"Workshop on Grothendieck-Teichmüller Theories"

July 28, 2016, 8:45-9:25 at Chern Institute of Mathematics, Nankai University.

"The Kashiwara-Vergne problem and the Goldman-Turaev Lie bialgebra, I"

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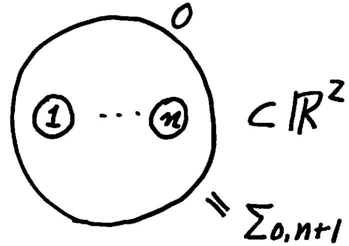
joint work with A. Alekseev, Y. Kuno and F. Naef

(Part II, Today 15:20-16:00)

$n \geqq 2$



$\mathbb{K}$: field of characteristic 0, $x_1, x_2, \cdots, x_n$: n letters

$L_n :=$ completed free Lie $\langle x_1, x_2, \cdots, x_n \rangle$

$A_n :=$ completed free Associative $\langle x_1, x_2, \cdots, x_n \rangle$

completed tensor product

$\Delta : A_n \to A_n \hat{\otimes} A_n$ coproduct, $\Delta x_i = x_i \otimes 1 + 1 \otimes x_i$

$L_n = \{ a \in A_n \,;\, \Delta a = a \otimes 1 + 1 \otimes a \} \subset A_n$

$|A_n| := A_n / [A_n, A_n]$, $[A_n, A_n] :=$ closure of linear span of $\{ab - ba \,;\, a, b \in A_n\}$

$|\ | : A_n \to |A_n|$, $a \mapsto |a|$, quotient map.

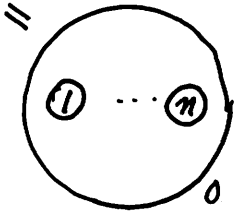
$$(A_n)_{\geq 1} \underset{\log}{\overset{\exp}{\rightleftarrows}} 1 + (A_n)_{\geq 1} \quad (\cong)$$

$$e^a = \exp(a) := \sum_{m=0}^{\infty} \frac{1}{m!} a^m$$

$$\log b := \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (b-1)^m$$

$n \geq 2$

$\Sigma_{0,n+1} \approx$



The Kashiwara-Vergne (KV) problem of type $(0, n+1)$

Find an element $F \in TAut(L_n)$ satisfying the conditions

(KV1) $F(x_1 + x_2 + \cdots + x_n) = \log(e^{x_1} e^{x_2} \cdots e^{x_n}) \in L_n$

(KV2) $\exists h(z) \in \mathbb{K}[[z]]$ $j(F^{-1}) = \left| -h\left(\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} h(x_i) \right| \in |A_n|$

- The KV problem (reformulated by Alekseev-Torossian) ------ Grothendieck-Teichmüller Lie algebra
  $=$ The KV problem of type $(0, 3)$
- $\exists$ solution to the KV problem (Alekseev-Meinrenken, Alekseev-Torossian)
  $\xRightarrow[\text{gluing solutions}]{}$ $\exists$ solution to the KV problem of type $(0, n+1)$ $(\forall n \geq 2)$
- Recall $A_n \underset{\text{special expansion}}{\cong} (\mathbb{K}\pi_1(\Sigma_{0,n+1}))^{\wedge}$ ← completion

Main Theorem (AKKN)

$\{$solutions to the KV problem of type $(0, n+1)\}$ / (some small equiv. relation) $\cong \mathbb{K}^n$

$\cong$ $\{$special expansions of $\pi_1(\Sigma_{0,n+1})$ which induces a formal description of (a regular homotopy version of) the Turaev cobracket$\}$

- Massuyeau gave a similar formal description of the Turaev cobracket on $\Sigma_{0,n+1}$ using the Kontsevich integral instead of solutions to the KV problem

◎ <u>Tangential automorphisms</u> $\mathrm{TAut}(A_n)$ <u>and</u> $\mathrm{TAut}(L_n)$

- $t\mathrm{Der}(A_n) := \left((A_n)_{\geq 1}\right)^{\oplus n}$ as $\mathbb{K}$-vector space
  "tangential derivations"
  $\rho : t\mathrm{Der}(A_n) \to \mathrm{Der}(A_n)$, $u = (u_1, \cdots, u_n) \mapsto \rho(u) \overset{\text{def}}{=} (\underbrace{x_i \mapsto [x_i, u_i] = x_i u_i - u_i x_i}_{\text{derivation}})$

  Lie algebra structure $\underset{\text{induced by}}{\Longleftarrow}$ $\hat{\rho} : t\mathrm{Der}(A_n) \to ((A_n)_{\geq 1})^{\oplus n} \rtimes \mathrm{Der}(A_n)$ semi-direct product
  $u \mapsto (u_1, \cdots, u_n, \rho(u))$

  $0 \to \bigoplus_{i=1}^{n} \mathbb{K}[[x_i]] \hookrightarrow t\mathrm{Der}(A_n) \xrightarrow{\rho} \rho(t\mathrm{Der}(A_n)) \to 0$ central extension of Lie algebras

  $t\mathrm{Der}(A_n)$: positively graded $\underset{\text{Baker-Campbell-Hausdorff series (BCH)}}{\Longrightarrow}$ $\boxed{\text{group structure on } t\mathrm{Der}(A_n)}$

  $\mathrm{TAut}(A_n) \underset{\text{set}}{=} t\mathrm{Der}(A_n)$
  we write $e^u \longleftarrow u$ "tangential automorphisms"

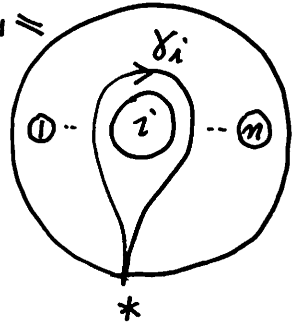
- $t\mathrm{Der}(L_n) := (L_n)^{\oplus n} \subset t\mathrm{Der}(A_n)$ Lie subalgebra $\quad \subset \mathrm{Aut}(A_n)$

$\underset{\text{BCH series}}{\Longrightarrow}$ group $\mathrm{TAut}(L_n)$, $\rho : \mathrm{TAut}(L_n) \to \mathrm{Aut}(L_n)$ group homomorphism

("some small equivalence relation" in Main Theorem
$= \mathrm{Ker}(\rho : \mathrm{TAut}(L_n) \to \mathrm{Aut}(L_n)) \cong \mathbb{K}^n$ (additive group))

◎ Tangential expansions $\pi = \pi_1(\Sigma_{0,n+1}) \longrightarrow A_n$

$\Sigma_{0,n+1} \approx$



$\pi := \pi_1(\Sigma_{0,n+1}, *) = \langle \gamma_1, \gamma_2, \cdots, \gamma_n \rangle$ free group of rank n

(Identification $x_i = [\gamma_i] \in H_1(\Sigma_{0,n+1} ; \mathbb{K})$, $1 \le i \le n$

$$A_n = \prod_{m=0}^{\infty} H_1(\Sigma_{0,n+1} ; \mathbb{K})^{\otimes m}$$

Definition $\theta : \pi \to A_n$ tangential (special) expansion

$\overset{\text{def}}{\Longleftrightarrow}$

1) $\theta : \pi \to$ (multiplicative group of A_n)
group homomorphism

2) $1 \le \forall i \le n$, $\exists \nu_i \in L_n$

$$\theta(\gamma_i) = e^{\nu_i} e^{x_i} e^{-\nu_i}$$

] tangential

3) (special)

$$\theta(\gamma_1 \gamma_2 \cdots \gamma_n) = e^{x_1 + x_2 + \cdots + x_n}$$

] special

examples

- $\theta^{\exp} : \pi \to A_n$, $\gamma_i \mapsto e^{x_i}$, $1 \le i \le n$, tangential, but not special
- Habegger-Masbaum Kontsevich integral $\Longrightarrow$ special expansions
- Kuno combinatorial construction of special expansions
- K analytic construction of special expansions $(\mathbb{K} = \mathbb{R})$

$$\rho(\mathrm{TAut}(L_n)) \overset{\sim}{=\!=\!=} \left\{ \begin{array}{l} \text{tangential expansions} \\ \pi \to A_n \end{array} \right\}$$

$$\cup \qquad \circlearrowright \qquad \cup$$

$$\left\{ \begin{array}{l} \rho(F) \in \rho(\mathrm{TAut}(L_n)) : \\ F \text{ satisfies (KV1)} \end{array} \right\} \overset{\sim}{=\!=\!=} \left\{ \begin{array}{l} \text{special expansions} \\ \pi \to A_n \end{array} \right\}$$

$\rho(F) \mapsto \rho(F)^{-1} \circ \theta^{\exp}$

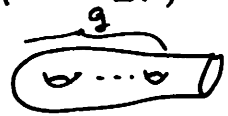
$\rightsquigarrow$ Main Theorem

<u>Theorem</u> (Kuno-K., Massuyeau-Turaev)

$\theta : \pi \to A_n$ special expansion

$\Longrightarrow$ $\theta : \left(\begin{array}{l} \text{completed Goldman Lie} \\ \text{algebra of } \Sigma_{0,n+1} \end{array} \right) \xrightarrow{\cong} (|A_n|_{\geq 1}, \{-,-\}_{KKS})$ Lie algebra isom.

Kirillov-Kostant-Souriau

- <u>original result</u> (Kuno-K.) for $\Sigma_{g,1}$ = 

- $(|A_n|_{\geq 2}, \{-,-\}_{KKS}) \hookrightarrow t\mathrm{Der}(A_n)$ Lie subalgebra

$$|x_{k_1} x_{k_2} \cdots x_{k_m}| \longmapsto (u_1, u_2, \dots, u_n), \quad u_i := \sum_{j=1}^{m} \delta_{i,k_j} x_{k_{j+1}} \cdots x_{k_m} x_{k_1} \cdots x_{k_{j-1}}$$

- (KV1) ---- the Goldman bracket

(KV2) ---- ? the Turaev cobracket

<u>background result</u> (K.) a regular homotopy version δ^+ of δ includes some restriction of <u>the Alekseev-Torossian divergence</u> $\mathrm{div} : t\mathrm{Der}(A_n) \to |A_n|$

◎ <u>AT divergence</u> $\mathrm{div} : t\mathrm{Der}(A_n) \to |A_n|$ <u>and AT group cocycle</u> $j : T\mathrm{Aut}(A_n) \to |A_n|$

$u = (u_1, \cdots, u_n) \in t\mathrm{Der}(A_n)$, $u_i = (u_i)^0 + \sum_{j=1}^{n} (u_i)^j x_j$, $(u_i)^0 \in \mathbb{K}$, $(u_i)^j \in A_n$

$\mathrm{div}(u) := \sum_{i=1}^{n} |x_i (u_i)^i| \in |A_n|$ Alekseev-Torossian divergence

$|A_n| \oplus \mathbb{K}$: $t\mathrm{Der}(A_n)$-module

$$\begin{array}{ccc} |A_n| \oplus \mathbb{K} & & t\mathrm{Der}(A_n) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (|a|, \lambda) & & u \end{array} \quad u \cdot (|a|, \lambda) := (|\rho(u)(a)| + \lambda\, \mathrm{div}(u), 0)$$

$u \mapsto e^u$

$|A_n| \oplus \mathbb{K}$: $T\mathrm{Aut}(\hat{A}_n)$-module, $e^u \cdot (|a|, \lambda) := \sum_{m=0}^{\infty} \frac{1}{m!} u^m \cdot (|a|, \lambda)$

$j : T\mathrm{Aut}(A_n) \to |A_n|$ Alekseev-Torossian group cocycle

$$e^u \cdot (0, 1) = (j(e^u), 1).$$

$$j(e^u) = \frac{e^u - 1}{u} \cdot \mathrm{div}(u) \quad \rightsquigarrow (KV2)$$

A regular homotopy version of the Turaev cobracket

$\hat{\pi} = \hat{\pi}(\Sigma_{0,n+1}) \overset{\text{def}}{=} \pi/\text{conj} = [S^1, \Sigma_{0,n+1}]$ free loops on $\Sigma_{0,n+1}$

$\ni |\alpha| \leftarrow\!\shortmid\ \alpha \in \pi$. $\qquad \mathbb{K}\hat{\pi} = \mathbb{K}\pi/[\mathbb{K}\pi, \mathbb{K}\pi] = |\mathbb{K}\pi|$

$\hat{\pi}^+ = \hat{\pi}^+(\Sigma_{0,n+1}) \overset{\text{def}}{=} \{\alpha : S^1 \to \Sigma_{0,n+1} : C^\infty \text{ immersions}\}/\text{regular homotopy}$

$\ni |\alpha|^+ \longleftarrow\!\shortmid\ \alpha \in$

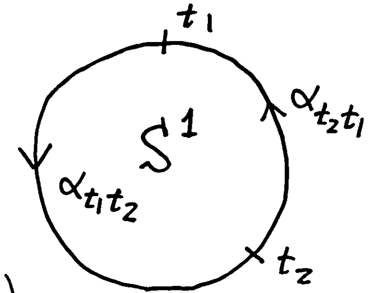
$\delta^+ : \mathbb{Z}\hat{\pi}^+ \to \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi}$ regular homotopy version of <u>the Turaev cobracket</u>

$\alpha : S^1 \to \Sigma_{0,n+1}$, generic immersion (<u>at worst</u>, trsv. double points)

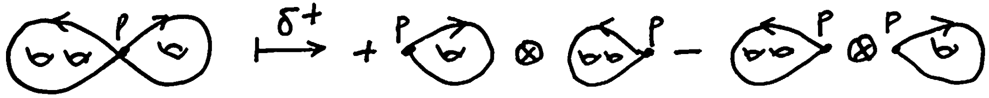
$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 ;\ t_1 \neq t_2, \alpha(t_1) = \alpha(t_2)\}$

$$\delta^+(\alpha) \overset{\text{def}}{=} \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1 t_2}| \otimes |\alpha_{t_2 t_1}| \in \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi}$$

where $\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \in \{\pm 1\}$ local intersection number



(<u>example</u>



)

- original version of the Turaev cobracket

$$\delta : \mathbb{Z}\hat{\pi}/\mathbb{Z}|1| \to (\mathbb{Z}\hat{\pi}/\mathbb{Z}|1|) \otimes (\mathbb{Z}\hat{\pi}/\mathbb{Z}|1|)$$

But we want to consider the augmentation $\varepsilon : \mathbb{Z}\hat{\pi} \to \mathbb{Z}$. $\sum \lambda_{|\alpha|} |\alpha| \mapsto \sum \lambda_{|\alpha|}$

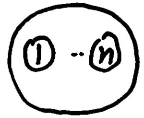
$f : T\Sigma_{0,n+1} \overset{\text{ori. pres}}{\cong} \Sigma_{0,n+1} \times \mathbb{R}^2 \xrightarrow{pr_2} \mathbb{R}^2$ framing forgetting C^∞ structure

$\Rightarrow \quad \hat{\pi}^+ \underset{f}{\cong} \hat{\pi} \times \mathbb{Z}, \quad |\alpha|^+ \mapsto (|\alpha|, \mathrm{rot}_f \alpha)$ rotation number of α w.r.to f

$f : \hat{\pi} \to \hat{\pi}^+, \quad |\alpha| \mapsto |\alpha|_f$, s.t. $\mathrm{rot}_f(|\alpha|_f) = 0$, section

(<u>example</u> $|1| \mapsto \infty$ (figure 8) ($\because$ $\mathrm{rot}_f(\infty) = 0$))

$\Rightarrow \quad \delta_f^+ : \mathbb{Z}\hat{\pi} \xrightarrow{f} \mathbb{Z}\hat{\pi}^+ \xrightarrow{\delta^+} \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi}$

We consider the framing coming from the standard embedding $\Sigma_{0,n+1} =$  $\hookrightarrow \mathbb{R}^2$

Then we write simply $\delta^+ := \delta_f^+ : \mathbb{Z}\hat{\pi} \to \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi}$

<u>Background result</u> (K.) (originally <u>Kuno-K</u>, <u>Massuyeau-Turaev</u> for δ)

$\theta : \pi \to A_n$ tangential expansion ($\Rightarrow \theta : (\mathbb{K}\hat{\pi})^\wedge \xrightarrow{\cong} |A_n|$) completion

$$\begin{tikzcd}
(\mathbb{K}\hat{\pi})^\wedge \arrow[r, "\delta^+"] \arrow[d, "\theta"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & (\mathbb{K}\hat{\pi})^\wedge \otimes (\mathbb{K}\hat{\pi})^\wedge \arrow[d, "\cong"', "\theta"] \\
|A_n| \arrow[r, dashed, "\exists ! \, \delta^{+,\theta}"] & |A_n| \otimes |A_n|
\end{tikzcd}$$

$\Rightarrow \quad \delta^{+,\theta} = \delta^{+,\theta}_{(-1)} + \delta^{+,\theta}_{(0)} + \cdots$

$\delta^{+,\theta}_{(m)}$: degree m component

$u \wedge v = u \otimes v - v \otimes u$

$$\delta^{+,\theta}_{(-1)}(|x_{k_1}\cdots x_{k_m}|)$$
$$= \sum_{a<b} \delta_{k_a k_b} \{ |x_{k_b} x_{k_{b+1}} \cdots x_{k_m} x_{k_1} \cdots x_{k_{a-1}}| \wedge |x_{k_{a+1}} \cdots x_{k_{b-1}}|$$
$$- |x_{k_{b+1}} \cdots x_{k_m} x_{k_1} \cdots x_{k_{a-1}}| \wedge |x_{k_{a+1}} \cdots x_{k_{b-1}} x_{k_b}| \} \in |A_n| \otimes |A_n|$$

in particular

AT-divergence

$$(1 \otimes \varepsilon) \circ \delta^{+,\theta}_{(-1)} \big|_{|A_n|_{\geq 2}} = \mathrm{div}\big|_{|A_n|_{\geq 2}} : |A_n|_{\geq 2} \hookrightarrow t\mathrm{Der}(A_n) \xrightarrow{\mathrm{div}} |A_n|$$

<u>Main Theorem</u> (Alekseev - K. - Kuno - Naef)

$F \in T\mathrm{Aut}(A_n)$ satisfying (KV1), $\theta = F^{-1} \circ \theta^{\exp}$

Then F : a solution to (KV2) modulo $\mathrm{Ker}(p : T\mathrm{Aut}(A_n) \to \mathrm{Aut}(A_n))$

$\Updownarrow$

$\delta^{+,\theta} = \delta^{+,\theta}_{(-1)}$ (formal description) --- no higher terms

◎ Basic ingredients in the proof of Main Theorem

--- non commutative Poisson geometry on the algebra A_n

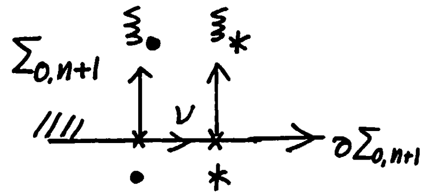
(1) <u>double divergence</u>

$$t\mathrm{Div} : t\mathrm{Der}(A_n) \to |A_n| \otimes |A_n|$$
$$t\mathrm{Div}\big|_{|A_n|_{\geq 2}} = \delta^{+,\theta}_{(-1)}\big|_{|A_n|_{\geq 2}}$$

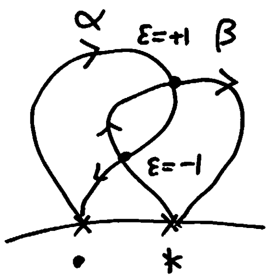
(2) <u>double brackets</u> $\leftrightarrow$ <u>topological counterpart</u> $\kappa : \mathbb{Z}\pi \otimes \mathbb{Z}\pi \to \mathbb{Z}\pi \otimes \mathbb{Z}\pi$

$$\pi := \pi_1(\Sigma_{0,n+1}, *) \overset{\text{Identify}}{\cong} \pi_1(\Sigma_{0,n+1}, \bullet)$$

$$\gamma \longmapsto \nu\gamma\bar{\nu} \quad (\xrightarrow{\text{read}})$$



$\kappa : \mathbb{Z}\pi \otimes \mathbb{Z}\pi \to \mathbb{Z}\pi \otimes \mathbb{Z}\pi$



$\alpha \in \pi_1(\Sigma_{0,n+1}, \bullet) \cong \pi$, $\beta \in \pi_1(\Sigma_{0,n+1}, *) = \pi$ in general position

$$\kappa(\alpha, \beta) \overset{\text{def}}{=} -\sum_{p \in \alpha\cap\beta} \varepsilon_p(\alpha, \beta)\, \beta_{*p}\alpha_{p\bullet}\nu \otimes \bar{\nu}\alpha_{\bullet p}\beta_{p*} \in \mathbb{Z}\pi \otimes \mathbb{Z}\pi$$

where $\varepsilon_p(\alpha,\beta) \in \{\pm 1\}$ local intersection number at p

$\alpha_{\bullet p}$ = the segment of α from $\bullet$ to p, $\alpha_{p\bullet}$, β_{*p}, β_{p*} segments

$$\begin{cases} \kappa(\alpha, \beta_1\beta_2) = \kappa(\alpha,\beta_1)(1\otimes\beta_2) + (\beta_1\otimes 1)\kappa(\alpha,\beta_2) =: \kappa(\alpha,\beta_1)\beta_2 + \beta_1\kappa(\alpha,\beta_2) \\ \kappa(\alpha_1\alpha_2, \beta) = \kappa(\alpha_1,\beta)(\alpha_2\otimes 1) + (1\otimes\alpha_1)\kappa(\alpha_2,\beta) =: \kappa(\alpha_1,\beta)*\alpha_2 + \alpha_1 * \kappa(\alpha_2,\beta) \end{cases}$$

κ : (non-symmetric) double bracket in the sense of van den Bergh

(Massuyeau-Turaev's observation) $\rightsquigarrow$ non-commutative Poisson geometry

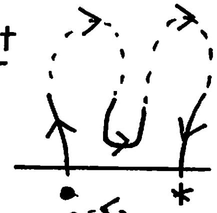
(3) <u>operation</u> $\mu : \mathbb{Z}\pi^+ \to \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\pi$ (originally introduced by Turaev)

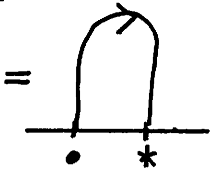
$\pi^+ := \{\ell : ([0,1], (\frac{d}{dt})_0, (\frac{d}{dt})_1) \to (\Sigma_{0,n+1}, \xi_\bullet, -\xi_*)\ C^\infty \text{ immersion}\}/\text{regular homotopy}$

$\pi^+ \ni [\ell]^+$

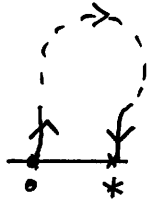
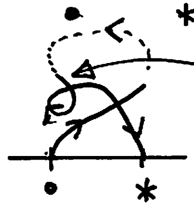
<u>group structure</u>

<u>product</u>



<u>unit</u> $1 =$ 

$\mathrm{rot}_f\, 1 = -\frac{1}{2}$

<u>inverse</u> $\Big($  $\Big)^{-1} =$ 

f : framing of $\Sigma_{0,n+1}$

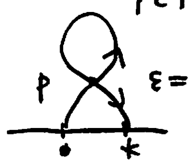
$\Longrightarrow \pi^+ \cong \pi \times \mathbb{Z}$, $[\ell]^+ \mapsto ([\ell], \mathrm{rot}_f\, \ell + \frac{1}{2})$ group isomorphism

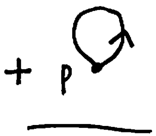
$\gamma : ([0,1], (\frac{d}{dt})_0, (\frac{d}{dt})_1) \to (\Sigma_{0,n+1}, \xi_\bullet, -\xi_*)$ generic immersion

$\Gamma_\gamma := \{\text{double points of } \gamma\}$

$\Gamma_\gamma \ni \forall p,\ 0 < {}^{\exists!}t_1^p < {}^{\exists!}t_2^p < 1$ s.t. $\gamma(t_1^p) = \gamma(t_2^p) = p$

$$\mu(\gamma) \overset{\text{def}}{=} -\sum_{p \in \Gamma_\gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))\, |\gamma_{t_1^p t_2^p}| \otimes \bar{\nu}\, \gamma_{0 t_1^p}\, \gamma_{t_2^p 1} \in \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\pi$$

<u>example</u> 

$\varepsilon = -1 \longmapsto +$ 

$\otimes$ 

- (alternating part of $(1\otimes|1|)\circ\mu$) $= \delta^+$
- $\mu(\gamma_1\gamma_2) = \mu(\gamma_1)(1\otimes\gamma_2) + (1\otimes\gamma_1)\mu(\gamma_2) + (|1|\otimes 1)\kappa(\gamma_1,\gamma_2)$

  μ : "coboundary" of $(|1|\otimes 1)\kappa$
- f : framing of $\Sigma_{0,n+1}$

  $\Rightarrow \pi \xrightarrow{f} \pi^+$ group homom, injective

  $\gamma \longmapsto \gamma_f$ s.t. $\mathrm{rot}_f(\gamma_f) = -\frac{1}{2}$

  $\Rightarrow \mu : \mathbb{Z}\pi \xrightarrow{f} \mathbb{Z}\pi^+ \xrightarrow{\mu} \mathbb{Z}\hat{\pi}\otimes\mathbb{Z}\pi$

- <u>proof of Main Theorem</u> :

  Reconstruction of $\mu : \mathbb{K}\pi \to \mathbb{K}\hat{\pi}\otimes\mathbb{K}\pi$

  in the context of non-commutative Poisson geometry

  $\longrightarrow$ Part II, Today 15:20-16:00