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"A tensorial description of the Turaev cobracket on genus 0 compact surfaces"

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Turaev cobracket

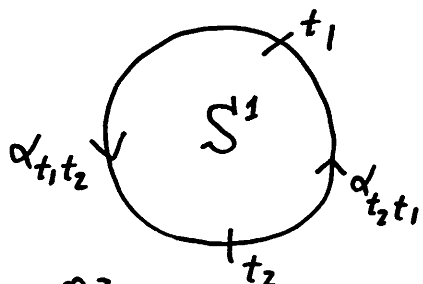
S : compact connected oriented surface

$\hat{\pi} = \hat{\pi}(S) := [S^1, S] = \pi_1(S)/_{\text{conj.}}$ the free homotopy set of free loops on S

$p \in S$, $|\ | : \pi_1(S,p) \to \hat{\pi}(S)$, $x \mapsto |x|$, forgetful map of the basepoint p

$\mathbb{1} := |1| \in \hat{\pi}(S)$ trivial loop.

$|\ |' : \mathbb{Z}\pi_1(S,p) \xrightarrow{|\ |} \mathbb{Z}\hat{\pi}(S) \xrightarrow{\text{quotient map}} \mathbb{Z}\hat{\pi}'(S) := \mathbb{Z}\hat{\pi}(S)/\mathbb{Z}\mathbb{1}$



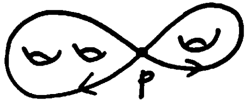
$\alpha \in \hat{\pi}$ in general position

$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 \,;\, t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$

$$\delta(\alpha) \overset{\text{def}}{=} \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot\alpha(t_1), \dot\alpha(t_2))\, |\alpha_{t_1t_2}|' \otimes |\alpha_{t_2t_1}|' \in (\mathbb{Z}\hat{\pi}'(S))^{\otimes 2}$$

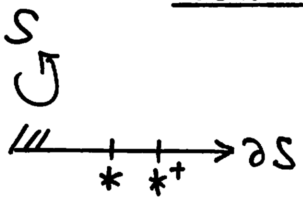
$\varepsilon(\dot\alpha(t_1), \dot\alpha(t_2)) \in \{\pm 1\}$ local intersection number

Turaev cobracket

$\delta\Big($  $\Big) =$ 

<u>Turaev</u> (1) δ is well-defined

(2) $\mathbb{Z}\hat{\pi}'(S)$: Lie bialgebra



<u>Coaction</u> <u>Assume</u> $\partial S \neq \phi$.

$*, *^+ \in \partial S$, $\pi := \pi_1(S, *)$ $(\cong \Pi S(*, *^+))$ a free group of finite rank

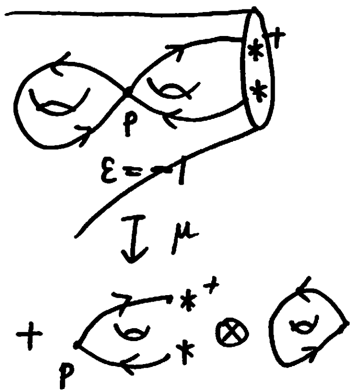
$\gamma \in \Pi S(*, *^+) \cong \pi$ in general position

$\Gamma_\gamma :=$ (the set of double points of γ) $\subset S$

$\Gamma_\gamma \ni p \qquad 0 \le t_1^p < t_2^p \le 1 \qquad \gamma(t_1^p) = \gamma(t_2^p) = p$

$$\mu(\gamma) \overset{\text{def}}{=} -\sum_{p \in \Gamma_\gamma} \varepsilon(\dot\gamma(t_1^p), \dot\gamma(t_2^p))(\gamma_{0 t_1^p}\gamma_{t_2^p 1}) \otimes |\gamma_{t_1^p t_2^p}|' \in \mathbb{Z}\pi \otimes \mathbb{Z}\hat{\pi}'$$

(inspired by Turaev's μ)



<u>Kuno-K.</u> (1) μ is well-defined

(2) $\mathbb{Z}\pi$: $\mathbb{Z}\hat{\pi}'$-bimodule

$$\delta(|x|) = \mathrm{alt} \circ (|\ |' \otimes 1)\,\mu(x) \qquad (\forall x \in \pi)$$

<u>where</u> $\mathrm{alt} : \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}' \circlearrowleft$ $\qquad \mathrm{alt}(u \otimes v) := u \otimes v - v \otimes u.$

<u>group-like expansion</u>

$\pi = \pi_1(S, *)$: free group of finite rank

$\mathbb{Q}\pi := \{\sum_{x\in\pi} a_x x \ ; \ a_x \in \mathbb{Q}, \ a_x = 0$ for all but finite $x \in \pi\}$ group ring

$\widehat{\mathbb{Q}\pi} := \varprojlim_{p\to\infty} \mathbb{Q}\pi/(I\pi)^p$, $I\pi := \{\sum a_x x \in \mathbb{Q}\pi : \sum a_x = 0\}$ augmentation ideal

completed group ring

$\Delta : \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi} \hat{\otimes} \widehat{\mathbb{Q}\pi}$, $x \in \pi \mapsto \Delta x = x \hat{\otimes} x$, coproduct

$\iota : \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi}$, $x \in \pi \mapsto \iota(x) = x^{-1}$, antipode $\rightsquigarrow$ $\widehat{\mathbb{Q}\pi}$: complete Hopf algebra.

$H := H_1(S : \mathbb{Q}) = (\pi/[\pi,\pi]) \otimes_{\mathbb{Z}} \mathbb{Q}$

$x \in \pi \mapsto [x] := (x \bmod [\pi,\pi]) \otimes 1 \in H$

$\hat{T} = \hat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$, the completed tensor algebra over H

$\hat{T}_{\geq p} := \prod_{m=p}^{\infty} H^{\otimes m}$, two-sided ideal in $\hat{T}$ $(p \geq 1)$

$\Delta : \hat{T} \to \hat{T} \hat{\otimes} \hat{T}$, $X \in H \mapsto \Delta X = X \hat{\otimes} 1 + 1 \hat{\otimes} X$ coproduct

$\iota : \hat{T} \to \hat{T}$, $X \in H \mapsto \iota(X) = -X$ antipode $\rightsquigarrow$ $\hat{T}$: complete Hopf algebra

<u>Definition</u> $\theta : \pi \to \hat{T}$ group-like expansion

$\overset{\text{def}}{\Longleftrightarrow}$
1) $\forall x \in \pi$ $\theta(x) = 1 + [x] +$ higher degree terms
2) $\forall x, \forall y \in \pi$ $\theta(xy) = \theta(x)\theta(y)$
3) $\forall x \in \pi$ $\Delta\theta(x) = \theta(x) \hat{\otimes} \theta(x)$ —

$\Longrightarrow \quad \theta : \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \hat{T}, \quad \sum a_x x \in \mathbb{Q}\pi \longmapsto \sum a_x \theta(x)$, isomorphism of complete Hopf algebras

$\widehat{\mathbb{Q}\hat{\pi}} := \varprojlim_{p\to\infty} \mathbb{Q}\hat{\pi} \big/ \left(\mathbb{Q}1 + |(I\pi)^p|\right)$ completion of $\mathbb{Q}\hat{\pi}$ w.r.to $I\pi$.

$\delta : \widehat{\mathbb{Q}\hat{\pi}} \longrightarrow \widehat{\mathbb{Q}\hat{\pi}} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}$ Turaev cobracket

$N : \hat{T} = \prod_{m=0}^{\infty} H^{\otimes m} \circlearrowleft$ linear endomorphism, <u>cyclic</u> <u>symmetrizer</u> (cyclicizer)

$N|_{H^{\otimes 0}} := 0$

$N(X_1 X_2 \cdots X_m) := \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1} \qquad (X_j \in H)$

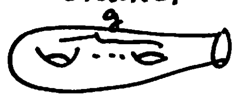
<u>Kuno-K.</u> $\forall \theta : \pi \to \hat{T}$ group-like expansion

$-N\theta : \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} N(\hat{T}), \quad |x| \in \hat{\pi} \longmapsto -N\theta(x)$,

is a linear isomorphism.

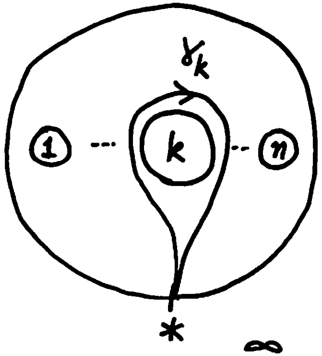
$\delta^{\theta} := (-N\theta)^{\hat{\otimes}2} \circ \delta \circ (-N\theta)^{-1} : N(\hat{T}) \to N(\hat{T}) \hat{\otimes} N(\hat{T})$

tensorial description of the Turaev cobracket

<u>Massuyeau-Turaev, Kuno-K.</u> If $S = \Sigma_{g,1} =$  and θ is "symplectic", then the lowest degree term of δ^{θ} equals Schedler's cobracket.

<u>K.</u> If $S = \Sigma_{0,n+1} =$  and θ is "special", then the lowest degree term of <u>a regular</u> <u>homotopy</u> <u>version</u> of δ^{θ} includes the divergence cocycle in the Kashiwara-Vergne problem.

$n \geq 1$, $S = \Sigma_{0,n+1} =$



$\pi = \langle \gamma_1, \gamma_2, \cdots, \gamma_n \rangle$ free group of rank n

$x_k := [\gamma_k] \in H$, $1 \leq k \leq n$

$$H = \bigoplus_{k=1}^{n} \mathbb{Q} x_k$$

$$\theta^{\mathrm{std}} : \pi \to \hat{T}, \quad \gamma_k \mapsto \exp(x_k) = \sum_{m=0}^{\infty} \frac{1}{m!} x_k^{\,m}$$

$$\delta^{\mathrm{std}} := \delta^{\theta^{\mathrm{std}}} : N(\hat{T}) \to N(\hat{T}) \hat{\otimes} N(\hat{T})$$

$$s(z) := \frac{1}{e^{-z}-1} + \frac{1}{z} = -\frac{1}{2} - \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} z^{2m-1} = -\frac{1}{2} - \frac{1}{12} z + \frac{1}{720} z^3 - \cdots$$

B_{2m} : Bernoulli number

Theorem (K.) $\forall m \geq 1$, $\forall k_1, \cdots, \forall k_m \in \{1, 2, \cdots, n\}$

$$\begin{aligned}
&\delta^{\mathrm{std}}(N(x_{k_1} x_{k_2} \cdots x_{k_m})) \\
&= \mathrm{alt}(N \hat{\otimes} N) \Bigg(\sum_{1 \leq i < j \leq m} \Big((1 \hat{\otimes} \iota) \Delta \Big(\varepsilon_{k_i k_j} x_{k_i} x_{k_j} - \delta_{k_i k_j} \frac{x_{k_i}^2}{e^{-x_{k_i}} - 1} \Big) \Big) (x_{k_{j+1}} \cdots x_{k_m} x_{k_1} \cdots x_{k_{i-1}} \hat{\otimes} x_{k_{i+1}} \cdots x_{k_{j-1}}) \\
&\qquad + \frac{1}{2} \sum_{i=1}^{m} x_{k_{i+1}} \cdots x_{k_m} x_{k_1} \cdots x_{k_{i-1}} \hat{\otimes} x_{k_i} \\
&\qquad - \sum_{i=1}^{m} \sum_{q=1}^{\infty} \frac{B_{2q}}{(2q)!} \sum_{p=0}^{2q-1} (-1)^p \binom{2q}{p} x_{k_1} \cdots x_{k_{i-1}} x_{k_i}^{p} x_{k_{i+1}} \cdots x_{k_m} \hat{\otimes} x_{k_i}^{2q-p} \Bigg)
\end{aligned}$$

where
$$\varepsilon_{k\ell} := \begin{cases} 1 & \text{if } k > \ell \\ 0 & \text{if } k \leq \ell \end{cases}$$

2 Lemmas ⊕ Massuyeau-Turaev's description of homotopy intersection form

⟨geometric⟩

Lemma 1 (conjectured by Kuno) If $x \in \pi_1(S,*)$ is represented by a simple loop with $\varepsilon(\dot{x}(0), \dot{x}(1)) = +1$, then

$$\mu(\log x) = \frac{1}{2}\cdot 1 \otimes |\log x|' + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} \sum_{p=0}^{2m-1} \binom{2m}{p} (-1)^p (\log x)^p \hat{\otimes} |(\log x)^{2m-p}|'$$

⟨algebraic⟩

$\dot{\rightsquigarrow} : \hat{T}_{\geq 1} \times \hat{T}_{\geq 1} \to \hat{T}_{\geq 1}$. Massuyeau-Turaev's pairing

$$x_{i_1} \cdots x_{i_{\ell-1}} x_{i_\ell} \dot{\rightsquigarrow} x_{j_1} x_{j_2} \cdots x_{j_m} := -\delta_{i_\ell j_1} x_{i_1} \cdots x_{i_{\ell-1}} x_{i_\ell} x_{j_2} \cdots x_{j_m} \quad (\ell, m \geq 1)$$

$\dot{\rightsquigarrow}$: associative with unit $x_0 := -\sum_{k=1}^{n} x_k \in H$

$u, v \in \hat{T}_{\geq 1}$, $u * v := \log((\exp u)(\exp v))$ Baker-Campbell-Hausdorff series

$\boxed{H} := x_1 * x_2 * \cdots * x_n \in \hat{T}_{\geq 1}$

$\boxed{H}^{-1} :=$ (the inverse element of $\boxed{H}$ with respect to $\dot{\rightsquigarrow}$) $\in \hat{T}_{\geq 1}$

Lemma 2

$$-\boxed{H}^{-1} + x_0 s(\boxed{H}) x_0 = x_0 - \sum_{k>\ell} x_k x_\ell + \sum_{k=1}^{n} s(x_k) x_k^2$$

Remarks (1) θ^{std} is group-like, but not "special"

(2) If $n=2$, there is an extra class of "special" expansions :

solutions to the Kashiwara-Vergne problem in a formulation by Alekseev-Torossion.

Conjecture If $n=2$ and θ is a solution to the Kashiwara-Vergne problem, then $\delta^\theta \overset{?}{=}$ the lowest degree term of δ^θ.

Problem For $\Sigma_{g,1}$, does there exist a "symplectic" expansion θ such that $\delta^\theta =$ Schedler's cobracket ?

--- positive genus Kashiwara-Vergne problem ? ? ?

(cf) Kuno's computation for $\Sigma_{1,1}$.)