for any $\xi_1(z)\frac{d}{dz}$ and $\xi_2(z)\frac{d}{dz} \in W_1^\sharp$. In fact, for $\xi(z)\frac{d}{dz} \in W_1^\sharp$,

$$\begin{aligned}
&d(\frac{dz^2}{(z-t)^2})(\xi(z)\frac{d}{dz}) = 2\frac{dz^2}{z-t}\frac{d}{dz}(\frac{\xi(z)-\xi(t)}{z-t}) \\
=&2\frac{dz^2}{z-t}\frac{d}{dz}(\frac{1}{z-t}(\xi(z)-\xi(t)-(z-t)\xi'(t)-\frac{1}{2}(z-t)^2\xi''(t))) \\
+&2\frac{dz^2}{z-t}\frac{d}{dz}(\xi'(t)+\frac{1}{2}(z-t)\xi''(t)) \\
\equiv&2\frac{dz^2}{z-t}\frac{d}{dz}(\xi'(t)+\frac{1}{2}(z-t)\xi''(t)) \quad \mod \mathbf{C}[t,z]dz^2 \\
=&2\frac{dt^{-1}dz^2}{z-t}\nabla_1^t(\xi(z)\frac{d}{dz}).
\end{aligned}$$

Hence $\nabla_1^t d(\frac{dz^2}{(z-t)^2}) \equiv 2(\nabla_1^t)^2\frac{dt^{-1}dz^2}{z-t} = 0 \mod \mathbf{C}[t,z]dtdz^2$.

Therefore the j-th term of

$$-d\Theta_{n;1} = \sum_{j=1}^{n}\oint_{t;z_1}\nabla_1^t d\left(\frac{dz_j{}^2}{(t-z_j)^2}\right)\prod_{i\neq j}\frac{dz_i{}^2}{(t-z_i)^2}$$

is regular on $\bigcap_{i\neq j}U_i$ for $j \geq 2$ and the first term vanishes. This means $d\Theta_{n;1} \in C^2(W_1; B^{n-2}(\mathfrak{U};\mathcal{Q}_n))$, i.e.,

$$d\Theta_{n;1} = 0 \in C^2(W_1; H^{n-2}(\mathfrak{U};\mathcal{Q}_n)),$$

as was to be shown.

We define a 1 cocycle $1_2 \otimes \theta_n \in C^1(L_0; 1_2 \otimes S^{n-1}(Q^\times/Q))$ by

$$\theta_n := 3\delta_2 q_0{}^{n-1} + 2(n-1)\delta_1 q_{-1} q_0{}^{n-2},$$

where $q_\nu := (z^{\nu-2}dz^2 \mod Q) \in Q^\times/Q$ and $\delta_k(z^{l+1}\frac{d}{dz}) := \delta_{k,l}$ (Kronecker's delta). The Schapiro isomorphism (3.3)

(5.6) $\quad H^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta;\mathcal{Q}_n)) \to H^1(L_0; 1_2 \otimes (Q^\times/Q)^{\otimes n-1})$

maps the class $\Theta_{n;1}$ to the class $(n-1)^{-1}1_2 \otimes \theta_n$.

LEMMA 5.7. *If* $n \geq 2$,

$$1_2 \otimes \theta_n \neq 0 \in H^1(L_0; 1_2 \otimes S^{n-1}(Q^\times/Q)).$$

Especially we obtain $\Theta_{n;1} \neq 0 \in H^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta; \mathcal{Q}_n))$.

PROOF: Assume $1_2 \otimes \theta_n = d(1_2 \otimes \alpha)$ for some

$$1_2 \otimes \alpha = 1_2 \otimes \sum \alpha_{i_{-1}i_0\cdots i_s} {q_1}^{i_{-1}} {q_0}^{i_0} \cdots {q_{-s}}^{i_s} \in (1_2 \otimes S^{n-1}(Q^\times/Q))^{e_0},$$

($\alpha_{i_{-1}i_0\cdots i_s} \in \mathbf{C}$). If $s_0 := \max\{s; \exists \alpha_{i_{-1}i_0\cdots i_s} \neq 0 \text{ and } i_s \geq 1\}$ is greater than 1, then $(z^{s_0+2}\frac{d}{dz})\alpha = \theta_n(z^{s_0+2}\frac{d}{dz}) = 0$, which contradicts the definition of s_0. Hence $s_0 \leq 1$, i.e., $\alpha = \sum_{k=0}^{n-3} a_k {q_1}^k {q_0}^{n-k-2} {q_{-1}}^{k+2}$ $(a_k \in \mathbf{C})$. Then we have

$$(z^3 \frac{d}{dz})\alpha = 3 \sum_{k=0}^{n-3} (k+3) a_k {q_1}^{k+1} {q_0}^{n-k-2} {q_{-1}}^{k+1},$$

which contradicts $(z^3\frac{d}{dz})\alpha = \theta_n(z^3\frac{d}{dz}) = 3{q_0}^{n-1}$. Consequently $1_2 \otimes \theta_n \neq 0 \in H^1(L_0; 1_2 \otimes S^{n-1}(Q^\times/Q))$, as was to be shown.

Consider the case $n = 2$. Then the Schapiro isomorphism (5.6) maps the class

$$(z_1 - z_2)^{-1}(\nabla_2^{z_2} {dz_1}^2 - \nabla_2^{z_1} {dz_2}^2) + (z_1 - z_2)^{-4}(\nabla_0^{z_1} - \nabla_0^{z_2}) {dz_1}^2 {dz_2}^2$$
$$\in H^1(W_1; \mathcal{Q}_2(\mathbf{C}^2 - \Delta))^{\mathfrak{S}_2}$$

to the class $1_2 \otimes \theta_2 \in H^1(L_0; 1_2 \otimes (Q^\times/Q))$. Lemma 5.2 for $n = 2$ follows from Lemma 5.7.

For the rest of this section we assume $n \geq 3$. Lemma 5.2 is reduced to the following

ASSERTION 5.8.

$$\Theta_{n;1} = (-1)^{n-1}\Theta_{n;2} \in C^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta; \mathcal{Q}_n)).$$

In fact, the cochain $\Theta_{n;1}$ (resp. $\Theta_{n;2}$) is invariant under any element in $\mathfrak{S}_n$ fixing the letter 1 (resp. the letter 2), and so the assertion implies the invariance of $\Theta_{n;1}$ under the whole $\mathfrak{S}_n$, i.e.,

$$\Theta_{n;1} \in H^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta; \mathcal{Q}_n))^{\mathfrak{S}_n}.$$

But Lemma 5.7 asserts $\Theta_{n;1} \neq 0 \in H^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta; \mathcal{Q}_n))$. Hence Lemma 5.2 follows.

To prove the assertion, we construct a 1 cochain of W_1 with values in $Z^{n-2}(\mathfrak{U} \cup \mathfrak{V}; \mathcal{Q}_n)$ for the union $\mathfrak{U} \cup \mathfrak{V}$. Here we identify $V_1 \in \mathfrak{V}$ with $U_2 \in \mathfrak{U}$.

The $(n-2)$-nerve of the covering $\mathfrak{U}\cup\mathfrak{V}$ is parametrized by the index set

$$\mathcal{I} := \left\{ \begin{array}{l} I =(I^{(1)}, I^{(2)}); \\ \quad 0 \le p = p(I) \le n-2 \\ \quad I^{(1)} = \{i_0, i_1, \dots, i_p\}, \quad 2 \le i_0 < i_1 < \cdots < i_p \le n \\ \quad I^{(2)} = \{i_{p+1}, \dots, i_{n-2}\}, \quad 3 \le i_{p+1} < \cdots < i_{n-2} \le n \end{array} \right\}$$

The index $I \in \mathcal{I}$ corresponds to the open set

$$U_I := U_{i_0} \cap \cdots \cap U_{i_p} \cap V_{i_{p+1}} \cap \cdots \cap V_{i_{n-2}}.$$

We associate each index $I \in \mathcal{I}$ with the unoriented graph Γ_I with n vertices corresponding the letters $1, 2, \cdots, n$ and $(n-1)$ unoriented edges

$$(1, i_0), \dots, (1, i_p), (2, i_{p+1}), \dots, (2, i_{n-2}).$$

For each $I \in \mathcal{I}$, $\alpha_I \in C^1(W_1; \mathcal{Q}_n(U_I))$ and $\varepsilon_I = \pm 1$ are defined as follows:

(1). If the graph Γ_I is not connected, then $\alpha_I := 0$ and $\varepsilon_I := 1$.

(2). If the graph Γ_I is connected and $2 \in I^{(1)}$, then we define

$$\begin{aligned} \alpha_I &= \alpha_{2i_1\dots i_p;i_{p+1}\dots i_{n-2}} \\ &:= \oint_{t;z_1} \nabla_1^t \frac{dz_1{}^2}{(t-z_1)^2} \Big(\prod_{i \in I^{(1)}} \frac{dz_i{}^2}{(t-z_i)^2} \Big) \Big(\prod_{i \in I^{(2)}} \frac{(2t + z_i - 3z_2)dz_i{}^2}{(z_i - z_2)^3} \Big) \\ \varepsilon_I &= \varepsilon_{2i_1\dots i_p;i_{p+1}\dots i_{n-2}} := (-1)^p \operatorname{sign} \begin{pmatrix} i_1, \cdots, i_{n-2} \\ 3, 4, \cdots, n \end{pmatrix}. \end{aligned}$$

The sign is well defined because $I^{(1)} \cap I^{(2)} = \emptyset$.

(3). If the graph Γ_I is connected and $2 \notin I^{(1)}$, then $\sharp(I^{(1)} \cap I^{(2)}) = 1$. Suppose $i_{\nu_1} = i_{\nu_2} = k$ and $\nu_1 \le p < \nu_2$. α_I is defined by

$$\begin{aligned} \alpha_I &= \alpha_{i_0\dots i_p;i_{p+1}\dots i_{n-2}} \\ &:= \oint_{t;z_1} \nabla_1^t \frac{dz_1{}^2}{(t-z_1)^2} \frac{(2t + z_2 - 3z_k)dz_2{}^2}{(z_k - z_2)^3} \Big(\prod_{i \in I^{(1)}} \frac{dz_i{}^2}{(t-z_i)^2} \Big) \\ &\qquad \times \Big(\prod_{i \in I^{(2)} - \{k\}} \frac{(2t + z_i - 3z_2)dz_i{}^2}{(z_i - z_2)^3} \Big). \end{aligned}$$

Since

$$\frac{2t + z_2 - 3z_k}{(t-z_k)^2 (z_k - z_2)^3} = \frac{2t + z_k - 3z_2}{(t-z_2)^2 (z_k - z_2)^3} - \frac{1}{(t-z_2)^2 (t-z_k)^2},$$

we have

$$\alpha_I = \alpha_{i_0 \dots i_p ; i_{p+1} \dots i_{n-2}} = \alpha_{2 i_0 \overset{\nu_1}{\widehat{\dots}} i_p ; i_{p+1} \dots i_{n-2}} - \alpha_{2 i_0 \dots i_p ; i_{p+1} \overset{\nu_2}{\widehat{\dots}} i_{n-2}}$$

on the open set $U_2 \cap U_I$. So, if I_1 and $I_2 \in \mathcal{I}$ are defined by

$$\text{(5.9)} \qquad \begin{aligned} I_1^{(1)} &= I^{(1)} \cup \{2\} - \{k\}, & I_1^{(2)} &= I^{(2)}, \\ I_2^{(1)} &= I^{(1)} \cup \{2\}, & I_2^{(2)} &= I^{(2)} - \{k\}, \end{aligned}$$

we have

$$\text{(5.10)} \qquad \alpha_I = \alpha_{I_1} - \alpha_{I_2} \quad \text{on } U_2 \cap U_I.$$

Since $p(I_1) = p$ and $p(I_2) = p - 1$, we have $(-1)^{\nu_1} \varepsilon_{I_1} = (-1)^{\nu_2 + 1} \varepsilon_{I_2}$. Thus we define

$$\text{(5.11)} \qquad \varepsilon_I = (-1)^{\nu_1} \varepsilon_{I_1} = (-1)^{\nu_2 + 1} \varepsilon_{I_2}.$$

Consider the $(n-2)$ cochain $f = \{f_I\}_{I \in \mathcal{I}} \in C^1(W_1; C^{n-2}(\mathfrak{U} \cup \mathfrak{V}; \mathcal{Q}_n))$ defined by

$$f_I := \varepsilon_I \alpha_I \in C^1(W_1; \mathcal{Q}_n(U_I)), \quad I \in \mathcal{I}.$$

Similarly the $(n-1)$ nerve of the covering $\mathfrak{U} \cup \mathfrak{V}$ is also parametrized by the index set

$$\mathcal{J} := \left\{ \begin{aligned} J =& (J^{(1)}, J^{(2)}); \\ & 0 \le p = p(J) \le n - 1 \\ & J^{(1)} = \{j_0, j_1, \dots, j_p\}, \quad 2 \le j_0 < j_1 < \dots < j_p \le n \\ & J^{(2)} = \{j_{p+1}, \dots, j_{n-1}\}, \quad 3 \le j_{p+1} < \dots < j_{n-1} \le n \end{aligned} \right\}$$

The open set U_J and the unoriented graph Γ_J are define in the same way as Γ_I. We shall prove the coboundary δf vanishes on U_J for each $J \in \mathcal{J}$: $(\delta f)_J = 0$.

(1). If the graph Γ_J is not connected, then no subgraph of Γ_J obtained by eliminating one edge is connected. Hence $(\delta f)_J = 0$.

(2). If the graph Γ_J is connected and $2 \in J^{(1)}$, then $\sharp(J^{(1)} \cap J^{(2)}) = 1$ and $j_0 = 2$. Suppose $j_{\nu_1} = j_{\nu_2} = k$ and $\nu_1 \le p < \nu_2$. We define $I \in \mathcal{I}$ by

$$I^{(1)} = J^{(1)} - \{2\}, \quad I^{(2)} = J^{(2)}$$

and I_1 and $I_2 \in \mathcal{I}$ by (5.9). We notice $p(I) = p - 1$ and $i_{\nu_1 - 1} = i_{\nu_2 - 1} = k$. It follows from (5.10) and (5.11)

$$f_I = \varepsilon_I (\alpha_{I_1} - \alpha_{I_2}) = -(-1)^{\nu_1} f_{I_1} - (-1)^{\nu_2} f_{I_2}.$$

Therefore, by a consideration of the connectivity of subgraphs of Γ_J, we obtain

$$(\delta f)_J = f_I + (-1)^{\nu_1} f_{I_1} + (-1)^{\nu_2} f_{I_2} = 0,$$

as was to be shown.

(3). If the graph Γ_J is connected and $2 \notin J^{(1)}$, then $\sharp(J^{(1)} \cap J^{(2)}) = 2$. Suppose $j_{\nu_1} = j_{\nu_2} = k$, $j_{\mu_1} = j_{\mu_2} = l$, $\nu_1 \leq p < \nu_2$, $\mu_1 \leq p < \mu_2$ and $3 \leq k < l \leq n$. We define $J_a \in \mathcal{J}$, $(a = 1, 2, 3, 4)$, by

$$(5.12) \qquad \begin{aligned} J_1^{(1)} &= J^{(1)} \cup \{2\} - \{l\}, & J_1^{(2)} &= J^{(2)}, \\ J_2^{(1)} &= J^{(1)} \cup \{2\}, & J_2^{(2)} &= J^{(2)} - \{l\}, \\ J_3^{(1)} &= J^{(1)} \cup \{2\} - \{k\}, & J_3^{(2)} &= J^{(2)}, \\ J_4^{(1)} &= J^{(1)} \cup \{2\}, & J_4^{(2)} &= J^{(2)} - \{k\}. \end{aligned}$$

The graph Γ_{J_a} is connected and $2 \in J_a^{(1)}$, and so we define I_a, I_{a1} and $I_{a2} \in \mathcal{I}$ as in (2). Clearly we have

$$(5.13) \qquad I_{11} = I_{31}, \quad I_{12} = I_{41}, \quad I_{21} = I_{32} \quad \text{and} \quad I_{22} = I_{42}.$$

As was proved in (2), we have

$$-f_{I_a} = (-1)^{\nu_1(a)} f_{I_{a1}} + (-1)^{\nu_2(a)} f_{I_{a2}},$$

where $\nu_1(a)$ and $\nu_2(a)$ are given by the 2×4 matrix

$$(\nu_b(a)) = \begin{pmatrix} \nu_1 + 1 & \nu_1 + 1 & \mu_1 & \mu_1 + 1 \\ \nu_2 & \nu_2 + 1 & \mu_2 & \mu_2 \end{pmatrix}.$$

It follows from (5.12) and (5.13)

$$(\delta f)_J = (-1)^{\nu_1} f_{I_3} + (-1)^{\mu_1} f_{I_1} + (-1)^{\nu_2} f_{I_4} + (-1)^{\mu_2} f_{I_2} = 0,$$

as was to be shown.

Consequently the 1 cochain $f = \{f_I\}_{I \in \mathcal{I}}$ of W_1 has its value in $Z^{n-2}(\mathfrak{U} \cup \mathfrak{V}; \mathcal{Q}_n)$, while

$$\begin{aligned} f|_{\mathfrak{U}} &= \varepsilon_{23\cdots n;}\alpha_{23\cdots n;} = \Theta_{n;1} \\ f|_{\mathfrak{V}} &= \varepsilon_{2;3\cdots n}\alpha_{2;3\cdots n} = (-1)^n \alpha_{2;3\cdots n}. \end{aligned}$$

Now we have

$$\begin{aligned} &\oint_{t;z_2} \nabla_1^t \frac{dz_1{}^2 dz_2{}^2}{(t - z_1)^2 (t - z_2)^2} \Big(\prod_{i=3}^n \frac{(2t + z_i - 3z_2) dz_i{}^2}{(z_i - z_2)^3}\Big) \\ =&\oint_{t;z_2} \nabla_1^t \frac{dz_1{}^2 dz_2{}^2}{(t - z_1)^2 (t - z_2)^2} \Big(1 + 2(t - z_2) \sum_{i=3}^n \frac{1}{z_i - z_2}\Big)\Big(\prod_{i=3}^n \frac{dz_i{}^2}{(z_i - z_2)^2}\Big) \\ =&\Theta_{n;2}. \end{aligned}$$

Computing the residue at $t = \infty$, we find that

$$\alpha_{2;3\cdots n} + \Theta_{n;2}$$
$$= (\oint_{t;z_1} + \oint_{t;z_2})\nabla_1^t \frac{dz_1{}^2 dz_2{}^2}{(t-z_1)^2(t-z_2)^2}(\prod_{i=3}^{n} \frac{(2t+z_i-3z_2)dz_i{}^2}{(z_i-z_2)^3})$$

is regular on $V_3 \cap V_4 \cap \cdots \cap V_n$. Therefore we obtain

$$\Theta_{n;1} = f|_{\mathfrak{U}} = f|_{\mathfrak{V}} = (-1)^{n-1}\Theta_{n;2} \in C^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta; \mathcal{Q}_n)).$$

This completes the proof of Assertion 5.8 and Lemma 5.2.

6. Sheaves of cohomology groups.

Let $\overline{x} \in (C, p, z) \in M_{g,\rho}$, $p_1 \in C_\rho$ and let w be a coordinate centered at the point p_1. We denote by $\overline{\chi}^{\overline{x}} = \overline{\chi}^{\overline{x}}_{p_1,w}$ the inverse of the isomorphism (2.8). Namely $\overline{\chi}^{\overline{x}}$ is the composite map

$$\overline{\chi}^{\overline{x}}_{p_1,w} : H^n(W_1; \bigwedge^n Q) \cong H^n(\mathbf{C}\{z\}\frac{d}{dz}; \bigwedge^n \mathbf{C}\{z\}dz^2)$$
$$\overset{w^*}{\cong} H^n(L(C^\times); (\mathcal{Q}_n)^{\mathfrak{S}_n}_{(p_1,\dots,p_1)}) \overset{(\mathrm{ev})^{-1}}{\to} H^n(L(C^\times) : \bigwedge^n Q(C_\rho))$$
$$= H^n((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^n T^*_{\overline{x}} M_{g,\rho}).$$

Similarly, for $x = (C, p, z, p_1) \in C_{g,\rho}$ and a coordinate w centered at p_1, we define an isomorphism $\chi^x = \chi^x_w$

$$\chi^x_w : H^n(L_0; \bigwedge^n Q^1) \to H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T^*_x C_{g,\rho})$$

as the inverse of the evaluation map in (2.10). Clearly the isomorphisms $\overline{\chi}^{\overline{x}}_{p_1,w}$ and χ^x_w preserve the multiplicative structures.

In this section we prove that the isomorphism χ^x_w does not depend on the coordinate w and that the isomorphism $\overline{\chi}^{\overline{x}}_{p_1,w}$ does not depend on the coordinate w and the point $p_1 \in C_\rho$. These imply

PROPOSITION 6.1. *The isomorphisms χ^x and $\overline{\chi}^{\overline{x}}$ induce the isomorphisms of complex analytic vector bundles*

$$\chi : C_{g,\rho} \times H^n(L_0; \bigwedge^n Q^1) \to \coprod_{x \in C_{g,\rho}} H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T^*_x C_{g,\rho})$$
$$\overline{\chi} : M_{g,\rho} \times H^n(W_1; \bigwedge^n Q) \to \coprod_{x \in M_{g,\rho}} H^n((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^n T^*_{\overline{x}} M_{g,\rho}).$$

In the succeeding sections we regard the vector bundles at the RHS's as trivial constant sheaves via the isomorphisms χ and $\overline{\chi}$. Then we have the following isomorphisms

$$(6.2)\qquad \begin{aligned} \chi &: H^n(L_0; \bigwedge^n Q^1) \cong H^0(C_{g,\rho}; H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T_x^* C_{g,\rho})) \\ \overline{\chi} &: H^n(W_1; \bigwedge^n Q) \cong H^0(M_{g,\rho}; H^n((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^n T_{\overline{x}}^* M_{g,\rho})). \end{aligned}$$

We prove first that the map χ_w^x does not depend on the coordinate w. From Corollary 4.12 the algebra $\bigoplus_{n\geq 0} H^n(L_0; \bigwedge^n Q^1)$ is generated by the classes ϵ and κ_n's. Hence it suffices to show that $\chi_w^x(\epsilon)$ and $\chi_w^x(\kappa_n)$ are independent of the coordinate w.

To investigate $\chi_w^x(\epsilon)$, we recall the notion of the residues of meromorphic quadratic differentials. Let U be a Riemann surface, p_1 a point of U, and λ an integer. As in §1, $Q^\lambda(U, p_1)$ denotes the space of meromorphic quadratic differentials on U with a pole only at the point p_1 of order $\leq \lambda$.

With respect to a coordinate w centered at $p_1 \in U$ we expand a meromorphic quadratic differential $q \in Q^2(U, p_1)$ to obtain

$$q = (a_{-2}w^{-2} + a_{-1}w^{-1} + \text{ regular terms})dw^2.$$

The complex number a_{-2} does not depend on the choice of coordinates w. We call it the *residue* of $q \in Q^2(U, p_1)$ at p_1 and denote

$$\operatorname{Res}_{p_1} q := a_{-2}.$$

Thus we have a natural extension of $L(U, p_1)$ modules

$$(6.3)\qquad 0 \to Q^1(U, p_1) \hookrightarrow Q^2(U, p_1) \overset{\operatorname{Res}}{\to} \mathbf{C} \to 0,$$

which we call the ***residual extension.***

With a point $x = (C, p, z, p_1) \in C_{g,\rho}$ we associate a meromorphic quadratic differential $q_0(x) \in Q^2(C_\rho, p_1)$ satisfying $\operatorname{Res}_{p_1} q_0(x) = 1$. Its coboundary $d(q_0(x)) \in C^1(L(C^\times, p_1); Q^2(C_\rho, p_1))$ has a value in the space $Q^1(C_\rho, p_1)$. The cohomology class defined by

$$[d(q_0(x))] \in H^1(L(C^\times, p_1); Q^1(C_\rho, p_1))$$

is independent of the choice of differentials $q_0(x) \in Q^2(C_\rho, p_1)$ and is mapped to $\epsilon \in H^1(L_0; Q^1)$ under the isomorphism (2.10). Thus $\chi_W^x(\epsilon) = [d(q_0(x))]$ is independent of the coordinate w.

Next we investigate the class $\chi_w^x(\kappa_n)$. Recall $H^q(L_0; 1_1 \otimes \bigwedge^p Q) = 0$ for $q \leq p$ and $H^q(W_1; \bigwedge^p Q) = 0$ for $q < p$. Hence, in a similar way to §2, we obtain

PROPOSITION 6.4. *For* $x = (C, p, z, p_1) \in C_{g,\rho}$, *we have*

$$H^q(L(C^\times, p_1); T^*_{p_1}C^\times \otimes \bigwedge^n T^*_{\overline{x}}M_{g,\rho}) = 0,$$

if $q \leq n$, *where* $\overline{x} = \pi_{g,\rho}(x) = (C, p, z) \in M_{g,\rho}$. *If* $q = n+1$, *the evaluation map at the point* p_1 *induces an isomorphism*

$$H^{n+1}(L(C^\times, p_1); T^*_{p_1}C^\times \otimes \bigwedge^n T^*_{\overline{x}}M_{g,\rho}) = H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q).$$

For an open Riemann surface U we denote by $F(U)$ the $L(U)$ module consisting of all complex analytic functions on U. From $H^0(L_0; 1_1 \otimes S^n F) = 0$ together with a similar argument to §2 we have

PROPOSITION 6.5. *For* $x \in (C, p, z, p_1) \in C_{g,\rho}$, *we have*

$$H^0(L(C^\times, p_1); T^*_{p_1}C^\times \otimes S^n F(C_\rho)) = 0 \quad \text{and}$$
$$H^1(L(C^\times, p_1); T^*_{p_1}C^\times \otimes S^n F(C_\rho)) = H^1(L_0; 1_1 \otimes S^n F).$$

The latter isomorphism is induced by the evaluation map at the point p_1.

For $x = (C, p, z, p_1) \in C_{g,\rho}$ and a coordinate w centered at p_1, we define

$$\chi^x_w : H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q) \to H^{n+1}(L(C^\times, p_1); T^*_{p_1}C^\times \otimes \bigwedge^n T^*_{\overline{x}}M_{g,\rho})$$
$$\chi^x_w : H^1(L_0; 1_1 \otimes S^n F) \to H^1(L(C^\times, p_1); T^*_{p_1}C^\times \otimes S^n F(C_\rho))$$

as the inverses of the evaluation maps in Propositions 6.4 and 6.5 respectively. Here $\overline{x} = \pi_{g,\rho}(x) = (C, p, z) \in M_{g,\rho}$. These map also preserve the multiplicative structures.

Now from Theorem 5.1(2) we have

$$\chi^x_w(\epsilon_1 \epsilon^n) = \chi^x_w(\eta_n)\chi^x_w(\kappa_n).$$

What is proved above implies that $\chi^x_w(\epsilon_1\epsilon^n)$ does not depend on the coordinate w. Considering the residue of meromorphic 1 forms, we find that $\chi^x_w(\eta_n)$ does not depend on the coordinate w. On the other hand the cup product by the class $\chi^x_w(\eta_n)$ is injective from Lemma 4.10. Therefore $\chi^x_w(\kappa_n)$ is independent of the coordinate w, and so is the map χ^x_w : $H^n(L_0; \bigwedge^n Q^1) \to H^n((\mathfrak{O}_\rho)_x; \bigwedge^n T^*_x C_{g,\rho})$. This completes the proof of the first half of Proposition 6.1. It should be remarked that the map

$\overline{\chi}^{\bar{x}}_{p_1,w} = \overline{\chi}^{\bar{x}}_{p_1}$ does not depend on the coordinate w because the algebra $\bigoplus_{n\geq 0} H^n(W_1; \bigwedge^n Q)$ is generated by the classes κ_n's.

In order to prove the second half, it suffices to show that the map $\overline{\chi}^{\bar{x}}_{p_1} = \overline{\chi}^{\bar{x}}_{p_1,w}$ does not depend on the point $p_1 \in C_\rho$. As was proved in §2, the restriction of the sheaf $\mathcal{F} := H^n(L(C^\times); \mathcal{Q}_n)^{\mathfrak{S}_n}$ to the diagonal $\Delta = \Delta(C_\rho) \subset (C_\rho)^n$ is a trivial constant sheaf. From the definition of the evaluation map (2.8), the map

$$\{\overline{\chi}^{\bar{x}}_{p_1}\}_{p_1 \in C_\rho} : C_\rho \times H^n(W_1; \bigwedge^n Q) \to \mathcal{F}|_\Delta$$

is locally constant, so that it is an isomorphism of sheaves over $C_\rho = \Delta(C_\rho)$. Hence the map $\overline{\chi}^{\bar{x}}_{p_1}$ does not depend on the point $p_1 \in C_\rho$. This completes the proof of Proposition 6.1.

We are able to define an analogue of teh fiber integral in a similar way to that in §4 and to reconstruct $\overline{\chi}(\kappa_n)$ by applying it to the class $\chi(\epsilon_1\epsilon^n)$. In the present paper, however, we adopted the indirect construction based on Theorem 5.1(2) for the economy of the number of pages.

Finally we remark

$$(6.6)\quad \chi(\epsilon_1\epsilon^n) = \chi(\eta_n)\pi_{g,\rho}{}^*\overline{\chi}(\kappa_n) \in H^0(C_{g,\rho}; H^{n+1}((\mathfrak{O}_\rho)_x; T^*_{p_1}C^\times \otimes (\pi_{g,\rho}{}^* \bigwedge^n T^* M_{g,\rho})_x)),$$

which is a key to establishing that the class κ_n corresponds to the n-th Morita Mumford class $e_n \in H^{n,n}(M_{g,\rho})$.

7. Construction of cohomology classes.

Let M be a complex analytic manifold acted on by a complex Lie algebra $\mathfrak{g}$. In this section we construct cohomology classes of the manifold M from the cohomology of the Lie algebra $\mathfrak{g}$ under a certain assumption. In the succeeding sections cohomology classes on the $\mathfrak{d}_\rho$ manifolds $M_{g,\rho}$ and $C_{g,\rho}$ are constructed in this way.

Let M be a (possibly infinite dimensional) complex analytic manifold on which a Lie algebra $\mathfrak{g}$ acts complex analytically. This means a homomorphism of Lie algebras

$$\mu : \mathfrak{g} \to \mathrm{Vect}(M),$$

called the *action*, is given, where $\mathrm{Vect}(M)$ denotes the complex Lie algebra of complex analytic vector fields on M. The kernel of the composite of the evaluation map ev_x at the point $x \in M$ and the action μ

$$\mathrm{ev}_x \circ \mu : \mathfrak{g} \to \mathrm{Vect}(M) \to T_x M$$

is denoted by $\mathfrak{g}_x$ and called the *isotropy subalgebra* of $\mathfrak{g}$ at the point $x \in M$.

Let $E \to M$ be a complex analytic vector bundle on which the algebra $\mathfrak{g}$ acts complex analytically and compatibly with the action μ. This means $\mathfrak{g}$ acts on each $\mathcal{O}_M(E)(O)$ $(O \overset{\text{open}}{\subset} M)$ such that

(1) each restriction map is $\mathfrak{g}$-equivariant, and

(2) the formula

$$X(f\sigma) = (Xf)\sigma + f(X\sigma), \quad X \in \mathfrak{g},\ f \in \mathcal{O}_M(O),\ \sigma \in \mathcal{O}_M(E)(O)$$

holds for any open subset $O \subset M$.

In the sequal we call such a vector bundle a $\mathfrak{g}$ *vector bundle over* M in short. The fiber E_x at $x \in M$ is a $\mathfrak{g}_x$ module in an obvious manner. Let $n \in \mathbb{N}_{\geq 0}$ be a fixed non-negative integer. We put an assumption:

$$(\mathrm{A}(n)) \qquad \forall x \in M \quad \forall n' < n \quad H^{n'}(\mathfrak{g}_x; E_x) = 0.$$

Under the assumption $(\mathrm{A}(n))$ we have an exact sequence of complex analytic 'vector bundles' over M

$$0 \to E \to \coprod_{x \in M} C^1(\mathfrak{g}_x; E_x) \to \coprod_{x \in M} C^2(\mathfrak{g}_x; E_x) \to \cdots$$

$$\cdots \to \coprod_{x \in M} C^{n-1}(\mathfrak{g}_x; E_x) \to \coprod_{x \in M} Z^n(\mathfrak{g}_x; E_x) \to \coprod_{x \in M} H^n(\mathfrak{g}_x; E_x) \to 0,$$

where Z^n means the n cocycles. This exzct sequence induces the n-fold composite of the connecting homomorphisms

$$D : H^0(M; \mathcal{O}_M(\coprod_{x \in M} H^n(\mathfrak{g}_x; E_x))) \to H^n(M; \mathcal{O}_M(E)).$$

The map D has a multiplicative property in the following sense:

LEMMA 7.1. *Let E_0, E_1 and E_2 be $\mathfrak{g}$ vector bundles over M satisfying the assumption $A(n_0)$, $A(n_1)$ and $A(n_0+n_1)$ respectively. Suppose a multiplication, i.e., a $\mathfrak{g}$ equivariant homomorphism of vector bundles $m : E_0 \otimes E_1 \to E_2$ is given. Then we have*

$$(Du_0) \cup (Du_1) = D(u_0 \cup u_1) \in H^{n_0+n_1}(M; \mathcal{O}_M(E_2))$$

for any $u_i \in H^0(M; \mathcal{O}_M(\coprod_{x\in M} H^{n_i}(\mathfrak{g}_x; E_{i,x})))$ $(i = 0, 1)$. Here the cup product $\cup$ means the composite of the usual cup product and the given multiplication m.

One deduces the lemma from the double complex of vector bundles over M

$$\coprod_{x\in M} C^{p_0}(\mathfrak{g}_x; E_{0,x}) \otimes C^{p_1}(\mathfrak{g}_x; E_{1,x}) \qquad (0 \le p_i \le n_i,\, i = 0, 1).$$

Concerning the functorial property of D we need the following two lemmata.

LEMMA 7.2. *Let E_0 and E_1 be $\mathfrak{g}$ vector bundles over M satisfying the same assumption $A(n)$, and $\Phi : E_0 \to E_1$ a $\mathfrak{g}$ equivariant homomorphism of vector bundles over M. Then we have the commutative diagram*

$$\begin{array}{ccc}
H^0(M; \mathcal{O}_M(\coprod_{x\in M} H^n(\mathfrak{g}_x; E_{0,x}))) & \xrightarrow{D} & H^n(M; \mathcal{O}_M(E_0)) \\
{\scriptstyle H^0(M;\coprod \Phi_x)}\downarrow & & \Phi\downarrow \\
H^0(M; \mathcal{O}_M(\coprod_{x\in M} H^n(\mathfrak{g}_x; E_{1,x}))) & \xrightarrow{D} & H^n(M; \mathcal{O}_M(E_1)).
\end{array}$$

LEMMA 7.3. *Let M_0 and M_1 be complex analytic manifolds acted on by the same Lie algebra $\mathfrak{g}$, and $f : M_0 \to M_1$ a $\mathfrak{g}$ equivariant complex analytic map. Suppose both the p-cotangent bundles $\bigwedge^p T^*M_i$ $(i = 0, 1)$ satisfy the assumption $A(n)$. As usual we abbreviate $\Omega^p_{M_i} := \mathcal{O}_{M_i}(\bigwedge^p T^*M_i)$ $(i = 0, 1)$. Then we have the commutative diagram*

$$\begin{array}{ccc}
H^0(M_1; \mathcal{O}_{M_1}(\coprod_{x_1\in M_1} H^n(\mathfrak{g}_{x_1}; \bigwedge^p T^*_{x_1}M_1))) & \xrightarrow{D} & H^n(M_1; \Omega^p_{M_1}) \\
f^*\downarrow & & f^*\downarrow \\
H^0(M_0; \mathcal{O}_{M_0}(\coprod_{x_0\in M_0} H^n(\mathfrak{g}_{x_0}; \bigwedge^p T^*_{x_0}M_0))) & \xrightarrow{D} & H^n(M_0; \Omega^p_{M_0}).
\end{array}$$

These two lemmata follow from the definition of D immediately.

Let $L \subset M$ be a closed subset. Under the assumption (A(n)) we can also define the map

$$D: H^0(M, L; \mathcal{O}_M(\coprod_{x\in M} H^n(\mathfrak{g}_x; E_x))) \to H^n(M, L; \mathcal{O}_M(E)).$$

by the same method as the original D. Clearly the same results as Lemmata 7.1, 7.2 and 7.3 hold for the new D.

Behind the definition of the map D there exists the notion of the $\mathfrak{g}$ equivariant cohomology, which is explained in §11.

Now we go back to the $\mathfrak{d}_\rho$ manifolds $M_{g,\rho}$ and $C_{g,\rho}$. From Corollaries 2.9 and 2.11 the $\mathfrak{d}_\rho$ vector bundle $\bigwedge^n T^*M$ satisfies the asssumption (A(n)) for each $n \in \mathbb{N}_{\geq 0}$ and $M = M_{g,\rho}$ and $C_{g,\rho}$. Hence the map

$$D: H^0(M; \mathcal{O}_M(\coprod_{x\in M} H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T_x^*M))) \to H^{n,n}(M)$$

is defined, where $H^{n,n}(M) = H^n(M; \Omega_M^n)$ as usual. Through the isomorphisms χ and $\overline{\chi}$ (6.2), we regard the vector bundle $\coprod_{x\in M} H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T_x^*M)$ as a trivial constant sheaf. thus we can define the composite maps

$$D\circ\chi: \bigoplus_{n\geq 0} H^n(L_0; \bigwedge^n Q^1) \to \bigoplus_{n\geq 0} H^{n,n}(C_{g,\rho})$$
$$D\circ\overline{\chi}: \bigoplus_{n\geq 0} H^n(W_1; \bigwedge^n Q) \to \bigoplus_{n\geq 0} H^{n,n}(M_{g,\rho}),$$

which are multiplicative by Lemma 7.1. In the next section we prove that $D\chi(\epsilon)$ is equal to the Euler class $e = c_1(T_{C_{g,\rho}/M_{g,\rho}}) \in H^{1,1}(C_{g,\rho})$ up to constant multiplier, and in §10 that $D\overline{\chi}(\kappa_n)$ is equal to the n-th Morita Mumford class $e_n \in H^{n,n}(M_{g,\rho})$ up to constant multiplier.

8. The Euler class of the relative tangent bundle.

In the following 2 sections we reconstruct the Euler class of the universal curve $C_{g,\rho} \to M_{g,\rho}$, i.e., the first Chern class of the relative tangent bundle

$$e := c_1(T_{C_{g,\rho}/M_{g,\rho}}) \in H^{1,1}(C_{g,\rho})$$

and its power

$$e^{n+1} \in H^{n+1,n+1}(C^{\times}_{g,\rho}, \overline{D^{\times}_{g,\rho}}), \quad (n \geq 1),$$

in our framework. The power e^{n+1} is essential to the original definition [Mo] [Mu] of the n-th Morita-Mumford characterstic class e_n (0.2). We define a vector bundle I over $C_{g,\rho}$ by

$$I := \coprod_{(C,p,z,p_1) \in C_{g,\rho}} Q^2(C_\rho, p_1).$$

The residual extension (6.3) induces an extension of vector bundles

(8.1) $$0 \to T^*C_{g,\rho} \hookrightarrow I \overset{\mathrm{Res}}{\to} C_{g,\rho} \times \mathrm{C} \to 0.$$

The class $D\chi(\epsilon) \in H^{1,1}(C_{g,\rho})$ is equal to the image of $1 \in H^0(C_{g,\rho}; \mathcal{O}_{C_{g,\rho}})$ under the connecting homomorphism

$$\delta^* : H^0(C_{g,\rho}; \mathcal{O}_{C_{g,\rho}}) \to H^1(C_{g,\rho}; \mathcal{O}_{C_{g,\rho}}(T^*C_{g,\rho}))$$

induced by the extension (8.1). We asserts

THEOREM 8.2.
$$\frac{\sqrt{-1}}{2\pi} D\epsilon = e \in H^{1,1}(C_{g,\rho}).$$

For the moment we recall a general theory on complex analytic line bundles. See [At] for details.

Let E be a complex analytic line bundle over a complex analytic manifold M. Consider a (canonical) extension of vector bundles over M

(8.3) $$0 \to T^*M \otimes E \overset{\sigma}{\to} J^1(E) \overset{\mathrm{ev}}{\to} E \to 0.$$

where $J^1(E)$ is the <u>holomorphic</u> 1-jet bundle of E and $\mathrm{ev} : J^1(E) \to E$ is the evaluation map. Let $j^1 : C^\infty(M, E) \to C^\infty(M, J^1(E))$ denote the jet extension map. If $\theta : J^1(E) \to T^*M \otimes E$ is a C^∞ homomorphism satisfying $\theta \circ \sigma = 1_{T^*M \otimes E}$, the composite

$$\theta \circ j^1 : C^\infty(M, E) \to C^\infty(M, J^1(E)) \to C^\infty(M, T^*M \otimes E)$$

gives the $(1,0)$ part of a C^∞ connection of type $(1,0)$ in E. This process gives a one-to-one correspondence between C^∞ splittings of the extension (8.3) and C^∞ connections of type $(1,0)$ in E.

Tensoring the dual E^* to the extension (8.3), we obtain an extension of vector bundles over M

$$(8.4) \qquad 0 \to T^*M \overset{\sigma\otimes E^*}{\to} J^1(E)\otimes E^* \overset{\mathrm{ev}\otimes E^*}{\to} M\times \mathbf{C} \to 0,$$

because E is a line bundle. The extension induces the connecting homomorphism

$$(8.5) \qquad \delta^* : H^0(M;\mathcal{O}_M) \to H^1(M;\mathcal{O}_M(T^*M)) = H^{1,1}(M).$$

PROPOSITION 8.6. ([AT]). *The image of* $1 \in H^0(M;\mathcal{O}_M)$ *under the map* δ^* *(8.5) is equal to the first Chern class* $-2\pi\sqrt{-1}c_1(E) \in H^{1,1}(M)$.

PROOF: Fix a C^∞ splitting of the extension (8.3). Let $\theta : J^1(E) \to T^*M \otimes E$ and $\psi : E \to J^1(E)$ be the C^∞ homomorphisms induced by the fixed splitting. There exist $(1,1)$ forms Ξ and Θ with trivial coefficients uniquely determined by the formulae

$$\begin{aligned}&\overline{\partial}_{J^1(E)}(\psi\circ s) - (\overline{T}^*M\otimes\psi)(\overline{\partial}_E s) = (\overline{T}^*M\otimes\sigma)(\Xi\otimes s) \quad \text{and}\\ &\overline{\partial}_{T^*M\otimes E}(\theta\circ(j^1 s)) - (\overline{T}^*M\otimes(\theta\circ j^1))(\overline{\partial}_E s) = \Theta\otimes s\end{aligned}$$

for any $s \in C^\infty(M,E)$. Here $\overline{\partial}_E$, $\overline{\partial}_{T^*M\otimes E}$ and $\overline{\partial}_{J^1(E)}$ denote the $\overline{\partial}$ operators of the holomorphic vector bundles E, $T^*M\otimes E$ and $J^1(E)$, respectively. Θ is the curvature of the connection corresponding to θ by definition. Hence the cohomology class of the $(1,1)$ form $\frac{1}{2\pi\sqrt{-1}}\Theta$ is equal to the first Chern class $c_1(E)$. We remark

$$(1-\sigma\theta)(j^1 s) = \psi\circ s$$

for any $s \in C^\infty(M,E)$. Hence

$$\begin{aligned}&(\overline{T}^*M\otimes\sigma)(\Xi\otimes s)\\ =&\overline{\partial}_{J^1(E)}(1-\sigma\theta)\circ j^1 s - (\overline{T}^*M\otimes(1-\sigma\theta)\circ j^1)(\overline{\partial}_E s)\\ =&(\overline{T}^*M\otimes\sigma)(-\overline{\partial}_{T^*M\otimes E}(\theta\circ j^1 s) + (\overline{T}^*M\otimes\theta\circ j^1)(\overline{\partial}_E s))\\ =&-(\overline{T}^*M\otimes\sigma)(\Theta\otimes s),\end{aligned}$$

which implies $[\frac{\sqrt{-1}}{2\pi}\Xi] = c_1(E) \in H^{1,1}(M)$. On the other hand, by definition, the cohomology class of $\Xi \in H^{1,1}(M)$ is equal to the desired class $\delta^*(1)$. This completes the proof.

Now we assume a complex Lie algebra $\mathfrak{g}$ acts on the manifold M complex analytically and transitively and E is a $\mathfrak{g}$ stable line subbundle of the tangent bundle TM (see §7). The transitivity means that the composite $\mathrm{ev}_x \circ \mu : \mathfrak{g} \to \mathrm{Vect}(M) \to T_xM$ is surjective for each $x \in M$. Then the isotropy algebra $\mathfrak{g}_x \subset \mathfrak{g}$ at $x \in M$ acts on the fiber $E_x (\cong \mathbf{C})$ of E at x. There exists a (canonical) linear map $\widetilde{\mu_x} : \mathfrak{g}_x \to \mathbf{C}$ given by

$$[X, v_x] = \widetilde{\mu_x}(X) v_x$$

for any $X \in \mathfrak{g}_x$ and $v_x \in E_x$. Furthermore we assume the map $\widetilde{\mu_x}$ is surjective for each $x \in M$. Denote $\mathfrak{h}_x := \ker(\widetilde{\mu_x} : \mathfrak{g}_x \to \mathbf{C})$ and define a vector bundle $\mathfrak{g}/\mathfrak{h}$ over M by

$$\mathfrak{g}/\mathfrak{h} := \coprod_{x \in M} \mathfrak{g}/\mathfrak{h}_x.$$

The natural projection $\mathfrak{g}/\mathfrak{h}_x \to \mathfrak{g}/\mathfrak{g}_x$ induces extensions of vector bundles over M

$$(8.7) \qquad 0 \to T^*M \to (\mathfrak{g}/\mathfrak{h})^* \xrightarrow{\tilde{\mu}} M \times \mathbf{C} \to 0 \quad \text{and}$$

$$(8.8) \qquad 0 \to T^*M \otimes E \to (\mathfrak{g}/\mathfrak{h})^* \otimes E \xrightarrow{\tilde{\mu} \otimes E} E \to 0.$$

If $X \in \mathfrak{g}$ and $s \in \mathcal{O}_M(E)_x$, the vector $[X, s](x) \in E_x$ depends only on (X and) the 1-jet of s. Hence the homomorphism

$$\begin{aligned} J^1(E) &\to (\mathfrak{g}/\mathfrak{h})^* \otimes E \\ s &\mapsto (X \mapsto [X, s](x)) \end{aligned}$$

can be defined, which gives a (canonical) isomorphism of the extension (8.3) onto the extension (8.8). Especially the extension (8.7) is isomorphic to the extension (8.4) and corresponds to the class $-2\pi\sqrt{-1}c_1(E) \in H^{1,1}(M)$.

We return now to the identification of the class $D\epsilon \in H^{1,1}(C_{g,\rho})$. Then $M = C_{g,\rho}$, $E = T_{C_{g,\rho}/M_{g,\rho}}$ and $\mathfrak{g} = \mathfrak{d}_\rho$ satisfy all the assumptions stated above. In fact, $\mathfrak{g}_x$ is given by $\mathfrak{g}_x = L(C^\times, p_1)$ for $x = (C, p, z, p_1) \in C_{g,\rho}$, the linear map $\widetilde{\mu_x} : \mathfrak{g}_x \to \mathbf{C}$ is given by

$$\begin{aligned} \widetilde{\mu_x} : & L(C^\times, p_1) \to \mathbf{C} \\ & X \overset{\text{loc.}}{=} f(w)\frac{d}{dw} \mapsto -f'(0), \end{aligned}$$

where w is a coordinate centered at $p_1 \in C^\times$, and so $\mathfrak{h}_x$ is equal to

$$L_1(C^\times, p_1) := \{X \in L(C^\times, p_1); X \text{ has a zero at } p_1 \text{ of order } \geq 2.\}.$$

LEMMA 8.9. *Under the above situation the extension (8.7) is isomorphic to the extension (8,1) by the Köthe duality (Theorem 1.4).*

PROOF: It follows from the Köthe duality (1.4)

$$(\mathfrak{g}/\mathfrak{h}_x)^* = (\mathfrak{d}_\rho/L_1(C^\times, p_1))^* \cong Q^2(C_\rho, p_1)$$

and the natural map $(\mathfrak{g}/\mathfrak{g}_x)^* \to (\mathfrak{g}/\mathfrak{h}_x)^*$ coincides with the inclusion $Q^1(C_\rho, p_1) \hookrightarrow Q^2(C_\rho, p_1)$ for $x = (C, p, z, p_1) \in C_{g,\rho}$. Hence it suffices to show that $\widetilde{\mu_x} : (\mathfrak{g}/\mathfrak{h}_x)^* \to \mathbb{C}$ is equal to $\mathrm{Res}_{p_1} : Q^2(C_\rho, p_1) \to \mathbb{C}$.

Let $q_0 \in Q^2(C_\rho, p_1)$ be a meromorphic quadratic differential satisfying $\mathrm{Res}_{p_1} q_0 = 1$. For an arbitrary $X \in \mathfrak{g}_x = L(C^\times, p_1)$, we have

$$\widetilde{\mu_x}(X) = \frac{-1}{2\pi\sqrt{-1}} \oint_{|w|=\delta \ll 1} q_0 \cdot X = \frac{1}{2\pi\sqrt{-1}} \oint_{|z|=\rho+\delta} q_0 \cdot X$$

from Stokes' Theorem. This implies $\mu_x \in (\mathfrak{g}_x/\mathfrak{h}_x)^* \cong Q^2(C_\rho, p_1)/Q^1(C_\rho, p_1)$ is equal to $q_0 \bmod Q^1(C_\rho, p_1)$, as was to be shown.

The class $D\chi(\epsilon)$ corresponds to the isomorphism class of the extension (8.7) $\cong$ (8.4) from Lemma 8.9. By Proposition 8.6 the extension (8.4) is equal to $-2\pi\sqrt{-1}c_1(E) = -2\pi\sqrt{-1}c_1(T_{C_{g,\rho}/M_{g,\rho}})$. Consequently we obtain

$$\frac{\sqrt{-1}}{2\pi} D\epsilon = c_1(T_{C_{g,\rho}/M_{g,\rho}}) = e \in H^{1,1}(C_{g,\rho}).$$

This completes the proof of Theorem 8.2.

Here it should be remarked there exists a one-to-one correspondence between C^∞ splittings of the extension (8.1) and C^∞ connections of type (1,0) in $T_{C_{g,\rho}/M_{g,\rho}}$. A C^∞ splitting of the extension (8.1) is equivalent to a C^∞ assignment

$$x \in C_{g,\rho} \mapsto q_0(x) \in Q^2(C_\rho, p_1)$$

satisfying $\mathrm{Res}_{p_1} q_0(x) = 1$ for all $x \in C_{g,\rho}$.

As an example, now we construct a canonical real analytic splitting of the residual extension (6.3) under the uniformization through the upper half plane $\mathsf{H} := \{z \in \mathbb{C}; \Im z > 0\}$. We consider the function

$$q(a, z) := \frac{(a - \bar{a})^2}{(z - a)^2 (z - \bar{a})^2}$$

for $a, z \in \mathbf{H}$. It is easily proved that

$$q(\gamma a, \gamma z) d(\gamma z)^2 = q(a, z) dz^2$$

for any $\gamma \in \mathrm{PSL}(2, \mathbf{R})$. If $\Gamma \subset \mathrm{PSL}(2, \mathbf{R})$ is a Fuchsian group, the Poincaré series

$$q_\Gamma(a) := \sum_{\gamma \in \Gamma} q(a, \gamma z) d(\gamma z)^2$$

converges uniformly on any compact subset of $\mathbf{H} - \Gamma \cdot a$. In fact, for $|z| \gg 1$,

$$|q(a, z)| \sim \frac{4(\Im a)^2}{|z|^4}, \quad \int\int_{|z| \geq 1} |z|^{-4} dx dy = \pi < +\infty,$$

and so $\displaystyle\int\int_{\mathbf{H} - \Gamma B_\delta} |q(a, z) dz d\bar{z}| < +\infty,$

where B_δ denotes the hyperbolic disk whose center is a and radius $\delta > 0$. For the rest, we may follow a usual argument of Poincaré series (see, e.g., [Kr] ch.III §§1-4).

By the construction $q_\Gamma(\gamma a) = q_\Gamma(a)$ for any $\gamma \in \Gamma$, and $q_\Gamma(a)$ has a pole of order 2 at $a \in \mathbf{H}$ with $\mathrm{Res}_a\, q_\Gamma(a) = 1$. Thus $q_\Gamma(a)$ gives a canonical splitting of the extension

$$0 \to Q^1(\mathbf{H}/\Gamma, a \bmod \Gamma) \to Q^2(\mathbf{H}/\Gamma, a \bmod \Gamma) \overset{\mathrm{Res}}{\to} \mathbf{C} \to 0.$$

9. The relative Euler class of the relative tangent bundle.
From now on we assume $\rho > 0$. In this section we study the pull back

$$\widehat{\chi}(\epsilon) := \iota_{g,\rho}{}^{*}\chi(\epsilon) \in H^0(C^{\times}_{g,\rho}; H^1(L(C^{\times}, p_1); T^*_{(C,p,z,p_1)}C^{\times}_{g,\rho}))$$

of the class

$$\begin{aligned}\chi(\epsilon) \in & H^0(C_{g,0}; H^1(L(C^{\times}, p_1); Q^1(C_0, p_1))) \\ &= H^0(C^{\times}_{g,0}; H^1(L(C^{\times}, p_1); T^*_{(C,p,z,p_1)}C^{\times}_{g,0}))\end{aligned}$$

by the natural map

$$\iota_{g,\rho} : C^{\times}_{g,\rho} \to C^{\times}_{g,0} = C_{g,0}, \quad (C, p, z, p_1) \mapsto (C, p, z, p_1).$$

From Theorem 8.2 and the naturality of the map D (§7) we have

$$\frac{\sqrt{-1}}{2\pi} D\widehat{\chi}(\epsilon) = c_1(T_{C^{\times}_{g,\rho}/M_{g,\rho}}) \in H^{1,1}(C^{\times}_{g,\rho}).$$

The behavior of $\widehat{\chi}(\epsilon)$ on the "boundary" $\overline{D^{\times}_{g,\rho}}$ is as follows.

PROPOSITION 9.1. *If $x = (C, p, z, p_1) \in \overline{D^{\times}_{g,\rho}}$,*

(1) *the coordinate z induces a canonical decomposition of $(\mathfrak{O}_\rho)_x (= L(C^{\times}, p_1))$ modules*

$$T^*_x C^{\times}_{g,\rho} = T^*_{\pi_{g,\rho}(x)} M_{g,\rho} \oplus T^*_{p_1} C^{\times} \tag{9.2}$$

(2) *under the decomposition (9.2), the class $\widehat{\chi}(\epsilon)_x$ corresponds to the class*

$$(0, \epsilon_1) \in H^1(L(C^{\times}, p_1); T^*_{\pi_{g,\rho}(x)} M_{g,\rho}) \oplus H^1(L(C^{\times}, p_1); T^*_{p_1} C^{\times}),$$

where the 1 cocycle $\epsilon_1 \in C^1(L(C^{\times}, p_1); T^*_{p_1} C^{\times})$ is defined by

$$\epsilon_1(X) := -f''(z(p_1))(dz)_{p_1} \in T^*_{p_1} C^{\times}$$

for $X \overset{\text{loc.}}{=} f(z)\frac{d}{dz} \in L(C^{\times}, p_1)$.

As a corollary of Proposition (9.1)(2), if $n \geq 1$ and $x \in \overline{D^{\times}_{g,\rho}}$, we have

$$(\widehat{\chi}(\epsilon)_x)^{n+1} = 0 \in H^{n+1}(L(C^{\times}, p_1); \textstyle\bigwedge^{n+1} T^*_{(C,p,z,p_1)} C^{\times}_{g,\rho}).$$

Since the assumption (A$(n+1)$) holds for the $\mathfrak{O}_\rho$ bundle $\bigwedge^{n+1} T^* C^{\times}_{g,\rho}$, the relative cohomology class

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^{n+1} D((\widehat{\chi}(\epsilon))^{n+1}) \in H^{n+1,n+1}(C^{\times}_{g,\rho}, \overline{D^{\times}_{g,\rho}})$$

is defined and coincides with the $(n+1)$-th power $e^{n+1} \in H^{n+1,n+1}(C^{\times}_{g,\rho}, \overline{D^{\times}_{g,\rho}})$ of the relative class $e = c_1(T_{C^{\times}_{g,\rho}/M_{g,\rho}}, \frac{d}{dz}) \in H^{1,1}(C^{\times}_{g,\rho}, \overline{D^{\times}_{g,\rho}})$. Thus we obtain

COROLLARY 9.3. *If* $n \geq 1$,

$$(\frac{\sqrt{-1}}{2\pi})^{n+1} D((\widehat{\chi}(\epsilon))^{n+1}) = e^{n+1} \in H^{n+1,n+1}(C^{\times}_{g,\rho}, \overline{D^{\times}_{g,\rho}}).$$

To prove Proposition 9.1 we recall the push-forward of extensions. For simplicity, let R be a unitary commutative ring, $0 \to A' \xrightarrow{i} A \xrightarrow{\pi} A'' \to 0$ an extension of left R modules, and $f : A' \to B'$ a left R homomorphism. Consider the fiber coproduct $B' \times_{A'} A$ defined by

$$B' \times_{A'} A := \mathrm{coker}((-f, i) : A' \to B' \oplus A).$$

Let $(b', a) \bmod A$ denote the element of $B' \times_{A'} A$ induced by $(b', a) \in B' \oplus A$. Then, in an obvious way, we obtain a natural homomorphism of extensions of left R modules

$$\begin{array}{ccccccccc} 0 & \to & A' & \xrightarrow{i} & A & \xrightarrow{\pi} & A'' & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & B' & \xrightarrow{i_B} & B' \times_{A'} A & \xrightarrow{\pi_B} & A'' & \to & 0. \end{array}$$

Fix a point $x = (C, p, z, p_1) \in C^{\times}_{g,\rho}$. Applying the above construction to the extension of $(\mathfrak{O}_\rho)_x (= L(C^{\times}, p_1))$ modules

$$0 \to Q(C^{\times}) \hookrightarrow Q^1(C^{\times}, p_1) \to T^*_{p_1} C^{\times} \to 0$$

and the $L(C^{\times}, p_1)$ homomorphism $Q(C^{\times}) \hookrightarrow Q(C_\rho)$, we obtain an extension of $L(C^{\times}, p_1)$ modules

$$0 \to Q(C_\rho) \to Q(C_\rho) \times_{Q(C^{\times})} Q^1(C^{\times}, p_1) \to T^*_{p_1} C^{\times} \to 0.$$

The cotangent map ${\iota_{g,\rho}}^* : T^*_{\overline{x}} M_{g,0} \to T^*_{\overline{x}} M_{g,\rho}$ induced by the natural map $\iota_{g,\rho} : M_{g,\rho} \to M_{g,0}, \quad (C, p, z) \mapsto (C, p, z)$ is equal to the inclusion $Q(C^{\times}) \hookrightarrow Q(C_\rho)$ under the Köthe duality. Hence the map ${\iota_{g,\rho}}^*$ induces an isomorphism of extensions of $L(C^{\times}, p_1)$ modules

(9.4)

$$\begin{array}{ccccccc} 0 \to & T^*_{(C,p,z)} M_{g,\rho} \to & T^*_{(C,p,z,p_1)} C^{\times}_{g,\rho} & \to & T^*_{p_1} C^{\times} & \to 0 \\ & \| & \downarrow & & \| & \\ 0 \to & Q(C_\rho) \to & Q(C_\rho) \times_{Q(C^{\times})} Q^1(C^{\times}, p_1) & \to & T^*_{p_1} C^{\times} & \to 0. \end{array}$$

Suppose $x = (C, p, z, p_1) \in \overline{D^{\times}_{g,\rho}}$. Let $q_1 = q_1(x) \in Q^1(C^{\times}, p_1)$ be a meromorphic quadratic differential which corresponds to $-(dz)_{p_1}$ under the restriction $Q^1(C^\times, p_1) = T^*_x C_{g,0} \to T^*_{p_1} C^\times$. In other words we may expand the differential q_1 by the coordinate z to obtain

$$q_1 = (\frac{1}{z - z(p_1)} + \text{ regular terms})dz^2.$$

The element

$$(-q_1, q_1) \bmod Q(C^\times) \in Q(C_\rho) \times_{Q(C^\times)} Q^1(C^\times, p_1)$$

is independent of the choice of $q_1 \in Q^1(C^\times, p_1)$. Furthermore, for $X \in L(C^\times, p_1)$,

$$(-\mathcal{L}(X)q_1, \mathcal{L}(X)q_1) \equiv \delta_{0,p_1}(X)(-q_1, q_1) \pmod{Q(C^\times)}.$$

Hence we obtain a canonical decomposition of $L(C^\times, p_1)$ modules

$$Q(C_\rho) \times_{Q(C^\times)} Q^1(C^\times, p_1) = Q(C_\rho) \oplus T^*_{p_1} C^\times. \tag{9.5}$$

LEMMA 9.6. *Under the isomorphism (9.4) the elelment* $(-q_1, q_1) \bmod Q(C^\times)$ *corresponds to* $-(dz)_{p_1} \in T^*_x C^\times_{g,\rho}$.

PROOF: Take an arbitrary $v \in T_x C^\times_{g,\rho}$. We have

$$\begin{aligned} \pi_{g,\rho_*} v &= X_\rho \bmod L(C^\times) \in T_{\bar{x}} M_{g,\rho} \\ \iota_{\rho_*} v &= X_0 \bmod L(C^\times, p_1) \in T_{(C,p,z,p_1)} C^\times_{g,0} \end{aligned}$$

for some $X_\rho \in \mathfrak{d}_\rho$ and $X_0 \in \mathfrak{d}_0$. Passing to $T_{(C,p,z)} M_{g,0}$, we have $X_\rho - X_0 \in L(C^\times)$ and $X_0 \in \mathfrak{d}_\rho$. It follows from the fact $q_1 \in Q(C_\rho)$ that

$$\frac{1}{2\pi\sqrt{-1}} \oint_{|z|=\rho+\delta} q_1 \cdot (X_\rho - X_0) = 0.$$

Thus, for $0 < \rho_1 < |z(p_1)| < \rho$,

$$\begin{aligned} &\langle -\pi_{g,\rho}{}^* q_1 + \iota_\rho{}^* q_1, v \rangle = -\langle q_1, X_\rho \rangle + \langle q_1, X_0 \rangle \\ =& -\frac{1}{2\pi\sqrt{-1}} \oint_{|z|=\rho+\delta} q_1 \cdot X_\rho + \frac{1}{2\pi\sqrt{-1}} \oint_{|z|=\rho_1} q_1 \cdot X_0 \\ =& -\frac{1}{2\pi\sqrt{-1}} \oint_{|z|=\rho+\delta} q_1 \cdot (X_\rho - X_0) \\ & -\frac{1}{2\pi\sqrt{-1}} (\oint_{|z|=\rho+\delta} - \oint_{|z|=\rho_1}) q_1 \cdot X_0 \\ =& -\frac{1}{2\pi\sqrt{-1}} \oint_{\text{around} p_1} q_1 \cdot X_0 = -(dz)_{p_1}(X_0) \\ =& \langle -(dz)_{p_1}, v \rangle, \end{aligned}$$

which proves the lemma.

Thus the proof of Proposition 9.1(1) is completed.

PROOF OF PROPOSITION 9.1(2): Let $q_0 \in Q^2(C^\times, p_1)$ be a meromorphic quadratic differential satisfying $\mathrm{Res}_{p_1} q_0 = 1$. We have

$$\widehat{\chi}(\epsilon) = (0, dq_0) \bmod Q(C^\times)$$

under the isomorphism (9.4). From the fact $q_0 \in Q(C_\rho)$, the cocycle $\widehat{\chi}(\epsilon)$ is cohomologous to the cocycle $(-dq_0, dq_0) \bmod Q(C^\times)$. For $X \overset{\text{loc.}}{=} f(z)\frac{d}{dz} \in L(C^\times, p_1)$,

$$\begin{aligned}
\mathcal{L}(X)q_0 =& \mathcal{L}(f(z)\frac{d}{dz})((\frac{1}{(z - z(p_1))^2} + \text{higher terms})dz^2) \\
=& (\frac{f''(z(p_1))}{z - z(p_1)} + \text{regular terms})dz^2 \\
\equiv& f''(z(p_1))q_1 \pmod{Q(C^\times)}.
\end{aligned}$$

Hence $\langle(-dq_0, dq_0) \bmod Q(C^\times), X\rangle = (-\mathcal{L}(X)q_0, \mathcal{L}(X)q_0) \bmod Q(C^\times) = f''(z(p_1))(-q_1, q_1) \bmod Q(C^\times)$ corresponds to $(0, \epsilon_1(X))$ under the isomorphism (9.5), which completes the proof of Proposition 9.1.

10. The Morita Mumford classes.

In §7 we constructed a cohomology class $D\overline{\chi}(\kappa_n) \in H^{n,n}(M_{g,\rho})$ for $n \in \mathbb{N}_{\geq 0}$. In this section we assume $\rho > 0$. We shall prove

THEOREM 10.1. *Suppose $g \geq 0$ and $\rho > 0$. If $n \geq 1$,*

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^n D\overline{\chi}(\kappa_n) = e_n \in H^{n,n}(M_{g,\rho}),$$

where e_n is the n-th Morita Mumford class (0.2).

As corollaries we obtain

COROLLARY 10.2. *If $\rho > 0$, the composite map*

$$D \circ \overline{\chi} : \bigoplus_{n\geq 0} H^n(W_1; \bigwedge^n Q) \to \bigoplus_{n\geq 0} H^{n,n}(M_{g,\rho})$$

$$(\text{resp.} \quad D \circ \chi : \bigoplus_{n\geq 0} H^n(L_0; \bigwedge^n Q^1) \to \bigoplus_{n\geq 0} H^{n,n}(C_{g,\rho}))$$

is a stable isomorphism onto the subalgebra generated by the Morita Mumford classes e_n's (resp. the Euler class e and the Morita Mumford classes e_n's).

COROLLARY 10.3. *There exist no algebraic relations among the classes κ_n's. Namely we have isomorphisms of* $\mathbb{C}$ *algebras*

$$\bigoplus_{n\geq 0} H^n(W_1; \bigwedge^n Q) = \mathbb{C}[\kappa_n; n \geq 1]$$
$$\bigoplus_{n\geq 0} H^n(L_0; \bigwedge^n Q^1) = \mathbb{C}[\epsilon, \kappa_n; n \geq 1].$$

To deduce the corrolaries from the theorem, we utilize the theorem of Miller [Mi] and Morita [Mo] quoted in (0.1).

The formula (6.6), which is an immediate consequence of Theorem 5.1(2),

$$\begin{aligned}(10.4) \quad \chi(\epsilon_1\epsilon^n) &= \chi(\eta_n)\pi_{g,\rho}{}^*\overline{\chi}(\kappa_n) \\ &\in H^0(C_{g,0}; H^{n+1}((\mathfrak{d}_0)_x; T^*_{p_1}C^\times \otimes (\pi_{g,0}{}^*\bigwedge^n T^*M_{g,0})_x)).\end{aligned}$$

is a key to the proof of Theorem 10.1. We begin it by investigating the class $\chi(\eta_n)$.

The pull back $\iota^*_{g,\rho}\chi(\eta_n)$ through the natural map $\iota_{g,\rho} : C^\times_{g,\rho} \to C^\times_{g,0} = C_{g,0}$ defines the class

$$\widehat{\chi}(\eta_n) \in H^0(C^\times_{g,\rho}, \overline{D^\times_{g,\rho}}; H^1((\mathfrak{d}_\rho)_x; T^*_{p_1}C^\times \otimes S^nF(C_\rho))).$$

Consider the (usual) fiber integral of the class

$$D\widehat{\chi}(\eta_n) \in H^1(C^\times_{g,\rho}, \overline{D^\times_{g,\rho}}; \mathcal{O}_{C^\times_{g,\rho}}(T^*C^\times_{g,\rho}/M_{g,\rho} \otimes S^nF(C_\rho))).$$

LEMMA 10.5.

$$\int_{\text{fiber}} D\widehat{\chi}(\eta_n) = 2\pi/\sqrt{-1} \in H^0(M_{g,\rho}; \mathcal{O}_{M_{g,\rho}}(S^nF(C_\rho))).$$

PROOF: It suffices to prove it for the case $n = 1$. To represent the class $\widehat{\chi}(\eta_1)$ explicitly, we introduce an "Elementarfunktion 1. Ordnung" of Behnke and Stein [BeSt].

LEMMA 10.6.[BEST]. *Let $C^\times$ be a once punctured compact Riemann surface. Then there exists a meromorphic section A of the bundle ${\mathrm{pr}_1}^* T^* C^\times$ over $C^\times \times C^\times$ such that*

(1) *A is complex analytic over $C^\times \times C^\times - \Delta$, where $\Delta = \Delta(C^\times) \subset C^\times \times C^\times$ is the diagonal.*
(2) *Near each point in the diagonal, A is locally represented by*

$$A = A(\zeta, z)d\zeta = \left(\frac{-1}{\zeta - z} + \mathit{holo.}\right) d\zeta.$$

The section A is called an Elementarfunktion 1. Ordnung in [BeSt]. Then we have in $H^1((\mathfrak{O}_0)_x; T^*_{p_1}C^\times \otimes F(C^\times))$

$$\chi^{(C,p,z,p_1)}(\eta_1) = [d((d\zeta)_{p_1} \otimes A(p_1, z))].$$

Let $\widetilde{A}$ be a C^∞ section of $T^*C^\times \otimes F(C_\rho)$ over $C^\times$ which is an extension of $A|_{\overline{D^\times_\rho} \times C_\rho}$. By definition we have

$$D\widehat{\chi}(\eta_1) = \bar{\partial}(A - \widetilde{A}) = -\bar{\partial}\widetilde{A} \in H^1(C^\times, \overline{D^\times_\rho}; T^*C^\times \otimes F(C_\rho)).$$

It follows from Stokes' theorem

$$\begin{aligned}(\int_{\text{fiber}} D\widehat{\chi}(\eta_1))(p_1) &= -\int_{C^\times} \bar{\partial}\widetilde{A}(p_1) = -\int_{C^\times} d\widetilde{A}(p_1) = \oint_{|\zeta|=\rho} \widetilde{A}(p_1) \\ &= \oint_{|\zeta|=\rho} A(p_1) = -2\pi\sqrt{-1}\,\mathrm{Res}_{p_1} A(\cdot, p_1) = 2\pi/\sqrt{-1}.\end{aligned}$$

This completes the proof of Lemma 10.5.

Taking the pullback of the formula (10.4) through the natural map $\iota_{g,\rho} : C^\times_{g,\rho} \to C^\times_{g,0} = C_{g,0}$, we obtain

$$\begin{aligned}(10.7) \quad \widehat{\chi}(\epsilon_1 \epsilon^n) &= \widehat{\chi}(\eta_n){\pi_{g,\rho}}^*\overline{\chi}(\kappa_n) \\ &\in H^0(C^\times_{g,\rho}, \overline{D^\times_{g,\rho}}; H^{n+1}((\mathfrak{O}_\rho)_x; T^*_{p_1}C^\times \otimes ({\pi_{g,\rho}}^* \textstyle\bigwedge^n T^*M_{g,\rho})_x)),\end{aligned}$$

where $\widehat{\chi}(\epsilon_1\epsilon^n)$ is the projection image of the class $\widehat{\chi}(\epsilon)^{n+1}$ defined in §9. From Corollary 9.3, the definition of the Morita Mumford class (0.2) and Lemma 10.5 we have

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^{n+1} \int_{\text{fiber}} D\widehat{\chi}(\epsilon_1\epsilon^n) = \int_{\text{fiber}} e^{n+1} = e_n, \quad \text{and}$$

$$\int_{\text{fiber}} D(\widehat{\chi}(\eta_n){\pi_{g,\rho}}^*\overline{\chi}(\kappa_n)) = \left(\int_{\text{fiber}} D\widehat{\chi}(\eta_n)\right) D\overline{\chi}(\kappa_n) = \frac{2\pi}{\sqrt{-1}} D\overline{\chi}(\kappa_n).$$

It follows from (10.7) that $e_n = (\sqrt{-1}/2\pi)^n D\overline{\chi}(\kappa_n)$, which completes the proof of Theorem 10.1.

11. Equivariant cohomology.

Finally we shall show how the results obtained in the preceding sections can be interpreted by an equivariant cohomology theory for Lie algebras [Ka1] §1.

As in §7, let M be a (possibly infinite dimensional) complex analytic manifold on which a complex Lie algebra $\mathfrak{g}$ acts complex analytically and let E be a $\mathfrak{g}$ vector bundle over M. Here we assume that the action of $\mathfrak{g}$ on M is transitive, i.e., that the composite $\mathrm{ev}_x \circ \mu : \mathfrak{g} \to \mathrm{Vect}(M) \to T_x M$ is surjective for each $x \in M$.

Since $\mathcal{O}_M(E)$ is a sheaf of $\mathfrak{g}$ modules, the cochain complex of sheaves over M

$$C^*(\mathfrak{g}; \mathcal{O}_M(E)) : M \overset{\text{open}}{\supset} O \mapsto C^*(\mathfrak{g}; \mathcal{O}_M(E)(O))$$

is defined, where $C^*(\mathfrak{g}; \cdot)$ is the standard cochain complex of the Lie algebra $\mathfrak{g}$ with values in a $\mathfrak{g}$ module $\cdot$ introduced in §2. We denote by $H^*_{\mathfrak{g}}(M, \mathcal{O}_M(E))$ the hypercohomology group of the cochain complex of sheaves over M with respect to the functor $\Gamma(M; \cdot)$ (= the sections of $\cdot$ over M) ([G,E] ch.0, §11.4, pp.32-) and call it *the* $\mathfrak{g}$ *equivariant cohomology group of* M *with values in the* $\mathfrak{g}$ *vector bundle* E. Namely we define

$$H^*_{\mathfrak{g}}(M; \mathcal{O}_M(E)) := H^*(\mathrm{Total}(\Gamma(M; \mathcal{C}^{*,*})))$$

for an injective right Cartan-Eilenberg resolution $\mathcal{C}^{*,*} = (\mathcal{C}^{i,j})_{i,j \geq 0}$ of the complex $C^*(\mathfrak{g}; \mathcal{O}_M(E))$ (cf. ibid. loc. cit.). Especially, if E is the n-cotangent bundle $\bigwedge^n T^*M$, we denote

$$H^{n,*}_{\mathfrak{g}}(M) := H^*_{\mathfrak{g}}(M; \mathcal{O}_M(\bigwedge^n T^*M))$$

and call it *the* $\mathfrak{g}$ *equivariant* $(n, *)$ *cohomology of* M.

There exist two spectral sequences converging to $H^*_{\mathfrak{g}}(M; \mathcal{O}_M(E))$

$$(11.1) \qquad {}'E_2^{p,q} = H^p(H^q(M; C^*(\mathfrak{g}; \mathcal{O}_M(E))))$$

$$(11.2) \qquad {}''E_2^{p,q} = H^p(M; H^q(\mathfrak{g}; \mathcal{O}_M(E))),$$

where we denote $H^*(\mathfrak{g}; \mathcal{O}_M(E))$ is the sheaf over M defined as the cohomology of the cochain complex of sheaves $C^*(\mathfrak{g}; \mathcal{O}_M(E))$.

We look at the natural map

$$\varphi^{p,q} : {}'E_2^{p,q} = H^p(H^q(M; C^*(\mathfrak{g}; \mathcal{O}_M(E)))) \to H^p(\mathfrak{g}; H^q(M; \mathcal{O}_M(E))).$$

Especially we have the natural map

$$H^n_{\mathfrak{g}}(M; \mathcal{O}_M(E)) \to H^0(\mathfrak{g}; H^n(M; \mathcal{O}_M(E))) = H^n(M; \mathcal{O}_M(E))^{\mathfrak{g}}.$$

Although the $\mathfrak{d}_\rho$ manifolds $M_{g,\rho}$ and $C_{g,\rho}$ are infinite dimensional, we have the following proposition proved in [Ka1](1.5) for the case when the $\mathfrak{g}$ manifold M and the $\mathfrak{g}$ vector bundle E are finite dimensional.

PROPOSITION 11.3. *Let $\mathfrak{g}$, M and E be as above. We assume that the Lie algebra $\mathfrak{g}$ is one of the following*

(1) *a finite dimensional Lie algebra,*
(2) *the Lie algebra consisting of all complex analytic vector fields on a finite dimensional complex manifold,*
(3) *a closed Lie subalgebra of a Lie algebra given in (2),*
(4) $\mathfrak{D}_\rho$ *introduced in §1.*

Furthermore we assume that M, E and $H^q(M;\mathcal{O}_M(E))$ for $q \neq 0$ are all finite dimensional. Then the natural map $\varphi^{p,q}$ is an isomorphism

$$\varphi^{p,q} : {}'E_2^{p,q} \cong H^p(\mathfrak{g}; H^q(M;\mathcal{O}_M(E))).$$

As an application, we have

EXAMPLE 11.4: Let U be an open Riemann surface and S a finite subset of U. We denote by $L(U,S)$ the Lie algebra of complex analytic vector fields on U which have zeroes at all points in S. $L(U,S)$ is one of Lie algebras given in Proposition 11.3. Let $E \to M$ be an complex analytic vector bundle over a finite dimensional Stein manifold M. Suppose the Lie algebra $L(U,S)$ acts on the sheaf of topological linear spaces $\mathcal{O}_M(E)$ continuously. From Proposition 11.3 folllows

$$H^*_{L(U,S)}(M;\mathcal{O}_M(E)) = H^*(L(U,S);\mathcal{O}_M(E)(M)).$$

Hence we obtain a spectral sequence ${}''E_2^{p,q} = H^p(M;\mathcal{H}^q)$ converging to $H^{p+q}(L(U,S);\mathcal{O}_M(E)(M))$, where $\mathcal{H}^q$ is a sheaf over M whose stalk at $x \in M$ is given by

$$\mathcal{H}^q_x = H^q(L(U,S);\mathcal{O}_M(E)_x).$$

We call this sequence *the Rešetnikov spectral sequence* (see [Ka] §9).

Next we investigate the second spectral sequence ${}''E_2^{p,q}$ (11.2). The map D defined in §7 is concerned with this sequence. If $x \in O \subset M$, the evaluation map $\mathrm{ev}_x : \mathcal{O}_M(E)(O) \to E_x$ is a $\mathfrak{g}_x$ homomorphism, where $\mathfrak{g}_x$ is the isotropy algebra of $\mathfrak{g}$ at the point x. This implies the evaluation homomorphism

$$(\mathrm{ev}_x)_* : H^n(\mathfrak{g};\mathcal{O}_M(E))_x \to H^n(\mathfrak{g}_x;E_x) \tag{11.5}$$

is defined. Especially we have a natural map

$$H^n_{\mathfrak{g}}(M;\mathcal{O}_M(E)) \to {}''E_2^{0,n} \overset{\mathrm{ev}_*}{\to} H^0(M;\mathcal{O}_M(\coprod_{x\in M} H^n(\mathfrak{g}_x;E_x))).$$

Thus the two cohomology groups $H^*(\mathfrak{g}_x; E_x)$ and $H^*(M; \mathcal{O}_M(E))$ are connected by the $\mathfrak{g}$ equivariant cohomology group $H^*_{\mathfrak{g}}(M; \mathcal{O}_M(E))$.

Taking into consideration the finite dimensional case studied by Bott [B] (although the $\mathfrak{d}_\rho$ manifolds $M_{g,\rho}$ and $C_{g,\rho}$ are infinite dimensional in our case), we may regard the $\mathfrak{g}$ module $(\mathcal{O}_M(E))_x$ as the (co-)induced module of the $\mathfrak{g}_x$ module E_x ($x \in M$). Hence we put a general hypothesis that the evaluation homomorphism (11.5) is an isomorphism

$$(11.6) \qquad (\mathrm{ev}_x)_* : H^*(\mathfrak{g}; \mathcal{O}_M(E))_x \overset{\cong}{\to} H^*(\mathfrak{g}_x; E_x)$$

for all $x \in M$. It could be regarded as a certain kind of the Frobenius reciprocity laws, i.e., the Shapiro isomorphisms. Through the isomorphism (11.6) the vector bundle $\coprod_{x \in M} H^*(\mathfrak{g}_x; E_x)$ possesses the natural structure of a sheaf over M and we have an isomorphism

$$\text{``}E_2^{p,q} \cong H^p(M; H^q(\mathfrak{g}_x; E_x)),$$

where the RHS means the cohomology of M with values in the sheaf $\coprod_{x \in M} H^*(\mathfrak{g}_x; E_x)$. Thus, under the hypothesis (11.6), the assumption (A(n)) implies that the term '$E_2^{p,q}$ vanishes for $q < n$, so that

$$H^q_{\mathfrak{g}}(M; \mathcal{O}_M(E)) = \begin{cases} 0, & \text{if } q < n, \\ H^0(M; H^n(\mathfrak{g}_x; E_x)), & \text{if } q = n. \end{cases}$$

The map D introduced in §7

$$D : H^0(M; H^n(\mathfrak{g}_x; E_x)) \to H^n(M; \mathcal{O}_M(E))$$

is nothing but the composite of the above isomorphism and the natural map $H^n_{\mathfrak{g}}(M; \mathcal{O}_M(E)) \to H^n(M; \mathcal{O}_M(E))$.

Consider the $\mathfrak{d}_\rho$ manifolds $M = M_{g,\rho}$ and $C_{g,\rho}$ and the $\mathfrak{d}_\rho$ vector bundles $\bigwedge^n T^*M$, where we assume $\rho > 0$. Then the hypothesis (11.6) is

$$(11.7) \quad H^*(\mathfrak{d}_\rho; \mathcal{O}_M(E)(\bigwedge^n T^*M))_x \cong H^*((\mathfrak{d}_\rho)_x; \bigwedge^n T^*_x M)$$
$$\text{for all } n \geq 0 \text{ and all } x \in M = M_{g,\rho} \text{ and } C_{g,\rho}.$$

The hypothesis (11.7) seems to be true. But at present the author has no proof for the assertion (11.7).

Under the hypothesis (11.7), the sheaf structure on the vector bundle $\coprod_{x \in M} H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T^*_x M)$ induced by (11.7) coincides with that induced by the isomorphisms χ and $\overline{\chi}$ introduced in (6.2). Thus we have

$$\chi : H^n(L_0; \bigwedge^n Q^1) \cong H^0(C_{g,\rho}; H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T^*_x C_{g,\rho}))$$
$$\overline{\chi} : H^n(W_1; \bigwedge^n Q) \cong H^0(M_{g,\rho}; H^n((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^n T^*_{\overline{x}} M_{g,\rho})).$$

From Corollaries 2.9 and 2.11 the $\mathfrak{d}_\rho$ vector bundles $\bigwedge^n T^*M_{g,\rho}$ and $\bigwedge^n T^*C_{g,\rho}$ satisfy the assumption (A(n)). Consequently we conclude from Corollaries 10.2 and 10.3

THEOREM 11.8. *If the hypothesis (11.7) holds good, we have*

$$(1) \qquad \bigoplus_{p\geq q} H^{p,q}_{\mathfrak{d}_\rho}(M_{g,\rho}) = \mathbb{C}[e_n; n \geq 1]$$

$$(2) \qquad \bigoplus_{p\geq q} H^{p,q}_{\mathfrak{d}_\rho}(C_{g,\rho}) = \mathbb{C}[e, e_n; n \geq 1]$$

for all $g \geq 0$ and $\rho > 0$, where $e = c_1(T_{C_{g,\rho}/M_{g,\rho}}) \in H^{1,1}(C_{g,\rho})$ and $e_n \in H^{n,n}(M_{g,\rho})$ is the n-th Morita Mumford class ($n \in \mathbb{N}_{\geq 1}$).

This gives an affirmative evidence for the conjecture: the stable cohomology algebra of the moduli of compact Riemann surfaces would be generated by the Morita Mumford classes e_n's.

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