, hook]
\end{tikzcd}$$

injective (essentially due to Dehn-Nielsen)

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- Putman's generators of $\mathcal{I}(S)$ $\leftarrow$ Dehn twists
- Dehn twist formula (stated above)

$\tau : \mathcal{I}(S) \hookrightarrow \widehat{\mathbb{Q}\hat{\pi}}(S)(3)$ geometric Johnson homomorphism

$\widehat{\mathbb{Q}\hat{\pi}}(S)(3)$: too large

reduce the target of τ using ① coproduct, ② the Turaev cobracket.

① <u>coproduct</u>

$$\Delta : \mathbb{Q}\widehat{\Pi S}(*_0, *_1) \to \mathbb{Q}\widehat{\Pi S}(*_0, *_1) \hat{\otimes} \mathbb{Q}\widehat{\Pi S}(*_0, *_1)$$

$$\gamma \in \Pi S(*_0, *_1) \longmapsto \gamma \hat{\otimes} \gamma \qquad \text{coproduct}$$

(If $*_0 = *_1$, Δ = the usual coproduct on the completed groupring)

$$L^+(S, E) := \{ u \in \mathbb{Q}\hat{\hat{\pi}}(S)(3) : (\sigma(u) \hat{\otimes} \sigma(u))\Delta = \Delta \sigma(u) \}$$

$$\subset \mathbb{Q}\hat{\hat{\pi}}(S)(3) \quad \text{Lie subalgebra}$$

$\tau(\mathcal{I}(S)) \subset L^+(S, E)$ ($\because$ $^\forall$diffeomorphism preserves $\Pi S(*_0, *_1)$)

In the case $S = \Sigma_{g,1} =$  ,

- $\tau : \mathcal{I}(\Sigma_{g,1}) \hookrightarrow L^+(\Sigma_{g,1}, \{*\})$ is equivalent to Massuyeau's total Johnson map
- $\mathrm{gr}(L^+(\Sigma_{g,1}, \{*\})) := \bigoplus_{m=3}^{\infty} (L^+(\Sigma_{g,1}, \{*\}) \cap \mathbb{Q}\hat{\hat{\pi}}(m)) / (L^+(\Sigma_{g,1}, \{*\}) \cap \mathbb{Q}\hat{\hat{\pi}}(m+1))$

  $\mathrm{gr}(L^+(\Sigma_{g,1}, \{*\})) \cong \mathfrak{h}^+_{g,1}$ Morita's Lie algebra.

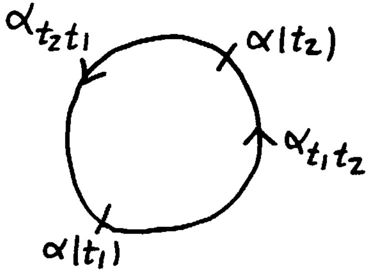
- $\mathrm{gr}(\tau) : \mathrm{gr}(\mathcal{I}(\Sigma_{g,1})) \to \mathrm{gr}(L^+(\Sigma_{g,1}, \{*\}))$

  is equivalent to the Johnson homomorphism.

② **Turaev cobracket** $\quad 1 \in \hat{\pi}$ constant loop $\quad (\mathbb{Z}1 < \mathrm{Center}\,\mathbb{Z}\hat{\pi})$

$$\delta : \mathbb{Z}\hat{\pi}/\mathbb{Z}1 \to (\mathbb{Z}\hat{\pi}/\mathbb{Z}1) \otimes (\mathbb{Z}\hat{\pi}/\mathbb{Z}1)$$

$\alpha \in \hat{\pi}$ in general position

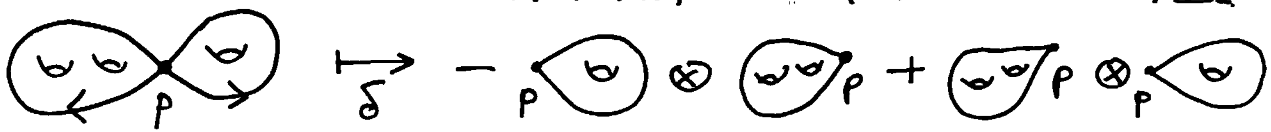


$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 \,;\, t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$ double points

$$\delta(\alpha) \overset{\mathrm{def}}{=} \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1 t_2}|' \otimes |\alpha_{t_2 t_1}|'$$

$\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \in \{\pm 1\}$ local intersection number

$$|\ |' : \mathbb{Z}\pi_1(S, \alpha(t_1)) \xrightarrow{|\ |} \mathbb{Z}\hat{\pi}(S) \xrightarrow{\text{quotient}} \mathbb{Z}\hat{\pi}/\mathbb{Z}1$$



Theorem (Turaev)

δ : well-defined.

$(\mathbb{Z}\hat{\pi}/\mathbb{Z}1, [\,,\,], \delta)$: Lie bialgebra

Compatibility Axiom of a Lie bialgebra

$$\delta([u, v]) = \mathrm{ad}(u)(\delta v) - \mathrm{ad}(v)(\delta u) \qquad \forall u, \forall v$$

$\Longrightarrow$ $\mathrm{Ker}\,\delta$: Lie subalgebra

δ extends to

$$\delta : \widehat{\mathbb{Q}\hat{\pi}} \to \widehat{\mathbb{Q}\hat{\pi}} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}$$

Moreover, we have

$$\delta(\widehat{\mathbb{Q}\hat{\pi}}(n)) \subset \sum_{p+q=n-2} \widehat{\mathbb{Q}\hat{\pi}}(p) \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}(q) \quad (\forall n \geq 1)$$

(<u>Theorem</u> (Kuno-K.) $\forall *_0, \forall *_1 \in E \subset \partial S$

$\widehat{\mathbb{Q}\Pi S}(*_0, *_1)$: $\widehat{\mathbb{Q}\hat{\pi}}$ - bimodule

$\Downarrow$ $\forall$ diffeomorphism preserves the self-intersections of any curves on S

<u>Corollary</u> $\tau(\mathcal{I}(S)) \subset \operatorname{Ker} \delta$ —

In the case $S = \Sigma_{g,1}$,

(<u>Theorem</u> (Massuyeau-Turaev, Kuno-K, independently)

$\mathrm{gr}(\delta) =$ Schedler's cobracket

<u>Corollary</u> $\mathrm{gr}(\delta)$ covers the Morita traces $\mathrm{Tr} : \mathfrak{h}_{g,1}^{+} \twoheadrightarrow \bigoplus_{k=1}^{\infty} \mathrm{Sym}^{2k+1} H_1(\Sigma_{g,1}; \mathbb{Q})$ —

(cf. Morita proved $\mathrm{Tr} \circ \mathrm{gr}(\tau) = 0$)