UTMS 93–29 June 29, 1993

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by

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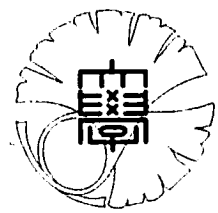

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Moduli space and Complex analytic Gel'fand Fuks cohomology of Riemann surfaces.

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Abstract. Let $C^\times$ be a once punctured compact Riemann surface and $L(C^\times)$ the Lie algebra consisting of all complex analytic vectors on $C^\times$. We determine the q-th cohomology group of $L(C^\times)$ with values in the complex analytic quadratic differentials on the p-fold product space $(C^\times)^p$ for the case $p \geq q$. The cohomology group vanishes for $p > q$, and, for $p = q$, it forms a trivial constant sheaf on the dressed moduli $M_{g,\rho}$ of compact Riemann surfaces of genus g. (Furthermore the stalk does not depend on the genus g.) This induces a natural map of the cohomology group for $p = q$ into the (p,p) cohomology of the moduli $M_{g,\rho}$. We prove the map is a stable isomorphism onto the subalgebra generated by the Morita Mumford classes $e_n = \kappa_n$'s, which gives an affirmative evidence for the conjecture: the stable cohomology algebra of the moduli of compact Riemann surfaces would be generated by the Morita Mumford classes.

INTRODUCTION.

Let M_g denote the moduli space of compact Riemann surfaces of genus g. In [H] Harer proved that the q-th rational cohomology group $H^q(M_g;\mathbf{Q})$ does not depend upon the genus g if $q < g/3$. This enables us to consider the stable cohomology group of the moduli of Riemann surfaces

$$\lim_{g\to\infty} H^*(M_g;\mathbf{Q}).$$

As was established by Miller [Mi] and Morita [Mo] independently, a polynomial algebra in countable many generators e_n's ($n \in \mathbf{N}_{\geq 1}$) is imbedded in the stable cohomology group

$$\mathbf{Q}[e_n; n \geq 1] \subset \lim_{g\to\infty} H^*(M_g;\mathbf{Q}). \tag{0.1}$$

Here $e_n \in H^{2n}(M_g;\mathbf{Q})$ is the n-th Morita Mumford characteristic class defined as follows [Mo] [Mu]. Let $C_g \to M_g$ denote the universal family of compact Riemann surfaces of genus g and let $e \in H^2(C_g)$ be the Euler class (the first Chern class) of the relative tangent bundle T_{C_g/M_g}.

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1991 Mathematical Subject Classification. Primary 14H15. Secondary 57R32.

The n-th Morita Mumford class e_n is defined as the fiber integral of the $(n+1)$-th power of the class e:

$$e_n := \int_{\text{fiber}} e^{n+1} \in H^{2n}(M_g; \mathbf{Q}). \tag{0.2}$$

It is natural to conjecture the injection (0.1) would be isomorphic [H1] [H2]. The purpose of the present paper is to give an affirmative evidence for the conjecture.

Let $L(U)$ denote the topological Lie algebra consisting of all complex analytic vector fields on an open Riemann surface U with the Fréchet topology of uniform convergence on compact sets. Fix an integer $g \geq 0$ and a positive real number $\rho > 0$. We denote by $M_{g,\rho}$ the dressed moduli of compact Riemann surfaces of genus g. Namely $M_{g,\rho}$ is the space consisting of all triples (C, p, z), where C is a compact Riemann surface of genus g, p is a point of C, and z is a complex coordinate of a neighbourhood U of p satisfying the conditions

$$z(p) = 0 \quad \text{and} \quad z(U) \supset \{z \in \mathbf{C}; |z| \leq \rho\}.$$

In view of the Harer stability [H] the rational cohomology group of $M_{g,\rho}$ is isomorphic to that of M_g in degree $< g/3$. As is observed in [BMS], the Lie algebra $\mathfrak{d}_\rho$ defined by

$$\mathfrak{d}_\rho := \varinjlim_{\rho_1 \downarrow \rho} L(\{z \in \mathbf{C}; 0 < |z| < \rho_1\})$$

acts on the space $M_{g,\rho}$ infinitesimally and transitively (§1).

Fix an arbitrary point $x = (C, p, z) \in M_{g,\rho}$ for the moment. In §1 it is proved that the isotropy algebra $(\mathfrak{d}_\rho)_x$ is equal to the Lie algebra $L(C - \{p\})$ and the cotangent space $T_x^* M_{g,\rho}$ is canonically isomorphic to the $(\mathfrak{d}_\rho)_x$ module consisting of all complex analytic quadratic differentials on $C - \{|z| \leq \rho\}$. In view of the nuclear theorem the p-fold alternating tensor product $\bigwedge^p T_x^* M_{g,\rho}$ is equal to the $(\mathfrak{d}_\rho)_x$ module of complex analytic quadratic differentials on the p-fold product space $(C - \{|z| \leq \rho\})^p$. Our first main result states as follows (Theorem 2.1):

THEOREM 0.3. *For any $x = (C, p, z) \in M_{g,\rho}$ we have*

$$H^q((\mathfrak{d}_\rho)_x; \bigwedge^p T_x^* M_{g,\rho}) = 0 \quad \text{if } p > q, \tag{1}$$

$$\bigoplus_{p \geq 0} H^p((\mathfrak{d}_\rho)_x; \bigwedge^p T_x^* M_{g,\rho}) = \mathbf{C}[\kappa_n; n \geq 1], \tag{2}$$

where $\kappa_n \in H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T_x^* M_{g,\rho})$ *is defined for* $n \geq 1$.

A similar result for the universal family

$$C_{g,\rho} := \coprod_{(C,p,z)\in M_{g,\rho}} C - \{|z| \leq \rho\}$$

is obtained (Theorem 2.1(2)(4)). In that theorem we have to add one class $\epsilon \in H^1((\mathfrak{d}_\rho)_x; T_x^* C_{g,\rho})$ to the generators κ_n's. The class ϵ corresponds to the Euler class $e = c_1(T_{C_{g,\rho}/M_{g,\rho}})$ under our framework (Theorem 8.2). In §2, utilizing a general theory of the cohomology of the Lie algebra $L(U)$, i.e., the complex analytic Gel'fand Fuks cohomology for open Riemann surfaces U [Ka], Theorem 0.3 is reduced to the case $g = 0$. In §3 and §4 the theorem for $g = 0$ (modulo the algebraic independency of the classes κ_n's) is established.

When the point $x = (C, p, z) \in M_{g,\rho}$ runs over the whole $M_{g,\rho}$, the cohomology group $H^p((\mathfrak{d}_\rho)_x; \bigwedge^p T_x^* M_{g,\rho})$ forms a trivial constant sheaf over the moduli $M_{g,\rho}$, and the class κ_n extends to a nowhere zero global section of the sheaf. From Theorem 0.3(1) we can define a natural map

$$D : H^0(M_{g,\rho}; H^p((\mathfrak{d}_\rho)_x; \bigwedge^p T_x^* M_{g,\rho})) \to H^{p,p}(M_{g,\rho}).$$

As is proved in §10, the class κ_n corresponds to the n-th Morita Mumford class e_n under the map D. It follows from the result of Miller and Morita quoted above there exist no algebraic relations among the classes κ_n's. Consequently the map D is a stable isomorphism onto the subalgebra of $\bigoplus_{p\geq 0} H^{p,p}(M_{g,\rho})$ generated by the Morita Mumford classes. Our cohomology group $H^p((\mathfrak{d}_\rho)_x; \bigwedge^p T_x^* C_{g,\rho})$ and especially the n-th power of the class ϵ are related to "the infinite dimensional Chern Weil theory" of Feigin and Tsygan [FT] through the fundamental exact sequence [Ka1]. Our present work, however, has no dependence on their theory.

What is mentioned above can be interpreted by an equivariant cohomology theory for Lie algebras as follows. Since the Lie algebra $\mathfrak{d}_\rho$ acts on the moduli $M_{g,\rho}$ complex analytically, the sheaf $\Omega^n_{M_{g,\rho}}$ of germs of n-forms on $M_{g,\rho}$ is a sheaf of $\mathfrak{d}_\rho$ modules ($n \in \mathbf{N}_{\geq 0}$). We denote by $H^{n,*}_{\mathfrak{d}_\rho}(M_{g,\rho})$ the hypercohomology of the cochain complex of sheaves

$$C^*(\mathfrak{d}_\rho; \Omega^n_{M_{g,\rho}}) : M_{g,\rho} \overset{\text{open}}{\supset} O \mapsto C^*(\mathfrak{d}_\rho; \Omega^n_{M_{g,\rho}}(O))$$

and call it *the* $\mathfrak{d}_\rho$ *equivariant* $(n,*)$ *cohomology* (§11). Here, for a $\mathfrak{d}_\rho$ module N, $C^*(\mathfrak{d}_\rho; N)$ means the standard cochain complex of the Lie

algebra $\mathfrak{d}_\rho$ with values in the module N. There exist two spectral sequences converging to the equivariant cohomology

$$
\begin{aligned}
{}'E_2^{p,q} &= H^p(H^q(M_{g,\rho}; C^*(\mathfrak{d}_\rho; \Omega^n_{M_{g,\rho}}))) \quad \text{and} \\
{}''E_2^{p,q} &= H^p(M_{g,\rho}; H^q(C^*(\mathfrak{d}_\rho; \Omega^n_{M_{g,\rho}}))),
\end{aligned} \tag{0.4}
$$

where $H^q(\mathfrak{d}_\rho; \Omega^n_{M_{g,\rho}})$ is the sheaf over $M_{g,\rho}$ defined as the cohomology of the cochain complex of sheaves $C^*(\mathfrak{d}_\rho; \Omega^n_{M_{g,\rho}})$. The map D is concerned with the second spectral sequence ${}''E_2$.

Taking into consideration the finite dimensional case studied by Bott [B] (although $\dim M_{g,\rho} = \infty$ in the present case), we may regard the $\mathfrak{d}_\rho$ module $(\Omega^n_{M_{g,\rho}})_x$ as the (co-)induced module of the $(\mathfrak{d}_\rho)_x$ module $\bigwedge^n T^*_x M_{g,\rho}$ ($x \in M_{g,\rho}$). Hence we put a general hypothesis

$$
H^*(\mathfrak{d}_\rho; \Omega^n_{M_{g,\rho}})_x \cong H^*((\mathfrak{d}_\rho)_x; \bigwedge^n T^*_x M_{g,\rho}) \tag{0.5}
$$

for all $x \in M_{g,\rho}$. The hypothesis (0.5) seems to be true. In fact it could be regarded as a certain kind of the Frobenius reciprocity laws, i.e., the Shapiro isomorphisms. But at present the author has no proof of the assertion (0.5).

Under the hypothesis (0.5), Theorem 0.3(1) implies that the term ${}''E_2^{p,q}$ (0.4) vanishes for $q < n$, so that

$$
H^{n,n}_{\mathfrak{d}_\rho}(M_{g,\rho}) = H^0(M_{g,\rho}; H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T^*_x M_{g,\rho})). \tag{0.6}
$$

The map D is the composite of the isomorphism (0.6) and the natural map $H^{n,n}_{\mathfrak{d}_\rho}(M_{g,\rho}) \to H^{n,n}(M_{g,\rho})$. Thus we have (Theorem 11.8)

COROLLARY 0.7. *If the hypothesis (0.5) holds good,*

$$
\bigoplus_{p \geq q} H^{p,q}_{\mathfrak{d}_\rho}(M_{g,\rho}) = \mathbf{C}[e_n; n \geq 1].
$$

This suggests that it is reasonable to conjecture that the stable cohomology algebra of the moduli of compact Riemann surfaces is generated by the Morita Mumford classes e_n's.

The author would like to thank all those who gave him help and advice, including Professors A. Hattori, K. Saito, S. Morita, T. Yoshida, T. Oda, K. Iwasaki, Doctors K. Ohba and K. Ahara. Especially he would like to express his gratitude to his supervisor Prof. Yukio Matsumoto for his constant encouragement and patience. This paper is dedicated to the memory of the author's loving mother Chieko Kawazumi.

CONTENTS:

LIST OF SYMBOLS:

§1: D_ρ, $\overline{D_\rho}$, $D_\rho^\times$, $L(U)$, $L(U, p_1)$, $\mathfrak{d}_\rho$, $M_{g,\rho}$, $C^\times$, C_ρ, $C_{g,\rho}$, $C_{g,\rho}^\times$, $\overline{D_{g,\rho}^\times}$, $\pi_{g,\rho}$, K, $Q(U)$, $Q^\lambda(U, p_1)$.

§2: $C^*(\mathfrak{g}; N)$, $H^*(\mathfrak{g}; N)$, $C^*(\mathfrak{g})$, $H^*(\mathfrak{g})$, W_1, L_0, Q, Q^λ, $\mathcal{Q}_n$, A_τ, Δ.

§3: 1_ν, T_ν, F, K, $W_1^\sharp$, $L_0^\sharp$, $T_\nu^\sharp$, $F^\sharp$, $K^\sharp$, $Q^\sharp$, $I(N)$, $Q^\times$, i_{n+1}, $\epsilon_1 \epsilon^n$.

§4: $\mathcal{KQ}_n$, O_n, $(K \otimes \bigwedge^n Q)^\times$, ψ_n, $\oint_t$, $\int_{\text{fiber}}$, κ_n, ∇_1, ∇_2, η_n.

§5: $\mathfrak{U} = \{U_i\}_{i=1}^n$, $\mathfrak{V} = \{V_1\} \cup \{V_j\}_{j=3}^n$, $\Theta_{n;1}$, $\Theta_{n;2}$, $\oint_{t;z_1}$, $\oint_{t;z_2}$, θ_n.

§6: $\overline{\chi}_{p_1,w}^{\overline{x}} = \overline{\chi}_{p_1}^{\overline{x}} = \overline{\chi}^{\overline{x}}$, $\chi_{p_1}^x = \chi^x$, $\overline{\chi}$, χ, Res_{p_1}, $F(U)$.

§7: μ, $\mathfrak{g}_x$, $(\text{A}(n))$, D.

§8: e, I, $\mathfrak{h}_x$, $\mathfrak{g}/\mathfrak{h}$, μ_x, $L_1(C^\times, p_1)$.

§9: $\widehat{\chi}(\epsilon)$, $\iota_{g,\rho}$, $\widehat{\chi}(\epsilon)$.

§10: $\widehat{\chi}(\eta_n)$, $\widehat{\chi}(\epsilon_1 \epsilon^n)$.

§11: $H_{\mathfrak{g}}^*(M; \mathcal{O}_M(E))$, $H_{\mathfrak{g}}^{n,*}(M)$.

1. Homogeneity of the dressed moduli of Riemann surfaces.

We review the infinitesimal homogeneity of the dressed moduli $M_{g,\rho}$ of Riemann surfaces [BMS] following [ADKP], and represent the cotangent space of $M_{g,\rho}$ as a space of quadratic differentials through a generalized Köthe duality [Koe].

First we fix our notations. Let $L(U)$ denote the topological Lie algebra consisting of all complex analytic vector fields on an open Riemann surface U with the Fréchet topology of uniform convergence on compact sets. The closed subalgebra consisting of all vector fields which have a zero at a (fixed) point $p_1 \in U$ is denoted by $L(U, p_1)$. Let $\rho \geq 0$ be a non-negative real number. Set

$$D_\rho := \{z \in \mathbf{C}; |z| < \rho\}, \quad \overline{D_\rho} := \{z \in \mathbf{C}; |z| \leq \rho\},$$
$$D_\rho^\times := \{z \in \mathbf{C}; 0 < |z| < \rho\} \quad \text{and} \quad \overline{D_\rho^\times} := \{z \in \mathbf{C}; 0 < |z| \leq \rho\}$$

The topological Lie algebra $\mathfrak{d}_\rho$ is defined by

$$\mathfrak{d}_\rho := \varinjlim_{\rho_1 \downarrow \rho} L(D_{\rho_1}^\times),$$

which is endowed with the inductive limit locally convex topology [G, TVS] [Ko] [Ko1].

Fix an integer $g \geq 0$. We denote by $M_{g,\rho}$ the dressed moduli of compact Riemann surfaces of genus g. Namely $M_{g,\rho}$ is the space consisting of all triples (C, p, z), where C is a compact Riemann surface of genus g, p is a point of C, and z is a complex coordinate of a neighbourhood U of p satisfying the conditions

$$z(p) = 0 \quad \text{and} \quad z(U) \supset \overline{D_\rho}. \tag{1.1}$$

For a point $(C, p, z) \in M_{g,\rho}$, we denote

$$C^\times := C - \{p\} \quad \text{and} \quad C_\rho := C - z^{-1}(\overline{D_\rho}).$$

Furthermore we denote

$$C_{g,\rho} := \bigcup_{(C,p,z) \in M_{g,\rho}} C_\rho$$
$$C_{g,\rho}^\times := \bigcup_{(C,p,z) \in M_{g,\rho}} C^\times$$
$$\overline{D_{g,\rho}^\times} := \bigcup_{(C,p,z) \in M_{g,\rho}} \overline{D_\rho^\times} = M_{g,\rho} \times \overline{D_\rho^\times} \subset C_{g,\rho}^\times.$$

We have inclusions $C_{g,\rho} \subset C^{\times}_{g,\rho}$ and $\overline{D^{\times}_{g,\rho}} \subset C^{\times}_{g,\rho}$ and a natural projection $\pi_{g,\rho} : C^{\times}_{g,\rho} \to M_{g,\rho}$ $(C, p, z, p_1) \mapsto (C, p, z)$. Each of these spaces has a natural fiber structure over $M_{g,\rho}$ through the projection $\pi_{g,\rho}$.

Now we introduce the infinitesimal action of the topological Lie algebra $\mathfrak{d}_\rho$ on the space $M_{g,\rho}$. (see [ADKP](3.19)).

Fix $X \in \mathfrak{d}_\rho$ and $(C, p, z) \in M_{g,\rho}$. For some $\rho_1 > \rho_2 > \rho$, $z(U)$ includes $\overline{D_{\rho_1}}$, and the integral $\exp \epsilon X : D^{\times}_{\rho_2} \to D^{\times}_{\rho_1}$ of the vector field X exists for sufficient small complex number ϵ. A complex analytic path

$$\epsilon \mapsto (C(\epsilon), p(\epsilon), z(\epsilon)) \in M_{g,\rho}$$

is defined by

$$\begin{aligned} &C(\epsilon) := (C^{\times} \amalg D_{\rho_2})/\sim_\epsilon \\ &\qquad z \sim_\epsilon (\exp \epsilon X)(z), \quad z \in D_{\rho_2} \\ &p(\epsilon) := \text{the image of } 0 \in D_{\rho_2} \text{ on } C(\epsilon) \\ &z(\epsilon) := \text{the coordinate of } C(\epsilon) \text{ induced by the identity } D_{\rho_2} \hookrightarrow D_{\rho_2}. \end{aligned}$$

Differentiating the path $(C(\epsilon), p(\epsilon), z(\epsilon))$, we obtain the map

$$P = P_{(C,p,z)} : \mathfrak{d}_\rho \to T_{(C,p,z)} M_{g,\rho}, \quad X \mapsto \frac{d}{d\epsilon}\Big|_{\epsilon=0} (C(\epsilon), p(\epsilon), z(\epsilon)).$$

P defines the desired infinitesimal action of $\mathfrak{d}_\rho$ on $M_{g,\rho}$.

Similarly the algebra $\mathfrak{d}_\rho$ acts on the space $C^{\times}_{g,\rho}$. In fact, for $X \in \mathfrak{d}_\rho$ and $(C, p, z, p_1) \in C^{\times}_{g,\rho}$, (i.e., $(C, p, z) \in M_{g,\rho}$ and $p_1 \in C^{\times} = C - \{p\}$,) we can define a complex analytic path

$$\epsilon \mapsto (C(\epsilon), p(\epsilon), z(\epsilon), p_1(\epsilon)) \in C^{\times}_{g,\rho}$$

by the same formulae for $(C(\epsilon), p(\epsilon), z(\epsilon))$ and

$$p_1(\epsilon) := \text{the image of } p_1 \in C^{\times} \text{ on } C(\epsilon).$$

Differentiating the path $(C(\epsilon), p(\epsilon), z(\epsilon), p_1(\epsilon))$, we obtain the map

$$\begin{aligned} P = P_{(C,p,z,p_1)} : \mathfrak{d}_\rho &\to T_{(C,p,z,p_1)} C^{\times}_{g,\rho} \\ X &\mapsto \frac{d}{d\epsilon}\Big|_{\epsilon=0} (C(\epsilon), p(\epsilon), z(\epsilon), p_1(\epsilon)). \end{aligned}$$

P defines the desired action of $\mathfrak{d}_\rho$ on $C^{\times}_{g,\rho}$. These actions are compatible to the fiber structure on the space $C^{\times}_{g,\rho}$ over $M_{g,\rho}$.

LEMMA 1.2.([ADKP] PROPOSITION 3.19). *The sequences*

$$0 \to L(C^\times) \hookrightarrow \mathfrak{d}_\rho \xrightarrow{P} T_{(C,p,z)}M_{g,\rho} \to 0 \tag{1}$$

$$0 \to L(C^\times, p_1) \hookrightarrow \mathfrak{d}_\rho \xrightarrow{P} T_{(C,p,z,p_1)}C_{g,\rho} \to 0 \tag{2}$$

are exact. Here the algebras $L(C^\times)$ *and* $L(C^\times, p_1)$ *are regarded as subalgebras of* $\mathfrak{d}_\rho$ *through the coordinate* z.

PROOF: Denote by M_g the moduli space of compact Riemann surfaces of genus g and Θ_C the sheaf of germs of complex analytic vector fields on $C \in M_g$. Then we have the following morphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & L(C^\times) \oplus L(\overline{D_\rho}) & \to & \mathfrak{d}_\rho & \to & H^1(C;\Theta_C) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & L(\overline{D_\rho}) & \to & T_{(C,p,z)}M_{g,\rho} & \to & T_C M_g & \to & 0, \end{array}$$

where $L(\overline{D_\rho}) = \varinjlim_{\rho_1 \downarrow \rho} L(D_{\rho_1})$. The exactness of the upstairs sequennce follows from the facts $\{C^\times, D_{\rho_1}\}$ is a Stein covering of C and $H^0(C;\Theta_C) = 0$. The left vertical is the second projection, the central is the map P, and the right is the Kodaira Spencer isomorphism. Chasing the diagram, we reach (1). (2) follows from (1) immediately.

Consequently we obtain a representation of the cotangent spaces $T^*_{(C,p,z)}M_{g,\rho}$ and $T^*_{(C,p,z,p_1)}C_{g,\rho}$.

COROLLARY 1.3.

$$T^*_{(C,p,z)}M_{g,\rho} = (\mathfrak{d}_\rho / L(C^\times))^*$$
$$T^*_{(C,p,z,p_1)}C_{g,\rho} = (\mathfrak{d}_\rho / L(C^\times, p_1))^*$$

Here and throughout this paper the asterisque $*$ *means the strong dual.*

To identify the dual spaces $(\mathfrak{d}_\rho / L(C^\times))^*$ and $(\mathfrak{d}_\rho / L(C^\times, p_1))^*$ we introduce a generalized Köthe duality [Koe] [Ko2]. Fix a point $(C, p, z) \in M_{g,\rho}$. The disk $\overline{D_\rho}$ is regarded as a subset of C through the coordinate z. Let E be a complex analytic line bundle over C. For an open set U in C, we denote by $E(U)$ the space of complex analytic sections of E on U:

$$E(U) = H^0(U; \mathcal{O}_C(E)).$$

$E(U)$ is a Fréchet space with respect to the topology of uniform convergence on compact sets. For a closed set F in C, we denote by $E(F)$ the inductive limit locally convex space

$$E(F) = \varinjlim_{F \subset U} E(U),$$

where U runs over all the open neighbourhoods of F in C. Similarly we define the locally convex space $E(\overline{D_\rho^\times})$ by

$$E(\overline{D_\rho^\times}) = \varinjlim_{\rho_1 \downarrow \rho} E(D_{\rho_1}^\times).$$

The canonical bundle of C, namely, the cotangent bundle of C is denoted by K. We have $(K^{-1})(\overline{D_\rho^\times}) = \mathfrak{d}_\rho$.

THEOREM 1.4. (A GENERALIZED KÖTHE DUALITY). *The map*

$$\eta : (E^{-1} \otimes K)(C_\rho) \to (E(\overline{D_\rho^\times})/E(C^\times))^*$$
$$s \mapsto (f \mapsto \frac{1}{2\pi\sqrt{-1}} \oint f \cdot s)$$

is a topological isomorphism. Here $f \cdot s$ intends a 1-form defined on an annulus $\{\rho < |z| < \rho + \varepsilon\}(0 < \varepsilon \ll 1)$, and the line integral $\oint$ is carried out on the circle $\{|z| = \rho + \varepsilon/2\}$.

Denote by $Q(U)$ the Fréchet space of complex analytic quadratic differentials on a Riemann surface U:

$$Q(U) = H^0(U; \mathcal{O}_U((T^*C)^{\otimes 2})) = H^0(U; \mathcal{O}_U(K^{\otimes 2})).$$

Let $\lambda \in \mathbf{Z}$ and $p_1 \in U$. We denote by $Q^\lambda(U, p_1)$ the Fréchet space of meromorphic quadratic differentials on U with a pole only at p_1 of order $\leq \lambda$.

$$Q^\lambda(U, p_1) = H^0(U; \mathcal{O}_U(K^{\otimes 2} \otimes [p_1]^{\otimes \lambda})).$$

COROLLARY 1.5. *We have topological isomorphisms*

$$T^*_{(C,p,z)} M_{g,\rho} = Q(C_\rho) \quad \text{and} \quad T^*_{(C,p,z,p_1)} C_{g,\rho} = Q^1(C_\rho, p_1).$$

By Stokes' Theorem these isomorphisms are $(\mathfrak{d}_\rho)_{(C,p,z)} = L(C^\times)$ and $(\mathfrak{d}_\rho)_{(C,p,z,p_1)} = L(C^\times, p_1)$ equivariant, respectively.

PROOF OF THEOREM 1.4: The given map is well defined because of Stokes' Theorem. From a general theory of locally convex spaces [Ko1] Theorem 12, p.377, we have topological isomorphisms

$$E(\overline{D_\rho})^* = \varinjlim_{\rho_1 \downarrow \rho} (E(D_{\rho_1})^*) \tag{1.6}$$

$$(E^{-1} \otimes K)(\overline{C_{\rho_1}})^* = \varinjlim_{\rho_2 \uparrow \rho_1} ((E^{-1} \otimes K)(C_{\rho_2})^*) \tag{1.7}$$

It suffices to prove the theorem for a line bundle $E \otimes [p]^{\otimes n}$ $(n \ll -1)$, where $[p]$ is the line bundle induced by the divisor p. Hence we may assume $H^0(E) = H^0(C; \mathcal{O}_C(E)) = 0$. Consider an exact sequence of Fréchet spaces

$$0 \to E(C^\times) \oplus E(D_{\rho_1}) \to E(D^\times_{\rho_1}) \to H^1(E) \to 0$$

for $\rho_1 > \rho$ satisfying $(C, p, z) \in M_{g,\rho_1}$. We have a closed lift $\tilde{H} \subset E(D^\times_{\rho_1})$, since $H^1(E) = H^1(C; \mathcal{O}_C(E))$ is finite dimensional (see [G, TVS] I, §12). A topological isomorphism

$$E(D^\times_{\rho_1}) = E(C^\times) \oplus E(D_{\rho_1}) \oplus \tilde{H}$$

follows from Open Mapping Theorem. Passing to the limit $\rho_1 \to \rho$ and using (1.6), we obtain

$$(1.8) \quad \varinjlim_{\rho_1 \downarrow \rho} (E(D^\times_{\rho_1})/E(C^\times))^* = E(\overline{D_\rho})^* \oplus \tilde{H}^* = (E(\overline{D^\times_\rho})/E(C^\times))^*.$$

Since $H^1(E^{-1} \otimes K) = H^0(E)^* = 0$, we have an exact sequence

$$(1.9) \quad 0 \to H^0(E^{-1} \otimes K) \to (E^{-1} \otimes K)(C_{\rho_2}) \to \frac{(E^{-1} \otimes K)(\{\rho_2 < |z| < \rho_1\})}{(E^{-1} \otimes K)(D_{\rho_1})} \to 0.$$

The right space is topologically isomorphic to $E(\overline{D_{\rho_2}})^*$ because of the original Köthe duality ([Koe] or [Ko2] Theorem 3.1, p.115). $E(\overline{D_{\rho_2}})$ is reflexive ([Ko1] Theorem 6, p.372). From Open Mapping Theorem the sequence (1.9) is topologically exact. Therefore its dual

$$0 \to E(\overline{D_{\rho_2}}) \to (E^{-1} \otimes K)(C_{\rho_2})^* \to H^0(E^{-1} \otimes K)^* \to 0$$

is exact as linear spaces by Hahn-Banach Theorem. From Mittag-Leffler Lemma ([P] or [Ko] III Lemma 10.1, p.314)

$$\varprojlim_{\rho_2 \uparrow \rho_1} {}^1E(\overline{D_{\rho_2}}) = \varinjlim_{\rho_2 \uparrow \rho_1} {}^1E(D_{\rho_2}) = 0$$

follows. Hence we obtain an exact sequence of linear spaces

$$0 \to E(D_{\rho_1}) \to (E^{-1} \otimes K)(\overline{C_{\rho_1}})^* \to H^0(E^{-1} \otimes K)^* \to 0,$$

where we utilize (1.7). Assembling it with the sequence associated to $H^*(E)$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & E(D_{\rho_1}) & \to & E(D_{\rho_1}^{\times})/E(C^{\times}) & \to & H^1(E) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & E(D_{\rho_1}) & \to & (E^{-1}\otimes K)(\overline{C_{\rho_1}})^* & \to & H^0(E^{-1}\otimes K)^* & \to & 0. \end{array}$$

The right vertical arrow is the Serre duality. Therefore

$$\eta : E(D_{\rho_1}^{\times})/E(C^{\times}) \to (E^{-1}\otimes K)(\overline{C_{\rho_1}})^* \tag{1.10}$$

is a continuous bijection between Fréchet spaces, namely, a topological isomorphism. Dualizing η, passing to the limit $(\rho_1 \to \rho)$ and utilizing (1.8), we have

$$(E^{-1}\otimes K)(C_\rho) \cong \varinjlim_{\rho_1 \downarrow \rho} (E(D_{\rho_1}^{\times})/E(C^{\times}))^* = (E(\overline{D_\rho^{\times}})/E(C^{\times}))^*,$$

which completes the proof of Theorem 1.3.

2. Complex analytic Gel'fand Fuks cohomology.

In the following 3 sections, we fix arbitraty points $\overline{x} = (C, p, z) \in M_{g,\rho}$ and $x = (C, p, z, p_1) \in C_{g,\rho}$, and prove

THEOREM 2.1.

(1) $H^q((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^p T^*_{\overline{x}} M_{g,\rho}) = 0 \quad \text{if } p > q,$

(2) $H^q((\mathfrak{d}_\rho)_x; \bigwedge^p T^*_x C_{g,\rho}) = 0 \quad \text{if } p > q,$

(3) $\bigoplus_{p\geq 0} H^p((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^p T^*_{\overline{x}} M_{g,\rho}) = \mathbf{C}[\kappa_n; n \geq 1]/\textit{relations},$

(4) $\bigoplus_{p\geq 0} H^p((\mathfrak{d}_\rho)_x; \bigwedge^p T^*_x C_{g,\rho}) = \mathbf{C}[\epsilon, \kappa_n; n \geq 1]/\textit{relations},$

where $\kappa_n \in H^n((\mathfrak{d}_\rho)_{\overline{x}}; \bigwedge^n T^*_{\overline{x}} M_{g,\rho})$ *is defined in* §4 *and* $\epsilon \in H^1((\mathfrak{d}_\rho)_x; T^*_x C_{g,\rho})$ *is defined in* §3.

Here and throughout this paper we mean by $\bigwedge^n$ the completed n-fold alternating tensor product.

From Lemma 1.2 we have to discuss the cohomology theory of the Lie algebra of complex analytic vector fields on open Riemann surfaces, i.e., the complex analytic Gel'fand Fuks cohomology theory for open Riemann surfaces.

We fix our notation relating to the cohomology theory of Lie algebras. Let $\mathfrak{g}$ be a complex topological Lie algebra. We mean by a $\mathfrak{g}$ *module* a complex topological vector space which $\mathfrak{g}$ acts on continuously. The standard continuous cochain complex of the topological Lie algebra $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$ module N is denoted by

$$C^*(\mathfrak{g};N)=\bigoplus_{p\geq 0}C^p(\mathfrak{g};N),$$

where $C^p(\mathfrak{g};N)$ is the linear space of continuous alternating multilinear mappings $c:\mathfrak{g}^{\otimes p}\to N$. The cohomology group of the complex $C^*(\mathfrak{g};N)$ is called the (continuous) cohomology group of $\mathfrak{g}$ with coefficients in N and denoted by $H^*(\mathfrak{g};N)$. When N is the trivial $\mathfrak{g}$ module $\mathbf{C}$, we abbreviate them to $C^*(\mathfrak{g})$ and $H^*(\mathfrak{g})$ respectively. (For details, see for example [HS].)

Let U be a connected open Riemann surface satisfying $b_1(U)<\infty$. As in §1 let $L(U)$ denote the topological Lie algebra consisting of all complex analytic vector fields on an open Riemann surface U, $L(U,p_1)$, $p_1\in U$, the subalgebra $L(U,p_1):=\{X\in L(U);X(p_1)=0\}$ of $L(U)$, $Q(U)$ the $L(U)$ module consisting of all complex analytic quadratic differentials on U, and $Q^\lambda(U,p_1)$ the $L(U,p_1)$ module consisting of all meromorphic quadratic differentials on U with a pole only at p_1 of order $\leq\lambda$. In the case $U=\mathbf{C}$ and $p_1=0\in\mathbf{C}$, we abbreviate

$$W_1:=L(\mathbf{C}),\quad L_0:=L(\mathbf{C},0),\quad Q:=Q(\mathbf{C})\quad\text{and}\quad Q^\lambda:=Q^\lambda(\mathbf{C},0).$$

Fix an integer $n\in\mathbf{N}$. In this section we prove

PROPOSITION 2.2. *$H^q(L(U);\bigwedge^n Q(U))=0$, if $q<n$, and there exists a natural isomorphism*

$$H^n(L(U);\bigwedge^n Q(U))=H^n(W_1;\bigwedge^n Q).$$

We define a sheaf $\mathcal{Q}_n$ over U^n by

$$\mathcal{Q}_n:=\mathcal{O}_{U^n}(\bigotimes_{i=1}^n \mathrm{pr}_i^*(T^*U)^{\otimes 2}),\tag{2.3}$$

where $\mathrm{pr}_i:U^n\to U$ is the i-th projection $(1\leq i\leq n)$. The symmetric group $\mathfrak{S}_n$ acts on the sheaf $\mathcal{Q}_n$ by the permutation of the i-th components $(1\leq i\leq n)$ twisted by the sign. In view of the nuclear theorem, we have

$$\mathcal{Q}_n(U^n)=Q(U)^{\otimes n}\quad\text{and}\quad\mathcal{Q}_n(U^n)^{\mathfrak{S}_n}=\bigwedge^n Q(U).$$

Taking the $\mathfrak{S}_n$ invariant part of the Rešetnikov spectral sequence (Example (11.4), [Ka]§9) converging to $H^*(L(U); Q(U)^{\otimes n})$, we obtain a spectral sequence

$$(2.4) \qquad E_2^{p,q} = H^p(U^n; H^q(L(U); \mathcal{Q}_n))^{\mathfrak{S}_n} \Rightarrow H^{p+q}(L(U); \bigwedge^n Q(U)),$$

Here $H^q(L(U); \mathcal{Q}_n)$ is a sheaf over U^n whose stalk at $z \in U^n$ is the q-th cohomology of $L(U)$ with values in the stalk $(\mathcal{Q}_n)_z$ of the sheaf $\mathcal{Q}_n$. Let $(\mathfrak{S}_n)_z$ denote the isotropy group of $\mathfrak{S}_n$ at the point $z \in U^n$. If $H^q(L(U); \mathcal{Q}_n)^{\mathfrak{S}_n}$ is the subsheaf of $H^q(L(U); \mathcal{Q}_n)$ whose stalk at $z \in U^n$ is

$$H^q(L(U); \mathcal{Q}_n)^{\mathfrak{S}_n}{}_z = H^q(L(U); (\mathcal{Q}_n)_z)^{(\mathfrak{S}_n)_z},$$

then the E_2 term (2.4) is given by

$$E_2^{p,q} = H^p(U^n; H^q(L(U); \mathcal{Q}_n)^{\mathfrak{S}_n})^{\mathfrak{S}_n}$$

(see [Ka1]§4).

We introduce a natural stratification on U^n. For an increasing sequence of integers

$$(2.5) \qquad \mathfrak{r} : 0 = r_0 < r_1 < \cdots < r_l = n,$$

we denote by $A_\mathfrak{r}$ the locally closed subset of U^n

$$A_\mathfrak{r} := \left\{ \begin{array}{r} z = (z_1, \ldots, z_n) \in U^n; z_i = z_{r_k}, \text{if } r_{k-1} < i < r_k, \text{ and} \\ z_{r_k} \neq z_{r_j}, \text{if } k \neq j. \end{array} \right\}$$

Clearly each subset $A_\mathfrak{r}$ is connected and

$$(2.6) \qquad U^n = \bigcup_{\sigma \in \mathfrak{S}_n} \bigcup_{\mathfrak{r}} \sigma(A_\mathfrak{r}).$$

If $r_1 = n$, $A_\mathfrak{r}$ is the diagonal set $\Delta = \Delta(U) \subset U^n$.

The stalk at $z \in A_\mathfrak{r}$ of the sheaf $H^*(L(U); \mathcal{Q}_n)^{\mathfrak{S}_n}$ is given by

$$\begin{aligned} &H^*(L(U); \mathcal{Q}_n)^{\mathfrak{S}_n}{}_z \\ =&\bigwedge{}^*(\Sigma^3 H_1(U, \{z_1, \ldots, z_n\})) \otimes \bigotimes_{k=1}^{l} H^*(W_1; \bigwedge{}^{r_k - r_{k-1}} Q) \end{aligned}$$

from [Ka] Theorem 5.3 (see also [Ka1] Lemmata 2.3 and 4.6). Here $\bigwedge^* \Sigma^3 H_1$ is the free graded commutative algebra generated by the graded

vector space $\Sigma^3 H_1$ concentrated to degree 2 and isomorphic to the first complex valued singular homology of the pair $(U, \{z_1, \ldots, z_n\})$ as a complex vector space. If $q < p$, $H^q(W_1; \bigwedge^p Q) = 0$ from [Ka1] Theorem 4.7. Consequently $H^q(L(U); \mathcal{Q}_n)^{\mathfrak{S}_n} = 0$, if $q < n$, and

$$H^n(L(U); \mathcal{Q}_n)^{\mathfrak{S}_n}{}_z = \bigotimes_{k=1}^{l} H^*(W_1; \bigwedge^{r_k - r_{k-1}} Q) \tag{2.7}$$

for $z \in A_{\mathfrak{r}}$. We abbreviate $\mathcal{F} := H^n(L(U); \mathcal{Q}_n)^{\mathfrak{S}_n}$. Substituting them to the spectral sequence (2.4), we obtain the first half of Proposition 2.2 and a natural isomorphism

$$H^n(L(U); \bigwedge^n Q(U)) = H^0(U^n; \mathcal{F})^{\mathfrak{S}_n}.$$

Fix a base point $p_1 \in U$ and consider the evaluation map

$$\begin{aligned} \text{ev} : H^n(L(U); \bigwedge^n Q(U)) &= H^0(U^n; \mathcal{F})^{\mathfrak{S}_n} \\ &\to \mathcal{F}_{(p_1, \ldots, p_1)} \cong H^n(W_1; \bigwedge^n Q). \end{aligned} \tag{2.8}$$

It must be proved that the map ev is isomorphic. From (2.7) the topological structure of the sheaf $\mathcal{F}$ depends only on the topology of U and the stratification $A_{\mathfrak{r}}$'s. Therefore we may assume that U admits an complex analytic immersion into the complex line $i : U \looparrowright \mathbf{C}$. Then $Q(U)$ is a $W_1 = L(\mathbf{C})$ module through the immersion i. We remark $H^n(W_1; \bigwedge^n Q(U)) = H^0(U^n; \mathcal{F})^{\mathfrak{S}_n}$. It follows from [Ka1] Lemma 2.3 the composite

$$H^n(W_1; \bigwedge^n Q) \xrightarrow{i_*} H^n(W_1; \bigwedge^n Q(U)) \xrightarrow{\text{ev}} H^n(W_1; \bigwedge^n Q)$$

is the identity map. Thus it suffices to show the injectivity of the map ev (2.8).

Let $\mathcal{F}''$ denote the sheaf given by extending $\mathcal{F}|\Delta$ to the whole U^n with zero outside the dialgonal Δ. If $\mathcal{F}'$ is the kernel of the natural projection $\mathcal{F} \to \mathcal{F}''$, we have

$$H^0(U^n; \mathcal{F}')^{\mathfrak{S}_n} = 0.$$

In fact, any section $s \in H^0(U^n; \mathcal{F}')^{\mathfrak{S}_n}$ satisfies $s|_\Delta = 0$ by definition. Hence s vanishes on a neighbourhood of the diagonal Δ. Take any sequence $\mathfrak{r}$ (2.5). The closure of the subset $A_{\mathfrak{r}}$ includes the diagonal Δ. Since $A_{\mathfrak{r}}$ is connected and the restriction $\mathcal{F}'|_{A_{\mathfrak{r}}} = \mathcal{F}|_{A_{\mathfrak{r}}}$ is a (locally)

constant sheaf, s vanishes on A_τ. From (2.6) s vanishes on the whole U^n.

Consequently the map

$$H^0(U^n;\mathcal{F})^{\mathfrak{S}_n} \to H^0(U^n;\mathcal{F}'')^{\mathfrak{S}_n} = (\mathcal{F}_{(p_1,\dots,p_1)})^{\pi_1(U,p_1)} \subset \mathcal{F}_{(p_1,\dots,p_1)}$$

is injective, which is equal to the map ev (2.8). This completes the proof of Proposition 2.2.

Using the Rešetnikov spectral sequence and [Ka] Theorem 5.3 again, we obtain the following from Corollaries 1.3 and 1.5.

COROLLARY 2.9. *For $x = (C,p,z) \in M_{g,\rho}$, we have*

$$H^q(\mathfrak{d}_{\rho_x};\bigwedge^n T_x^* M_{g,\rho}) = \begin{cases} 0, & \text{if } q < n, \\ H^n(W_1;\bigwedge^n Q), & \text{if } q = n. \end{cases}$$

One deduces the following from [Ka1] Theorem 4.9 in a similar way to $M_{g,\rho}$.

PROPOSITION 2.10. *Let U be a connected open Riemann surface satisfying $b_1(U) < \infty$ and $p_1 \in U$. If $q < n$,*

$$H^q(L(U,p_1);\bigwedge^n Q^1(U,p_1)) = 0,$$

and the evaluation map at the point p_1 induces an isomorphism

$$H^n(L(U,p_1);\bigwedge^n Q^1(U,p_1)) = H^n(L_0;\bigwedge^n Q^1).$$

COROLLARY 2.11. *For $x = (C,p,z,p_1) \in C_{g,\rho}$, we have*

$$H^q(\mathfrak{d}_{\rho_x};\bigwedge^n T_x^* C_{g,\rho}) = \begin{cases} 0, & \text{if } q < n, \\ H^n(L_0;\bigwedge^n Q^1), & \text{if } q = n. \end{cases}$$

3. Some properties of the algebra $\bigoplus_{n\geq 0} H^n(L_0; \bigwedge^n Q^1)$.

By Corollaries 2.9 and 2.11 the proof of Theorem 2.1 is reduced to the calculation of the cohomology groups of the Lie algebras $W_1 = L(\mathbf{C})$ and $L_0 = L(\mathbf{C}, 0)$ with values in the quadratic differentials $\bigwedge^n Q$ and $\bigwedge^n Q^1$. The main purpose of this section is to prove

THEOREM 3.1. *The algebra $\bigoplus_{n\geq 0} H^n(L_0; \bigwedge^n Q^1)$ is isomorphic to the polynomial algebra over the algebra $\bigoplus_{n\geq 0} H^n(W_1; \bigwedge^n Q)$ with a single generator ϵ, i.e.,*

$$\bigoplus_{n\geq 0} H^n(L_0; \bigwedge^n Q^1) = (\bigoplus_{n\geq 0} H^n(W_1; \bigwedge^n Q))[\epsilon].$$

Here $\epsilon \in H^1(L_0; Q^1)$ is given by $\epsilon = d(\frac{1}{z^2}dz^2)$.

It is an analogue of [H2] Theorem 7.1:

$$\lim_{g\to\infty} H^*(C_{g,\rho}) = (\lim_{g\to\infty} H^*(M_{g,\rho}))[e]$$

in our algebraic model. Here $e \in H^2(C_{g,\rho})$ is the Euler class of the relative tangent bundle $T_{C_{g,\rho}/M_{g,\rho}}$. As is proved in §8, the class ϵ corresponds to the class e under our framework.

First we introduce typical examples of L_0 and W_1 modules. Let $\nu \in \mathbf{Z}$.

(1). The 1 dimensional complex vector space $1_\nu = \mathbf{C}1_\nu$ with the preferred base 1_ν is acted on by the Lie algebra L_0 in the following way.

$$(\xi(z)\frac{d}{dz}) \cdot 1_\nu = \nu\xi'(0)1_\nu \quad (\xi(z)\frac{d}{dz} \in L_0).$$

The L_0 module 1_ν is naturally isomorphic to the ν-cotangent space $(T_0^*\mathbf{C})^{\otimes\nu}$ of the complex line $\mathbf{C}$ at the origin 0.

(2). Through the Lie derivative the Fréchet space T_ν consisting of all complex analytic ν-covariant tensor fields on $\mathbf{C}$ forms a W_1 module:

$$T_\nu := H^0(\mathbf{C}; \mathcal{O}_{\mathbf{C}}((T^*\mathbf{C})^{\otimes\nu})).$$

By definition $Q = T_2$. If $\nu = 0$ and 1, we denote $F := T_0$ and $K := T_1$.

There exist some algebraic variants of W_1, L_0 and T_ν:

$$W_1^\sharp := \mathbf{C}[z]\frac{d}{dz} \quad \text{(the polynomial vector fields)},$$
$$L_0^\sharp := \{X \in W_1^\sharp; X(0) = 0\}, \quad \text{and}$$
$$T_\nu^\sharp := \mathbf{C}[[z]]dz^\nu \quad \text{(the formal tensor fields)}.$$

Similarly we denote $F^\sharp := T_0^\sharp$, $K^\sharp := T_1^\sharp$ and $Q^\sharp := T_2^\sharp$. Clearly we have $W_1^\sharp \subset W_1$, $L_0^\sharp \subset L_0$ and $T_\nu \subset T_\nu^\sharp$. These inclusions induce isomorphisms

$$
\begin{aligned}
H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}) &= H^*(L_0^\sharp; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_\nu^\sharp) \\
H^*(W_1; \bigotimes_{i=1}^n T_{\nu_i}) &= H^*(W_1^\sharp; \bigotimes_{i=1}^n T_\nu^\sharp)
\end{aligned}
\tag{3.2}
$$

for any integers $\nu_0, \nu_1, \ldots, \nu_n$ (see [Ka1] Lemma 2.3. In [Ka1], $W_1^\sharp = W_1{}^{\text{alg.}}$, $L_0^\sharp = L_0{}^{\text{alg.}}$ and so on.) Hence we have no essential difference between these algebraic variants and the originals.

We shall give some consideration about the Schapiro isomorphism for the pair of Lie algebras $(W_1^\sharp, L_0^\sharp)$. If N is an $L_0^\sharp$ module, let $I(N)$ denote the co-induced W_1 module derived from N:

$$I(N) := \mathrm{Hom}_{\mathrm{left}U(L_0^\sharp)}(U(W_1^\sharp), N).$$

Here $U(\cdot)$ means the universal enveloping algebra. Then there exists a natural isomorphism

$$H^*(W_1^\sharp; I(N)) \cong H^*(L_0^\sharp; N)$$

called *the Schapiro isomorphism.* Easily one deduces an $W_1^\sharp$ isomorphism $I(1_\nu) = T_\nu^\sharp$ for any integer ν. Hence, if N is a $W_1^\sharp$ module, we have a natural isomorphism $I(1_\nu \otimes N) = T_\nu^\sharp \otimes N$ and the Schapiro isomorphism

$$H^*(W_1^\sharp; T_\nu^\sharp \otimes N) \cong H^*(L_0^\sharp; 1_\nu \otimes N). \tag{3.3}$$

It should be remarked that the evaluation map

$$\mathrm{ev} : T_\nu^\sharp = \mathbf{C}[[t]]dt^\nu \to 1_\nu, \quad f(t)dt^\nu \mapsto f(0)1_\nu$$

is an $L_0^\sharp$ homomorphism. So we have the composite map

$$H^*(W_1^\sharp; T_\nu^\sharp \otimes N) \overset{\mathrm{res}}{\to} H^*(L_0^\sharp; T_\nu^\sharp \otimes N) \overset{\mathrm{ev}\otimes N}{\to} H^*(L_0^\sharp; 1_\nu \otimes N),$$

which coincides with the Schapiro isomorphism (3.3). The Hochschild Serre spectral sequence of the pair $(W_1^\sharp, L_0^\sharp)$ with coefficients in a $W_1^\sharp$ module N implies a cohomology exact sequence

$$\cdots \to H^q(W_1^\sharp; N) \overset{\mathrm{res}}{\to} H^q(L_0^\sharp; N) \overset{d_1}{\to} H^q(L_0^\sharp; 1_1 \otimes N) \to \cdots. \tag{3.4}$$

LEMMA 3.5. *If N is a $W_1^\sharp$ module, the diagram*

$$\begin{array}{ccc} H^*(W_1^\sharp; F^\sharp \otimes N) & \xrightarrow{d\otimes N} & H^*(W_1^\sharp; K^\sharp \otimes N) \\ \downarrow & & \downarrow \\ H^*(L_0^\sharp; N) & \xrightarrow{-d_1} & H^*(L_0^\sharp; 1_1 \otimes N) \end{array}$$

commutes, where the vertical arrows are the Schapiro isomorphisms (3.3) and $d : F^\sharp \to K^\sharp$ is a $W_1^\sharp$ homomorphism given by $f(z) \mapsto f'(z)dz$.

PROOF: A q cocycle $c \in Z^q(W_1^\sharp; F^\sharp \otimes N)$ has an expansion

$$c = \sum_{i=0}^{\infty} z^i c_i, \quad c_i \in C^q(W_1^\sharp; N).$$

Taking the constant term of the cocycle condition $d_{W_1^\sharp} c = 0$, we have

$$d_{W_1^\sharp} c_0 + \delta_{-1} \cup c_1 = 0,$$

where $\delta_{-1} \in C^1(W_1^\sharp)$ is given by $\delta_{-1}(\xi(z)\frac{d}{dz}) = \xi(0)$. Hence

$$\begin{aligned} & d_1 \circ (\mathrm{ev} \otimes N)(c) = 1_1 \otimes \mathrm{int}(\frac{d}{dz})(d_{W_1^\sharp} c_0)|_{L_0^\sharp} \\ = & -1_1 \otimes \mathrm{int}(\frac{d}{dz})(\delta_{-1} \cup c_1)|_{L_0^\sharp} = -1_1 \otimes c_1|_{L_0^\sharp} = -((\mathrm{ev} \circ d) \otimes N)(c), \end{aligned}$$

as was to be shown.

As a corollary of this observation, we have

LEMMA 3.6. *The sequence (3.4) for the $W_1^\sharp$ module $\bigwedge^n Q^\sharp$ is resolved into the split exact sequences*

$$0 \to H^q(W_1^\sharp; \bigwedge^n Q^\sharp) \xrightarrow{\mathrm{res}} H^q(L_0^\sharp; \bigwedge^n Q^\sharp) \xrightarrow{d_1} H^q(L_0^\sharp; 1_1 \otimes \bigwedge^n Q^\sharp) \to 0.$$

PROOF: The inclusion

$$1\otimes : \bigwedge^n Q^\sharp \hookrightarrow F \otimes \bigwedge^n Q^\sharp, \quad u \mapsto 1 \otimes u$$

has a $W_1^\sharp$ left inverse given by $f(z) \otimes u \mapsto \frac{1}{n}(\sum_{i=1}^n f(z_i))u$. This means the short exact sequence

$$0 \to \bigwedge^n Q^\sharp \hookrightarrow F^\sharp \otimes \bigwedge^n Q^\sharp \xrightarrow{d} K^\sharp \otimes \bigwedge^n Q^\sharp \to 0$$

splits over $W_1^\sharp$ and so the induced cohomology exact sequence of $W_1^\sharp$ is resolved into the split exact sequences

$$0 \to H^q(W_1^\sharp; \bigwedge^n Q^\sharp) \to H^q(W_1^\sharp; F^\sharp \otimes \bigwedge^n Q^\sharp) \\ \to H^q(W_1^\sharp; K^\sharp \otimes \bigwedge^n Q^\sharp) \to 0.$$

Hence the lemma follows from Lemma 3.5.

From the lemma above together with the isomorphisms (3.2) follows

COROLLARY 3.7. *The Hochshild Serre spectral sequence of the pair* (W_1, L_0) *with values in the* W_1 *module* $\bigwedge^n Q$ *induces a split exact sequence*

$$0 \to H^q(W_1; \bigwedge^n Q) \xrightarrow{\text{res}} H^q(L_0; \bigwedge^n Q) \xrightarrow{d_1} H^q(L_0; 1_1 \otimes \bigwedge^n Q) \to 0.$$

Since the E_1 term of the Hochschild Serre spectral sequence of the pair $(\mathfrak{d}_0, L_0)$ is given by $E_1^{p,q} = H^q(L_0; \bigwedge^p Q^1)$, we have the transgression map

$$d_1 : H^q(L_0; \bigwedge^p Q^1) \to H^q(L_0; \bigwedge^{p+1} Q^1).$$

From the definition of the transgression one deduces the commutative diagram

$$\begin{array}{ccc} H^q(L_0; 1_1 \otimes \bigwedge^p Q) & \longleftarrow & H^q(L_0; \bigwedge^{p+1} Q^1) \\ {\scriptstyle d_1}\uparrow & & {\scriptstyle d_1}\uparrow \\ H^q(L_0; \bigwedge^p Q) & \longrightarrow & H^q(L_0; \bigwedge^p Q^1), \end{array}$$

where the upper arrow is derived from the quotient map $\bigwedge^{p+1} Q^1 \to \bigwedge^{p+1} Q^1 / \bigwedge^{p+1} Q = 1_1 \otimes \bigwedge^p Q$ and the lower from the inclusion $\bigwedge^p Q \hookrightarrow \bigwedge^p Q^1$.

Consequently the L_0 exact sequence

$$0 \to \bigwedge^n Q \hookrightarrow \bigwedge^n Q^1 \to 1_1 \otimes \bigwedge^{n-1} Q \to 0$$

induces a split exact sequence

$$(3.8)\quad 0 \to H^q(L_0; \bigwedge^n Q) \to H^q(L_0; \bigwedge^n Q^1) \to H^q(L_0; 1_1 \otimes \bigwedge^{n-1} Q) \to 0.$$

It follows from Corollary 3.7

$$H^q(L_0; \bigwedge^n Q^1) = H^q(W_1; \bigwedge^n Q) \oplus H^q(L_0; 1_1 \otimes \bigwedge^n Q) \oplus H^q(L_0; 1_1 \otimes \bigwedge^{n-1} Q).$$

As was proved in [Ka1] §4, if $q < n$,

$$H^q(W_1; \bigwedge^n Q) = 0 \quad \text{and} \quad H^q(L_0; \bigwedge^n Q^1) = 0. \tag{3.9}$$

Hence we obtain

$$H^q(L_0; 1_1 \otimes \bigwedge^n Q) = 0 \quad \text{if } q \leq n, \tag{3.10}$$

and

$$H^q(L_0; \bigwedge^n Q) = \begin{cases} H^n(W_1; \bigwedge^n Q), & \text{if } q = n, \\ 0, & \text{if } q < n. \end{cases}$$

Let $Q^\times$ denote the W_1 module

$$Q^\times := Q(\mathbf{C}^\times) = Q(\mathbf{C} - \{0\}).$$

The key to proving the theorem (3.1) is

LEMMA 3.11.

$$H^q(L_0; \bigwedge^n Q^2) = \begin{cases} H^n(W_1; \bigwedge^n Q), & \text{if } q = n, \\ 0, & \text{if } q < n. \end{cases}$$

Especially we have $H^q(L_0; \bigwedge^n Q) = H^q(L_0; \bigwedge^n Q^2)$ *for* $q \leq n$.

PROOF: From [Ka1] Theorem 3.4 follows

$$H^*(L_0; \bigwedge^n Q^2) = H^*(L_0; \bigwedge^n Q^\times).$$

If we define $u \in H^2(L_0; S^n F)$ by

$$u(\xi_1(z)\frac{d}{dz}, \xi_2(z)\frac{d}{dz}) := \sum_{i=1}^{n} \int_0^{z_i} \det \begin{pmatrix} \xi_1'(z) & \xi_2'(z) \\ \xi_1''(z) & \xi_2''(z) \end{pmatrix} dz$$

for $\xi_1(z)\frac{d}{dz}$ and $\xi_2(z)\frac{d}{dz} \in L_0$, and $\delta_0 \in H^1(L_0)$ by $\delta_0(\xi(z)\frac{d}{dz}) = \xi'(0)$, then we have

$$H^*(L_0; \bigwedge^n Q^\times) = \bigwedge^*(\delta_0, u) \otimes H^*(W_1; \bigwedge^n Q^\times).$$

In fact the E_2 terms of the Rešetnikov spectral sequences of both sides are isomorphic to each other by the decomposition theorem [Ka] Theorem 5.3.

As was proved in §2 ($U = \mathbf{C}^\times$, cf. (2.8)ff.), we have

$$H^q(W_1; \bigwedge^n Q^\times) = \begin{cases} H^n(W_1; \bigwedge^n Q), & \text{if } q = n, \\ 0, & \text{if } q < n. \end{cases}$$

This completes the proof of Lemma 3.11.

PROOF OF THEOREM 3.1: Lemma 3.11 implies the exact diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & H^{n-1}(\bigwedge^{n-1} Q^1) & & \\ & & & & \downarrow & & \\ 0 \to H^n(\bigwedge^n Q) & \to & & H^n(\bigwedge^n Q^1) & & \to & H^n(1_1 \otimes \bigwedge^{n-1} Q) \to 0, \\ & & \searrow & & \downarrow & & \\ & & & & H^n(\bigwedge^n Q^2) & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

where $H^*(\cdot) = H^*(L_0; \cdot)$, the horizontal is the exact sequence (3.8), the vertical is derived from the L_0 exact sequence

$$0 \to \bigwedge^n Q^1 \to \bigwedge^n Q^2 \to \bigwedge^n Q^2/\bigwedge^n Q^1 = \bigwedge^{n-1} Q^1 \to 0,$$

and the slanted arrow is isomorphic from Lemma 3.11. By definition the second vertical arrow is given by the cup product $\epsilon \cup$. Consequently we have a decomposition

$$\begin{aligned} H^n(L_0; \bigwedge^n Q^1) &= H^n(W_1; \bigwedge^n Q) \oplus \epsilon \cup H^{n-1}(L_0; \bigwedge^{n-1} Q^1) \\ &\cong H^n(W_1; \bigwedge^n Q) \oplus H^{n-1}(L_0; \bigwedge^{n-1} Q^1), \end{aligned}$$

which derives the theorem (3.1) inductively.

Now we recall the fundamental exact sequence ([Ka1] §4) for the L_0 module $1_1 \otimes \bigwedge^n Q$

$$\begin{aligned} (3.12) \quad 0 \to H^1(L_0; 1_1 \otimes S^n(Q^\times/Q)) &\to H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q) \\ &\xrightarrow{i_{n+1}} H^{n+1}_{L_0}(\mathbf{C}^n - \{0\}; 1_1 \otimes \mathcal{Q}_n)^{\mathfrak{S}_n} \to \cdots, \end{aligned}$$

where $\mathcal{Q}_n$ is the sheaf over $\mathbf{C}^n$ introduced in (2.3) ($U = \mathbf{C}$), $S^n(\cdot)$ means the completed n-fold symmetric tensor product, and $H^*_{\mathfrak{g}}(\cdot, \cdot)$ denotes the $\mathfrak{g}$ equivariant cohomology (§11, [Ka1] §1). We denote by $\epsilon_1 \epsilon^n$ the

image of the $(n+1)$th power ϵ^{n+1} under the projection homomorphism $H^{n+1}(L_0; \bigwedge^{n+1} Q^1) \to H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q)$. Substituting (3.1) to the sequence (3.8), we have a decomposition

(3.13)

$$H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q) = \bigoplus_{i=0}^{n} \epsilon_1 \epsilon^{n-i} H^i(W_1; \bigwedge^i Q) \cong \bigoplus_{i=0}^{n} H^i(W_1; \bigwedge^i Q).$$

Hence we obtain

COROLLARY 3.14. *The kernel of the map i_{n+1} in the sequence (3.12) is a 1 dimensional complex vector space spanned by the class $\epsilon_1 \epsilon^n$.*

This corollary plays an important role in studying the algebra $\bigoplus_{n \geq 0} H^n(W_1; \bigwedge^n Q)$ in §5.

REMARK 3.15: The following assertions are equivalent to each other:

(1.n)

$$H^*(L_0; \bigwedge^p Q) \xrightarrow{\cong} H^*(L_0; \bigwedge^p Q^2) \quad \text{for any } p \leq n.$$

(2.n)

$$H^*(L_0; \bigwedge^p Q^2 / \bigwedge^p Q) = 0 \quad \text{for } p \leq n.$$

(3.n)

$$H^*(L_0; S^p(Q^\times / Q)) = 0 \quad \text{for } p \leq n.$$

(4.n)

$$H^*(L_0; \bigwedge^p Q^1) = \bigoplus_{i=0}^{p} \epsilon^i H^{*-i}(L_0; \bigwedge^{p-i} Q) \quad \text{for } p \leq n.$$

By a straight computation we can prove the assertion (3.n) for $n \leq 6$. But I do not know whether the above holds for all integers n. It is an analogue of the Harer stability of the mapping class group [H] in our algebraic model.

REMARK 3.16: In [FT] Feigin and Tsygan constructed a natural map

$$\varphi : H^2(W_1, \mathrm{C}e_0; S^n {W_1}^*) \to H^{2n}(M_g)$$

as an analogue of the Weil homomorphism. Here $e_0 = z\frac{d}{dz} \in W_1$. Now we look at the domain of the map φ, $H^2(W_1, \mathrm{C}e_0; S^n {W_1}^*)$. First we remark there exists a natural W_1 isomorphism $Q^\times / Q = {W_1}^*$. Hence, by Corollary 3.14, the cohomology group $H^1(L_0; 1_1 \otimes S^n {W_1}^*)$ is spanned by the single element $\epsilon_1 \epsilon^n$ corresponding to the $(n+1)$-th power, of the

Euler class e. The Hochschild Serre spectral sequence (3.4) for the W_1 module $S^nW_1{}^*$ induces a map

$$H^1(L_0; 1_1 \otimes S^nW_1{}^*) \to H^2(W_1; S^nW_1{}^*),$$

which is isomorphic under the assertion stated in Remark 3.15. From the isomorphism $H^2(W_1, \mathbf{C}e_0; S^nW_1{}^*) \cong H^2(W_1; S^nW_1{}^*)$ we conclude that the domain of the map φ is spanned by the single element $\epsilon_1\epsilon^n$ under the assertion (3.15). Our construction given in the sequal, however, has no dependence on that of the map φ in [FT].

4. Algebraic fiber integrals.

As in the previous section let $Q^\times$ denote the W_1 module $Q(\mathbf{C}^\times) = Q(\mathbf{C} - \{0\})$. Now we shall introduce a map called the *algebraic fiber integral*

$$\int_{\text{fiber}} : \ker(H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q) \to H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q^\times)) \to H^n(W_1; \bigwedge^n Q),$$

and prove that the algebra $\bigoplus_{n\geq 0} H^n(W_1; \bigwedge^n Q)$ is generated by the classes $\kappa_n \in H^n(W_1; \bigwedge^n Q)$, $n \in \mathbf{N}_{\geq 1}$,(Theorem 4.11):

$$\bigoplus_{n\geq 0} H^n(W_1; \bigwedge^n Q) = \mathbf{C}[\kappa_n; n \geq 1]/\text{relations}.$$

The class κ_n is defined through this integral (4.3), and corresponds to the n-th Morita Mumford class $e_n \in H^{n,n}(M_{g,\rho})$ under our framework (§10). It follows from a theorem of Miller [Mi] and Morita [Mo] there exist no algebraic relations among the κ_n's (Corollary 10.3).

First we fix our notations. As in §2 we denote $\mathcal{Q}_n := \mathcal{O}_{\mathbf{C}^n}(\bigotimes_{i=1}^n \mathrm{pr}_i{}^*(T^*\mathbf{C})^{\otimes 2})$, where $\mathrm{pr}_i : \mathbf{C}^n \to \mathbf{C}$ is the i-th projection $(1 \leq i \leq n)$ $(U = \mathbf{C})$. In view of the nuclear theorem, we have

$$\begin{aligned} Q^{\otimes n} &= \mathcal{Q}_n(\mathbf{C}^n) \\ &= \{f(z_1, \ldots, z_n)dz_1{}^2 \cdots dz_n{}^2; f \text{ is complex analytic on } \mathbf{C}^n.\} \end{aligned}$$

The symmetric group $\mathfrak{S}_n$ acts on the sheaf $\mathcal{Q}_n$ by the permutation of the variables z_i twisted by the sign. Furthermore we set

$$O_n := \{(t, z_1, \ldots, z_n) \in \mathbf{C}^{n+1}; t \neq z_i \quad (1 \leq \forall i \leq n)\}$$

and introduce the sheaf $\mathcal{KQ}_n$ over the space $\mathbf{C} \times \mathbf{C}^n = \mathbf{C}^{n+1}$ given by

$$\mathcal{KQ}_n := \mathcal{O}_{\mathbf{C}^{n+1}}(\mathrm{pr}_0{}^* T^*\mathbf{C} \otimes \bigotimes_{i=1}^{n} \mathrm{pr}_i{}^*(T^*\mathbf{C})^{\otimes 2}),$$

where $\mathrm{pr}_0 : \mathbf{C}^{n+1} \to \mathbf{C}$ is the 0-th projection $(t, z) \mapsto t$ and $\mathrm{pr}_i : \mathbf{C}^{n+1} \to \mathbf{C}$ is the i-th projection $(t, z) \mapsto z_i$ $(1 \leq i \leq n)$. We have

$$\begin{aligned} K \otimes Q^{\otimes n} &= \mathcal{KQ}_n(\mathbf{C}^n) \\ &= \{f(t, z_1, \ldots, z_n) dt dz_1{}^2 \cdots dz_n{}^2 ; f \text{ is complex analytic on } \mathbf{C}^{n+1}.\} \end{aligned}$$

The symmetric group $\mathfrak{S}_n$ acts on the sheaf $\mathcal{KQ}_n$ by the permutation of the variables z_i twisted by the sign. We have $K \otimes \bigwedge^n Q = \mathcal{KQ}_n(\mathbf{C}^{n+1})^{\mathfrak{S}_n}$. A W_1 module $(K \otimes \bigwedge^n Q)^\times$ is defined by

$$(K \otimes \bigwedge^n Q)^\times := (\mathcal{KQ}_n(O_n))^{\mathfrak{S}_n}.$$

LEMMA 4.1. *The restriction to* $\{t = 0\}$ *induces an isomorphism*

$$\psi_n : H^*(W_1; (K \otimes \bigwedge^n Q)^\times / (K \otimes \bigwedge^n Q)) \xrightarrow{\cong} H^*(L_0; 1_1 \otimes (\bigwedge^n Q^\times / \bigwedge^n Q)).$$

PROOF: It suffices to show that the restriction to $\{t = 0\}$ induces isomorphisms

$$\begin{aligned} H^*(W_1; K \otimes \bigwedge^n Q) &\xrightarrow{\cong} H^*(L_0; 1_1 \otimes \bigwedge^n Q) \\ H^*(W_1; (K \otimes \bigwedge^n Q)^\times) &\xrightarrow{\cong} H^*(L_0; 1_1 \otimes \bigwedge^n Q^\times). \end{aligned}$$

The former is the Schapiro isomorphism (3.3). Over the identification

$$O_n \cong \mathbf{C} \times (\mathbf{C}^\times)^n \quad (t, z_1, \ldots, z_n) \mapsto (t, z_1 - t, \ldots, z_n - t)$$

there exists a natural isomorphism of sheaves

$$H^*(W_1; \mathcal{KQ}_n) = \mathbf{C} \times H^*(L_0; 1_1 \otimes \mathcal{Q}_n).$$

Hence the E_2 term of the Rešetnikov spectral sequence of the latter

$$\begin{aligned} \mathrm{res}_{t=0} : H^p(\mathbf{C} \times (\mathbf{C}^\times)^n; \mathbf{C} \times H^q(L_0; 1_1 \otimes \mathcal{Q}_n))^{\mathfrak{S}_n} \\ \to H^p((\mathbf{C}^\times)^n; H^q(L_0; 1_1 \otimes \mathcal{Q}_n))^{\mathfrak{S}_n} \end{aligned}$$

is isomorphic and so is the latter itself.

The map $\oint_t$ defined by

$$\oint_t : (K \otimes \bigwedge^n Q)^\times \to \bigwedge^n Q$$

$$\omega = f(t, z_1, \ldots, z_n) dt dz_1{}^2 \cdots dz_n{}^2 \mapsto \oint_t \omega$$

$$\oint_t \omega := \lim_{R \to +\infty} \left(\frac{1}{2\pi\sqrt{-1}} \int_{|z|=R} f(t, z_1, \ldots, z_n) dt\right) dz_1{}^2 \cdots dz_n{}^2$$

is a W_1 homomorphism. Clearly $\oint_t \omega = 0$ for any $\omega \in K \otimes \bigwedge^n Q$. Thus a W_1 homomorphism

$$\oint_t : (K \otimes \bigwedge^n Q)^\times / (K \otimes \bigwedge^n Q) \to \bigwedge^n Q$$

is obtained. We denote by $\int_{\text{fiber}}$ the composite map

$$\int_{\text{fiber}} := \oint_t \circ (\psi_n)^{-1} : H^*(L_0; 1_1 \otimes (\bigwedge^n Q^\times / \bigwedge^n Q)) \to H^*(W_1; \bigwedge^n Q)$$

and call it *the algebraic fiber integral.*

Since $H^n(L_0; 1_1 \otimes \bigwedge^n Q^\times) = 0$ by (3.9) and the decomposition theorem [Ka] Theorem 5.3, there exists an isomorphism

$$\begin{aligned} (4.2) \quad & H^n(L_0; 1_1 \otimes (\bigwedge^n Q^\times / \bigwedge^n Q)) \\ & = \ker(H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q) \to H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q^\times)). \end{aligned}$$

In the sequal we regard $H^n(L_0; 1_1 \otimes (\bigwedge^n Q^\times / \bigwedge^n Q))$ as a subspace of $H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q)$ through this isomorphism. We have $\epsilon_1 \epsilon^n \in H^n(L_0; 1_1 \otimes (\bigwedge^n Q^\times / \bigwedge^n Q))$ and so define

$$(4.3) \qquad \kappa_n := \int_{\text{fiber}} \epsilon_1 \epsilon^n \in H^n(W_1; \bigwedge^n Q).$$

As is proved in §10, the class κ_n corresponds to the n-th Morita Mumford class $e_n \in H^{n,n}(M_{g,\rho})$ under our framework.

To treat with the case $n = 1$ we define two 1 cocycles $\nabla_1 \in Z^1(W_1, K)$ and $\nabla_2 \in Z^1(W_1; Q)$ by

$$(4.4) \qquad \begin{aligned} \nabla_1(X) = \nabla_1^z(X) &:= \frac{1}{2}\xi^{(2)}(z) dz \\ \nabla_2(X) = \nabla_2^z(X) &:= \frac{1}{3!}\xi^{(3)}(z) dz^2 \end{aligned}$$

for $X = \xi(z)\frac{d}{dz} \in W_1$. By the Schapiro isomorphism (3.3) we have

$$H^1(W_1; K) \cong H^1(L_0; 1_1) \cong \mathbf{C}, \quad \text{and} \quad H^1(W_1; Q) \cong H^1(L_0; 1_2) \cong \mathbf{C}.$$

As is easily proved, the LHS's are generated by ∇_1 and ∇_2 respectively. If $n = 1$,

$$\frac{1}{6}\kappa_1 = \nabla_2 \in H^1(W_1; Q). \tag{4.5}$$

In fact, $\psi_1(2\nabla_1^t d(\frac{dz^2}{(z-t)^2})) = \epsilon_1\epsilon$, and so, from the Cauchy integral formula,

$$\frac{1}{6}\kappa_1(\xi(z)\frac{d}{dz}) = \frac{1}{6}\oint_t \frac{\xi''(t)dtdz^2}{(z-t)^2} = \frac{1}{6}\xi^{(3)}(z)dz^2 = \nabla_2(\xi(z)\frac{d}{dz})$$

for $\xi(z)\frac{d}{dz} \in W_1$.

Especially we obtain

$$H^1(W_1; Q) = \mathbf{C}\kappa_1 \cong \mathbf{C}. \tag{4.6}$$

From (3.13) follows

LEMMA 4.7.

$$H^n(L_0; 1_1 \otimes (\textstyle\bigwedge^n Q^\times / \bigwedge^n Q)) = \bigoplus_{i=1}^n \epsilon_1\epsilon^i H^{n-i}(W_1; \bigwedge^{n-i} Q).$$

PROOF: It suffices to show

$$H^n(L_0; 1_1 \otimes \textstyle\bigwedge^n Q^\times) = \epsilon_1 H^n(W_1; \bigwedge^n Q) \cong H^n(W_1; \bigwedge^n Q). \tag{4.8}$$

By the decomposition theorem [Ka] Theorem 5.3, the LHS is isomorphic to $H^n(W_1; \bigwedge^n Q^\times)$ through the cup product by ϵ_1. As was proved in §2 (2.8)ff., we have $H^n(W_1; \bigwedge^n Q^\times) \cong H^n(W_1; \bigwedge^n Q)$. This implies (4.8) and (4.7).

REMARK 4.9: From (4.8) we have

$$H^{n+1}(L_0; 1_1 \otimes (\textstyle\bigwedge^n Q^\times / \bigwedge^n Q))$$
$$= \ker(H^{n+2}(L_0; 1_1 \otimes \textstyle\bigwedge^n Q) \to H^{n+2}(L_0; 1_1 \otimes \bigwedge^n Q^\times)).$$

Hence we can define the algebraic fiber integral

$$\int_{\text{fiber}} : \ker(H^{n+1}(L_0; 1_1 \otimes \textstyle\bigwedge^n Q) \to H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q^\times))$$
$$\to H^{n+1}(W_1; \textstyle\bigwedge^n Q).$$

Define a 1 cocycle $\eta_n \in Z^1(L_0; 1_1 \otimes S^n F)$ by

$$\eta_n := -d\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{z_i}\right) \in Z^1(L_0; 1_1 \otimes S^n F).$$

Then we have

LEMMA 4.10. *Under the identification (4.2), for any* $u \in H^n(W_1; \bigwedge^n Q)$, *we have* $\eta_n u \in H^n(L_0; 1_1 \otimes (\bigwedge^n Q^\times / \bigwedge^n Q))$ *and*

$$\int_{\text{fiber}} \eta_n u = u \in H^n(W_1; \bigwedge^n Q).$$

PROOF: The first half is clear. Since

$$\frac{1}{n} \sum_{i=1}^n \frac{dt}{t - z_i}\Big|_{t=0} = -\frac{1}{n} 1_1 \otimes \sum_{i=1}^n \frac{1}{z_i} \in 1_1 \otimes S^n F,$$

it follows from the Cauchy integral theorem

$$\int_{\text{fiber}} \eta_n u = \oint_t \frac{1}{n} \left(\sum_{i=1}^n \frac{dt}{t - z_i} \right) u = u.$$

THEOREM 4.11. *The algebra* $\bigoplus_{n \geq 0} H^n(W_1; \bigwedge^n Q)$ *is generated by the classes* $\kappa_n \in H^n(W_1; \bigwedge^n Q)$, $n \in \mathbf{N}_{\geq 1}$, *as a commutative* $\mathbf{C}$*-algebra:*

$$\bigoplus_{n \geq 0} H^n(W_1; \bigwedge^n Q) = \mathbf{C}[\kappa_n; n \geq 1]/\textit{relations},$$

PROOF: By induction on n we prove that $H^n(W_1; \bigwedge^n Q)$ consists of polynomials of the κ_i's.

The assertion for $n = 1$ is already proved in (4.6).

Suppose $n \geq 2$. By Lemmas 4.10 and 4.7, for an arbitrary $u \in H^n(W_1; \bigwedge^n Q)$, we have

$$\eta_n u = \sum_{i=1}^n \epsilon_1 \epsilon^i u_i, \quad u_i \in H^{n-i}(W_1; \bigwedge^{n-i} Q).$$

Since $i \geq 1$, each u_i is a polynomial of the κ_j's from the inductive assumption. It follows from Lemma 4.10

$$u = \int_{\text{fiber}} \eta_n u = \sum_{i=1}^n \kappa_i u_i \in \mathbf{C}[\kappa_j; 1 \leq j \leq n]/\text{relations},$$

which completes the induction.

COROLLARY 4.12. *The algebra* $\bigoplus_{n \geq 0} H^n(L_0; \bigwedge^n Q^1)$ *is generated by the classes* $\epsilon \in H^1(L_0; Q^1)$ *and* $\kappa_n \in H^n(W_1; \bigwedge^n Q)$, $n \in \mathbf{N}_{\geq 1}$, *as a commutative* $\mathbf{C}$*-algebra:*

$$\bigoplus_{n \geq 0} H^n(L_0; \bigwedge^n Q^1) = \mathbf{C}[\epsilon, \kappa_n; n \geq 1]/\textit{relations}.$$

These together with Corollaries 2.9 and 2.11 complete the proof of Theorem 2.1.

5. Non-triviality of the class κ_n.

Now we shall investigate the class κ_n in detail. The purpose of this section is to prove

THEOREM 5.1. *If* $n \geq 1$,

$$\kappa_n \neq 0 \in H^n(W_1; \bigwedge^n Q) \tag{1}$$

$$\epsilon_1 \epsilon^n = \eta_n \kappa_n \in H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q). \tag{2}$$

This is a key lemma for establishing that the class κ_n corresponds to the n-th Morita Mumford class $e_n \in H^{n,n}(M_{g,\rho})$.

First we prove it for the case $n = 1$ (Theorem 10.1).

Recall (4.5): $\kappa_1 = 6\nabla_2 \neq 0$. Easily we have

$$d(1_1 \otimes z^{-3}dz^2) - \delta_1 1_1 \otimes z^{-2}dz^2 - 3\delta_2 1_1 \otimes z^{-1}dz^2 \in C^1(L_0; 1_1 \otimes Q)$$
$$1_1 \otimes \nabla_2 - \delta_2 1_1 \otimes z^{-1}dz^2 \in C^1(L_0; 1_1 \otimes Q).$$

Here $\delta_k \in C^1(L_0)$ is defined by $\delta_k(z^{l+1}\frac{d}{dz}) = \delta_{k,l}$ (Kronecker's delta). Hence in $H^2(L_0; 1_1 \otimes Q)$ we have

$$\begin{aligned}\epsilon_1 \epsilon &= -2d(\delta_1 1_1 \otimes z^{-2}dz^2) = 6d(\delta_2 1_1 \otimes z^{-1}dz^2) \\ &= 6d(1_1 \otimes z^{-1}\nabla_2) = \eta_1 \kappa_1,\end{aligned}$$

as was to be shown.

Let $n \geq 2$. As in §2 we abbreviate $\mathcal{Q}_n := \mathcal{O}_{\mathbf{C}^n}(\bigotimes_{i=1}^n \mathrm{pr}_i^*(T^*\mathbf{C})^{\otimes 2})$, where $\mathrm{pr}_i : \mathbf{C}^n \to \mathbf{C}$ is the i-th projection $(1 \leq i \leq n)$. The diagonal set of the product space $\mathbf{C}^n$ is denoted by

$$\Delta := \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n; z_1 = z_2 = \cdots = z_n\}.$$

In a similar way to [Ka1] §4 we construct *the fundamental exact sequence* for the W_1 module $\bigwedge^n Q$

$$\begin{aligned}0 \to H^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta; \mathcal{Q}_n))^{\mathfrak{S}_n} &\to H^n(W_1; \bigwedge^n Q) \\ &\to H^n_{W_1}(\mathbf{C}^n - \Delta; \mathcal{Q}_n)^{\mathfrak{S}_n} \to \cdots.\end{aligned}$$

Since $H^q(W_1; \bigwedge^n Q) = 0$ for $q < n$ (3.9), we may regard $H^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta; \mathcal{Q}_n))^{\mathfrak{S}_n}$ as a subspace of $H^n(W_1; \bigwedge^n Q)$.

LEMMA 5.2. *If $n \geq 2$, we have*

$$H^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta; \mathcal{Q}_n))^{\mathfrak{S}_n} \neq 0.$$

The latter half of this section is devoted to the proof of this lemma.

To begin the proof of Theorem 5.1, we introduce the notion of *support* for cohomology classes in $H^n(W_1; \bigwedge^n Q)$, $H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q)$ and $H^1(L_0; 1_1 \otimes S^n F)$. From (3.9) one deduces an isomorphism

$$H^n(W_1; \bigwedge^n Q) = H^n_{W_1}(\mathbf{C}^n; \mathcal{Q}_n)^{\mathfrak{S}_n} = H^0(\mathbf{C}^n; H^n(W_1; \mathcal{Q}_n)^{\mathfrak{S}_n})^{\mathfrak{S}_n}.$$

A cohomology class $u \in H^n(W_1; \bigwedge^n Q)$ is identified with a section of the sheaf $H^n(W_1; \mathcal{Q}_n)$ over $\mathbf{C}^n$ through this isomorphism. We define the *support* of the class u by that of the corresponding section of the sheaf $H^n(W_1; \mathcal{Q}_n)$. Since $H^q(L_0; 1_1 \otimes \bigwedge^n Q) = 0$ for $q \leq n$ (3.10) and $H^0(L_0; 1_1 \otimes S^n F) = 0$, we can define the *support* of the cohomology classes in $H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q)$ and $H^1(L_0; 1_1 \otimes S^n F)$ in a similar way. From the definition the support of the class $\eta_n \in H^1(L_0; 1_1 \otimes S^n F)$ is (included in) the subset $\bigcup_{i=1}^n \{z_i = 0\} \subset \mathbf{C}^n$.

Lemma 5.2 asserts that there exists a non-zero cohomology class $u \in H^n(W_1; \bigwedge^n Q)$ whose support is included in the diagonal set $\Delta \subset \mathbf{C}^n$. The cup product $\eta_n u \in H^{n+1}(L_0; 1_1 \otimes \bigwedge^n Q)$ has its support in the subset

$$\Delta \cap \bigcup_{i=1}^n \{z_i = 0\} = \{0\}.$$

It follows from Corollary 3.14 that $\eta_n u = a\epsilon_1 \epsilon^n$ for some $a \in \mathbf{C}$. Applying the algebraic fiber integral we obtain

$$u = a\kappa_n \in H^n(W_1; \bigwedge^n Q).$$

Since u is non-zero, we have $a \neq 0$ and $\kappa_n \neq 0$. The first half of Theorem 5.1 is proved.

Especially we obtain $\kappa_n = a^{-1}u \in H^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta; \mathcal{Q}_n))^{\mathfrak{S}_n}$ and

$$H^1(W_1; H^{n-2}(\mathbf{C}^n - \Delta; \mathcal{Q}_n))^{\mathfrak{S}_n} = \mathbf{C}\kappa_n \cong \mathbf{C}. \tag{5.3}$$

Substituting $u = \kappa_n$ to the above discussion, we have $\eta_n \kappa_n = a\epsilon_1 \epsilon^n$ and $\kappa_n = a\kappa_n$. From $a \neq 0$ and $\kappa_n \neq 0$, we obtain $a = 1$, i.e., $\epsilon_1 \epsilon^n = \eta_n \kappa_n$. This completes the proof of Theorem 5.1 modulo that of Lemma 5.2.

For the rest of this section we prove Lemma 5.2. Consider two Stein coverings $\mathfrak{U} = \{U_i\}_{i=2}^n$ and $\mathfrak{V} = \{V_1\} \cup \{V_j\}_{j=3}^n$ defined by

$$U_i := \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n; z_1 \neq z_i\}$$
$$U_j := \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n; z_2 \neq z_j\}.$$

Using the 1 cocycles ∇_1 and ∇_2 (4.4), we define two 1 cochains $\Theta_{n;1} \in C^1(W_1; C^{n-2}(\mathfrak{U}; \mathcal{Q}_n))$ and $\Theta_{n;2} \in C^1(W_1; C^{n-2}(\mathfrak{V}; \mathcal{Q}_n))$ by

$$\begin{aligned}\Theta_{n;1} &:= \oint_{t;z_1} \nabla_1^t \prod_{i=1}^n \frac{{dz_i}^2}{(t-z_i)^2} \\ &= 3\nabla_2^{z_1} \prod_{i=2}^n \frac{{dz_i}^2}{(z_1-z_i)^2} - 2\nabla_1^{z_1}\Big(\sum_{i=2}^n \frac{dz_1}{z_1-z_i}\Big) \prod_{i=2}^n \frac{{dz_i}^2}{(z_1-z_i)^2} \\ \Theta_{n;2} &:= \oint_{t;z_2} \nabla_1^t \prod_{i=1}^n \frac{{dz_i}^2}{(t-z_i)^2}.\end{aligned}$$

Here we denote by $\oint_{t;z_1}$ the line integral along a small loop about the point $t = z_1$:

$$\oint_{t;z_1} := \frac{1}{2\pi\sqrt{-1}} \int_{|t-z_1|=\delta<<1},$$

which is also W_1 equivariant. Since $\dim \mathfrak{U} = \dim \mathfrak{V} = n-2$, we may regard $\Theta_{n;1}$ (resp. $\Theta_{n;2}$) as an element of $C^1(W_1; H^{n-2}(\mathfrak{U}; \mathcal{Q}_n))$ (resp. $C^1(W_1; H^{n-2}(\mathfrak{V}; \mathcal{Q}_n))$).

LEMMA 5.4.

$$d\Theta_{n;1} = 0 \in C^2(W_1; H^{n-2}(\mathfrak{U}; \mathcal{Q}_n))$$
$$d\Theta_{n;2} = 0 \in C^2(W_1; H^{n-2}(\mathfrak{V}; \mathcal{Q}_n)).$$

PROOF: We prove only the first half $d\Theta_{n;1} = 0$. First we remark

$$\nabla_1^t d\left(\frac{dz^2}{(z-t)^2}\right)(\xi_1(z)\frac{d}{dz}, \xi_2(z)\frac{d}{dz}) \in \mathbf{C}[t,z]dtdz^2 \tag{5.5}$$