2014年5月28日(水) 15:50−16:50 117号室

数学講究XB

「2次元トポロジーと余括弧積」

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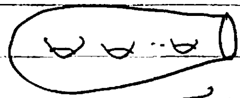
久野雄介氏(津田塾大・学芸)との共同研究

oriented $\mathbb{R}$ vector bundle of rank n

$\Leftarrow$ $GL_n(\mathbb{R}) \simeq_{\exp} BGL_n(\mathbb{R})$: ∞-dim'l Grassmannian ($\log$)

$\Leftarrow$ $\mathrm{Lie}\, GL_n(\mathbb{R}) = \mathfrak{gl}_n(\mathbb{R})$

2次元トポロジー $S =$  $\Rightarrow \pi_1(S)$: free group of rank $2g$

$g \geqq 1$

oriented S-bundle

$\Leftarrow \mathrm{Diff}_+(S, \partial S) := \{\varphi : S \to S : \text{ori. pres. diffeo}, \varphi|_{\partial S} = 1_{\partial S}\}$

$\simeq \pi_0 \mathrm{Diff}_+(S, \partial S)$ the mapping class group

(Teichmüller − Earle − Eells)

$B\,\mathrm{Diff}_+(S, \partial S) = B\pi_0 \mathrm{Diff}_+(S, \partial S) = T\mathbb{C}_g/\mathbb{M}_g \setminus (0\text{-section})$

($\pi : \mathbb{C}_g \to \mathbb{M}_g$: universal family
$\mathbb{M}_g$: the coarse moduli space of compact Riemann surface)

"Lie" $\pi_0 \mathrm{Diff}_+(S, \partial S) = ?$

- "正しい答" $= 0$ ($\because$ $\pi_0 \mathrm{Diff}_+(S, \partial S)$: discrete)
- Virasoro algebra (Beilinson − Manin − Schechtman, Kontsevich)
- Johnson − Morita theory

$H := H_1(S; \mathbb{Q}) \ (\cong \mathbb{Q}^{2g})$

$\mathcal{J}(S) := \mathrm{Ker}(\pi_0 \mathrm{Diff}_+(S, \partial S) \to \mathrm{Aut}(H))$

Torelli group

D. Johnson

$\exists\ \{\mathcal{J}(S)(n)\}_{n=1}^{\infty}$ "Johnson filtration"

canonical decreasing filtration (central)

$\tau : \mathfrak{gr}(\mathcal{G}(S)) \hookrightarrow \mathrm{Der}(\mathcal{L}(H))$ ($\mathcal{L}(H)$: the free Lie algebra over H)

Johnson homomorphism (⇐ purely algebraic construction)

injective Lie algebra homomorphism

森田茂之 $\mathfrak{H}_S$:= annihilator of the intersection form on H
$\subset \mathrm{Der}(\mathcal{L}(H))$ Lie subalgebra

- $\tau(\mathfrak{gr}(\mathcal{G}(S))) \subset \mathfrak{H}_S$
- $\exists\, \mathrm{Tr} : \mathfrak{H}_S \to S^*_{ym}(H)$ Morita trace
  $\mathrm{Tr} \circ \tau = 0$ (except degree 1)

$\overline{\tau(\mathfrak{gr}(\mathcal{G}(S)))}^{\text{Zariski closure}} =$ "Lie" $\pi_0 \mathrm{Diff}_+(S, \partial S)$

具体的な計算 (森田, 朝田・中村, Ham, 榎本-佐藤, 榎本-榎本
森田鈴木逆井 (degree 6))

榎本直也-佐藤隆夫 $\xrightarrow{\mathrm{Tr}} S^*_{ym}(H)$

$\exists\, \hat{\mathrm{Tr}} : \mathfrak{H}_S \to HH_0(T(H))$ ($T(H)$: the tensor alg over H, HH_0: the 0th Hochschild homology)

Enomoto-Satoh trace

$\hat{\mathrm{Tr}} \circ \tau = 0$

今日の話 ・τの幾何的再構成
・$\mathrm{Tr}, \hat{\mathrm{Tr}}$: "cobracket"で解釈できる

cobracket $\mathfrak{g}$: vector space$/\mathbb{Q}$

$(\mathfrak{g}, [\,,\,])$: Lie algebra$/\mathbb{Q}$

$[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ (bi) linear map bracket

skew

$$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g} \arrow[r, "{[\,,\,]}"] & \mathfrak{g} \\ \mathfrak{g}\otimes\mathfrak{g} \arrow[u, "1+(12)"] \arrow[ur, "0"'] \arrow[u, phantom, shift left=6, "\circlearrowleft"] & \end{tikzcd}$$

Jacobi

$$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[r, "1+(123)+(213)"] \arrow[ddr, "0"'] & \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[d, "{[\,,\,]\otimes 1_{\mathfrak{g}}}"] \\ & \mathfrak{g}\otimes\mathfrak{g} \arrow[d, "{[\,,\,]}"] \\ & \mathfrak{g} \end{tikzcd}$$

$(\mathfrak{g}, \delta)$: Lie coalgebra

$\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ cobracket

$$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g} \arrow[d, "1+(12)"'] & \mathfrak{g} \arrow[l, "\delta"'] \arrow[dl, "0"] \\ \mathfrak{g}\otimes\mathfrak{g} & \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} & \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[l, "1+(123)+(213)"'] \\ & \mathfrak{g}\otimes\mathfrak{g} \arrow[u, "\delta\otimes 1_{\mathfrak{g}}"'] \\ & \mathfrak{g} \arrow[u, "\delta"'] \arrow[uul, "0"] \end{tikzcd}$$

~~Yang-Baxter eq.~~

Definition (Drinfel'd)

$(\mathfrak{g},[\,,\,],\delta)$: Lie bialgebra$/\mathbb{Q}$

$\overset{\text{def}}{\iff}$ $\begin{cases} (\mathfrak{g},[\,,\,]) : \text{Lie algebra}/\mathbb{Q} \\ (\mathfrak{g},\delta) : \text{Lie coalgebra}/\mathbb{Q} \\ \text{compatibility condition : } \delta[X,Y] = \mathrm{ad}(X)(\delta Y) - \mathrm{ad}(Y)(\delta X) \quad (\forall X, \forall Y \in \mathfrak{g}) \end{cases}$

($\Longrightarrow$ $\operatorname{Ker}\delta \subset \mathfrak{g}$: Lie subalgebra)

Goldman-Turaev Lie bialgebra

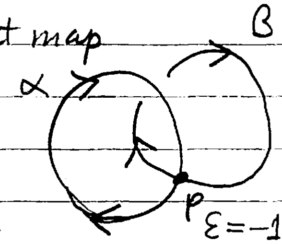
S : as above, $* \in \partial S$, $\pi := \pi_1(S,*)$

$\hat{\pi} := [S^1, S] = \pi_1(S,*)/\text{conj}$ the free homotopy set of free loops on S

$1 \in \hat{\pi}$: const. loop

$|\ \ |' : \mathbb{Z}\pi \to \mathbb{Z}\hat{\pi} \to \mathbb{Z}\hat{\pi}/\mathbb{Z}1$ quotient map

$\alpha, \beta \in \hat{\pi}$ in general position.



Goldman bracket

$$[\alpha,\beta] \overset{\text{def}}{=} \sum_{p \in \alpha\cap\beta} \varepsilon(p;\alpha,\beta)\,|\alpha_p\beta_p|' \in \mathbb{Z}\hat{\pi}/\mathbb{Z}1$$

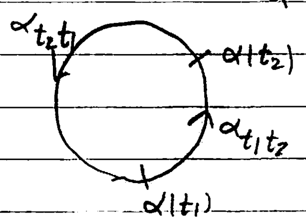
$\varepsilon(p;\alpha,\beta) \in \{\pm 1\}$ local intersection number

$(\mathbb{Z}\hat{\pi}/\mathbb{Z}1, [\,,\,])$ Lie algebra (Goldman)

Turaev cobracket

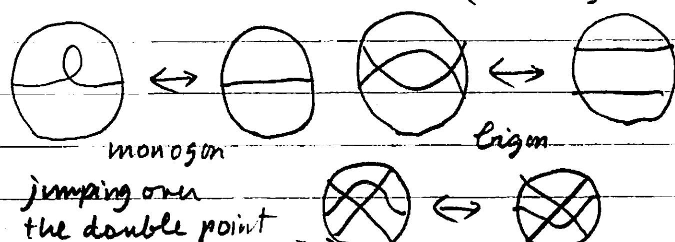
$$\delta(\alpha) \overset{\text{def}}{=} \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot\alpha(t_1),\dot\alpha(t_2))\,|\alpha_{t_1t_2}|' \otimes |\alpha_{t_2t_1}|' \in (\mathbb{Z}\hat{\pi}/\mathbb{Z}1)^{\otimes 2}$$

where $D_\alpha := \{(t_1,t_2) \in S^1\times S^1 ; t_1 \neq t_2, \alpha(t_1) = \alpha(t_2)\}$

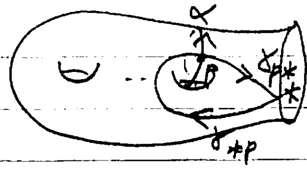


$(\mathbb{Z}\hat{\pi}/\mathbb{Z}1, [\,,\,], \delta)$ Lie bialgebra (Turaev)

レポート問題 これらを示せ (well-defined 性)



Action of $\mathbb{Z}\hat{\pi}/\mathbb{Z}1$ on $\mathbb{Z}\pi$



$\alpha\in\hat{\pi}$, $\gamma\in\pi$ in general position

$$\sigma(\alpha)(\gamma)\overset{\text{def}}{=}\sum_{p\in\alpha\cap\gamma}\varepsilon(p,\alpha,\gamma)\,\gamma_{*p}\,\alpha_p\,\gamma_{p*}\in\mathbb{Z}\pi$$

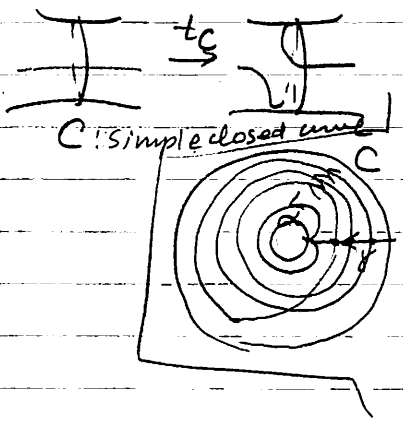
$\sigma:\mathbb{Z}\hat{\pi}/\mathbb{Z}1\longrightarrow \mathrm{Der}(\mathbb{Z}\pi)$ Lie algebra hom. (Kuno-K.)

$\Longrightarrow$ ($\wedge$: completion)

$$\begin{tikzcd}
\sigma:(\mathbb{Q}\hat{\pi}/\mathbb{Q}1)^{\wedge} \arrow[r, hook, "\text{inj}"] & \mathrm{Der}(\widehat{\mathbb{Q}\pi})\ni\log(\varphi_*) \\
& \mathcal{M}(S)\ni\varphi \arrow[ul, dashed, "\exists!\,\tau_{geom}"] \arrow[u] \arrow[ul, phantom, "\circlearrowleft"]
\end{tikzcd}$$

$\mathrm{gr}(\tau_{geom})=\tau$ (Johnson homom)

e.g. Dehn twist



Kuno-K.

$\tau_{geom}(t_C)=\frac{1}{2}|(\log C)^2|\in(\mathbb{Q}\hat{\pi}/\mathbb{Q}1)^{\wedge}$

(∵) $\sigma(C^n)(\gamma)=n\gamma\alpha^n$

$\sigma(f(C))(\gamma)=\gamma\alpha f'(\gamma)$ ($\forall f(x)$)

$(\log t_{C*})(\gamma)=\gamma\log\alpha$

$\alpha f'(\alpha)=\log\alpha$

$\Longrightarrow f(x)=\int\frac{1}{x}\log x\,dx=\frac{1}{2}(\log x)^2$ //

$\Longrightarrow \delta\circ\tau_{geom}(t_C)=0$

Theorem (Kuno-K.)

(1) $\delta\circ\tau_{geom}=0:\mathcal{M}(S)\longrightarrow((\mathbb{Q}\hat{\pi}/\mathbb{Q}1)^{\wedge})^{\hat{\otimes}2}$

(2) $\mathrm{gr}((\mathbb{Q}\hat{\pi}/\mathbb{Q}1)^{\wedge})=HH_0(T(H))$

(3) Morita trace は $\mathrm{gr}(\delta)$ から回復できる

Enomoto　Enomoto-Satoh trace は $\mathrm{gr}(\delta)$ から回復できない

Theorem (K.)

$(\mathbb{Q}\hat{\pi}/\mathbb{Q}1)^{\wedge}$ の正則 homotopy version から ES-trace がえられる

(背景 古田の cocycle: $\pi_0\mathrm{Diff}(S,\partial S)\longrightarrow H^1(S;\mathbb{Z})$)

(pants では 柏原-Vergne 問題, Grothendieck-Teichmüller群 と関係?)