研究集会「多様体のトポロジーの展望」2014年11月28日13:00-13:25, 東京大学大学院数理科学研究科 大講義室

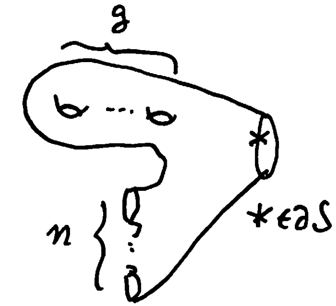
「Goldman-Turaev Lie 双代数のテンソル表示について」

河澄響矢 (東大・数理), 久野雄介 氏 (津田塾大・学芸) との共同研究

S : compact connected oriented surface with $\partial S \neq \phi \xRightarrow[\text{定理}]{\text{分類}} \exists g, \exists n \geq 0$

$\pi = \pi_1(S, *)$: free group of rank $2g+n$

$S = \Sigma_{g,n+1} =$



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§1 自由群の群環のテンソル表示 (古典的)

§2 Goldman Lie 代数のテンソル表示 (完全に記述できた)

§3 Turaev 余括弧積のテンソル表示について (不完全な結果しかない。「正種数柏原Vergne問題」?)

背景 (予稿参照: 写像類群の Johnson 準同型)

§1 ⇒ 自由群の自己同型群全体への Johnson 準同型の拡張

§2 ⇒ Dehn twist 公式 $t_C = \exp\left(\frac{1}{2}(\log C)^2\right)$

⇒ Johnson 準同型とは Torelli 群を完備 Goldman Lie 代数に幾何的に埋めこむことである

§3 ⇒ Johnson 準同型像の幾何的制約条件 (森田traces, 榎本・佐藤 traces の幾何的再構成)

§1. 自由群の群環のテンソル表示

π : free group of finite rank

$\mathbb{Q}\pi := \{ \sum_{x\in\pi} a_x x \, ; \, a_x \in \mathbb{Q}$, 有限個の x を除いて $a_x = 0 \}$ 群環

$I\pi := \mathrm{Ker}(\varepsilon : \mathbb{Q}\pi \to \mathbb{Q}, \ \sum a_x x \mapsto \sum a_x)$ 添加 ideal

$\widehat{\mathbb{Q}\pi} := \varprojlim_{p\to\infty} \mathbb{Q}\pi/(I\pi)^p$ 完備群環，$\{(I\pi)^p\}_{p=1}^{\infty}$ の定める位相を考える.

$\Delta : \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi} \hat{\otimes} \widehat{\mathbb{Q}\pi}$，$x\in\pi \mapsto \Delta x = x \hat{\otimes} x$，余積 $\Rightarrow$ $\widehat{\mathbb{Q}\pi}$: 完備 Hopf 代数

$\widehat{\mathbb{Q}\pi}$ の<u>テンソル表示</u> (古典的 : Magnus, Witt, ---, Quillen)

$H := (\pi/[\pi,\pi]) \otimes_{\mathbb{Z}} \mathbb{Q} = H_1(\pi ; \mathbb{Q})$, $x \in \pi \mapsto [x] := (x \bmod [\pi,\pi]) \otimes_{\mathbb{Z}} 1 \in H$

$\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$，完備テンソル代数

$\widehat{T}_{\geq p} := \prod_{m\geq p} H^{\otimes m}$, $p \geq 1$, を 0 の基本近傍系とする位相を考える.

$\Delta : \widehat{T} \to \widehat{T} \hat{\otimes} \widehat{T}$，$X \in H \mapsto \Delta X = X \hat{\otimes} 1 + 1 \hat{\otimes} X$，余積 $\Rightarrow$ $\widehat{T}$: 完備 Hopf 代数

<u>定義</u> $\theta : \pi \to \widehat{T}$ group-like expansion

$\overset{\text{def}}{\Longleftrightarrow}$
1) $\forall x \in \pi$, $\theta(x) = 1 + [x] +$ higher degree terms
2) $\forall x, \forall y \in \pi$, $\theta(xy) = \theta(x)\theta(y)$
3) $\forall x \in \pi$ $\Delta\theta(x) = \theta(x) \hat{\otimes} \theta(x)$

Johnson 準同型

$\Longrightarrow$ $\theta : \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \widehat{T}$, $\sum a_x x \in \mathbb{Q}\pi \mapsto \sum a_x \theta(x) \in \widehat{T}$，完備 Hopf 代数の同型

§3. Goldman Lie 代数のテンソル表示.

$S = \Sigma_{g,1}$ の場合のみを述べる. (一般の $\Sigma_{g,n+1}$ でも完全に記述できている)

$\pi = \pi_1(\Sigma_{g,1}, *)$, $* \in \partial\Sigma_{g,1}$

$\Sigma_{g,1} =$  $* \in \partial\Sigma_{g,1}$

<u>代数的準備</u>: <u>symplectic derivation algebra</u>

$H = H_1(\Sigma_{g,1}; \mathbb{Q})$, $\cdot : H \times H \to \mathbb{Q}$, 交叉数. <u>非退化</u>交代形式

$H \cong H^*$, $X \mapsto (Y \mapsto Y\cdot X)$, Poincaré duality

$\omega := \sum_{i=1}^{g} A_i B_i - B_i A_i \in H^{\otimes 2} \subset \hat{T}$ symplectic 形式, symplectic 基底 $\{A_i, B_i\}_{i=1}^{g} \subset H$ のとりかたによらない

$$\mathrm{Der}_\omega(\hat{T}) := \left\{ D : \hat{T} \to \hat{T} \text{ 連続}\mathbb{Q}\text{線型写像} : \begin{array}{l} \forall u\ \forall v \in \hat{T}\quad D(uv) = (Du)v + u(Dv) \ (\text{導分}) \\ D\omega = 0 \quad (\text{symplectic}) \end{array} \right\}$$

symplectic derivation algebra ($\cup$ Kontsevich's "associative")

$$\mathrm{Der}_\omega(\hat{T}) \xhookrightarrow{|_H} \mathrm{Hom}(H, \hat{T}) = H^* \otimes \hat{T} \overset{\text{P.d.}}{=} H \otimes \hat{T} = \hat{T}_{\geq 1}$$

像は $\prod_{m=1}^{\infty} (H^{\otimes m})^{\text{cyclic}}$: cyclic invariants に一致する.

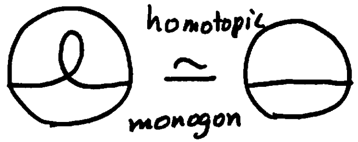
(∵) $X_j \in H$, $1 \leq j \leq m$, $X_1 \cdots X_m \in H^{\otimes m}$, $\hat{T}$ の導分とみなす.

$$(X_1 \cdots X_m)\omega = \sum_{i=1}^{g} (X_1 \cdots X_m)(A_i B_i - B_i A_i)$$

$$= \sum_{i=1}^{g} \{ (A_i \cdot X_1) X_2 \cdots X_m B_i - (B_i \cdot X_1) X_2 \cdots X_m A_i - A_i (B_i \cdot X_1) X_2 \cdots X_m + B_i (A_i \cdot X_1) X_2 \cdots X_m \}$$

$$= X_2 \cdots X_m X_1 - X_1 X_2 \cdots X_m \quad //$$

$N : \hat{T} \to \hat{T}$ cyclic symmetrizer (cyclicizer)

$N|_{H^{\otimes 0}} := 0$ $\Big(\leftrightarrow$ 

$\Big)$

$N(X_1 \cdots X_m) := \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1}$, $(X_j \in H)$

$\mathrm{Der}_{\omega}(\hat{T}) = N(\hat{T}) = N(\hat{T}_{\geq 1})$ 同一視

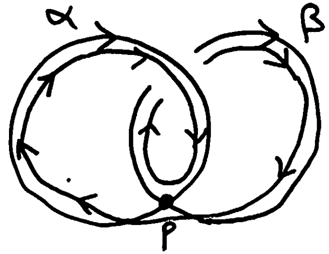
Goldman Lie 代数

S : 一般の $\Sigma_{g,n+1}$ を考える.

$\hat{\pi} = \hat{\pi}(S) = [S^1, S] = \pi_1(S)/_{\text{conj.}}$ free loops

$|\ \ | : \pi_1(S,p) \to \hat{\pi}(S)$ quotient map = forgetful map of a basepoint $p \in S$

$\alpha, \beta \in \hat{\pi}$ in general position



$$[\alpha,\beta] \stackrel{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha,\beta)\, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}$$

ただし $\varepsilon_p(\alpha,\beta) \in \{\pm 1\}$ 局所交叉数

$\alpha_p, \beta_p \in \pi_1(S,p)$ based loops along α, β

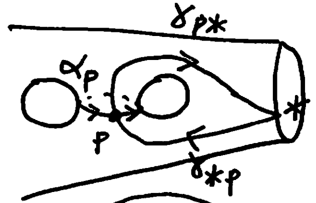
Goldman (1) $[\ ,\ ]$: well-defined

(2) $(\mathbb{Z}\hat{\pi}, [\ ,\ ])$: Lie algebra

$\mathbb{1} \in \hat{\pi}(S)$ constant loop. $\mathbb{1} \in \text{Center}\, \mathbb{Z}\hat{\pi}$ ($\because$ $\forall \alpha \cap \mathbb{1} = \phi$) $\rightsquigarrow \mathbb{Z}\hat{\pi}/\mathbb{Z}\mathbb{1}$: Lie algebra

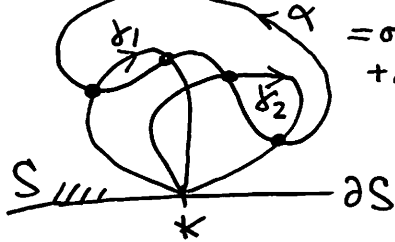
Completion $\widehat{\mathbb{Q}\hat{\pi}} := \varprojlim_{p \to \infty} \mathbb{Q}\hat{\pi} / (\mathbb{Q}\mathbb{1} + |(I\pi)^p|)$ completed Goldman Lie algebra

$* \in \partial S$, $\pi = \pi_1(S, *)$



$\alpha \in \hat{\pi}$, $\gamma \in \pi$ in general position

$$\sigma(\alpha)(\gamma) \overset{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon_p(\alpha, \gamma)\, \gamma_{*p}\, \alpha_p\, \gamma_{p*} \in \mathbb{Z}\pi$$



$\sigma(\alpha)(\gamma_1\gamma_2)$
$= \sigma(\alpha)(\gamma_1)\gamma_2$
$+ \gamma_1 \sigma(\alpha)(\gamma_2)$

Kuno-K. (1) σ: well-defined ($\Leftarrow * \in \partial S$)

(2) $\sigma : \mathbb{Z}\hat{\pi} \to \mathrm{Der}(\mathbb{Z}\pi)$ Lie algebra homomorphism

$\sigma(\mathbb{1}) = 0$ ($\because$ $\mathbb{1} \cap {}^\forall\gamma = \phi$)

$\sigma : \mathbb{Z}\hat{\pi}/\mathbb{Z}\mathbb{1} \to \mathrm{Der}(\mathbb{Z}\pi)$

$\overset{\text{completion}}{\Longrightarrow}$ $\sigma : \widehat{\mathbb{Q}\hat{\pi}} \to \mathrm{Der}(\widehat{\mathbb{Q}\pi})$ continuous Lie algebra homomorphism

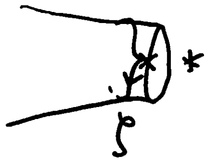
テンソル表示 ($S = \Sigma_{g,1}$ の場合のみを述べる)

Definition (Massuyeau)

$\theta : \pi \to \hat{T} \; (= \hat{T}(H_1(\Sigma_{g,1}; \mathbb{Q})))$ symplectic expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\theta : \pi \to \hat{T}$ group-like expansion

2) $\theta(\zeta) = e^{\omega} \left(= \sum_{m=0}^{\infty} \frac{1}{m!}\omega^m\right) \in \hat{T}$



ここで $\zeta \in \pi = \pi_1(\Sigma_{g,1}, *)$ 負の向きの境界 loop とする

($\Longrightarrow$ $(\widehat{\mathbb{Q}\pi}, \widehat{\mathbb{Q}\langle\zeta\rangle}) \xrightarrow[\cong]{\theta} (\hat{T}, \mathbb{Q}[[\omega]])$ 完備Hopf代数対の同型)

examples 1) (K.) $/\mathbb{R}$ harmonic Magnus expansion (⇝ $a_g \doteq \varphi : \mathcal{M}_g \to \mathbb{R}$ C^ω function $(g \geq 2)$)
K. S-W.Zhang Robin de Jong

2) (Massuyeau) the Le-Murakami-Ohtsuki functor

3) (Kuno) combinatorial construction

Theorem (Kuno-K.) $\theta : \pi \to \widehat{T}$ symplectic expansion

$\Longrightarrow$ (1) $-N\theta : \mathbb{Q}\hat{\pi} \xrightarrow{\cong} N(\widehat{T}) = \mathrm{Der}_\omega(\widehat{T})$, $|x| \in \hat{\pi} \mapsto -N\theta(x)$, Lie代数の同型

(2)
$$\begin{tikzcd}
\mathbb{Q}\hat{\pi} \otimes \widehat{\mathbb{Q}\pi} \arrow[r, "\sigma"] \arrow[d, "(-N\theta)\otimes\theta"', "\wr\|"] \arrow[dr, phantom, "\circlearrowleft"] & \widehat{\mathbb{Q}\pi} \arrow[d, "\theta"', "\wr\|"] \\
\mathrm{Der}_\omega(\widehat{T}) \otimes \widehat{T} \arrow[r, "\text{derivation}"] & \widehat{T}
\end{tikzcd}$$

一般の$\Sigma_{g,n+1}$への一般化：Massuyeau-Turaev, Kuno-K.

[MT] homotopy交叉形式のテンソル表示

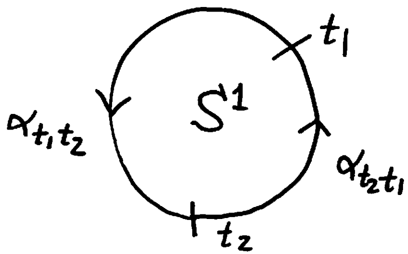
[KK] 完備Hopf代数封の交叉理論 ⇝ $t_C = \exp\left(\frac{1}{2}(\log C)^2\right)$

幾何的Johnson準同型

§3. Turaev 余括弧積のテンソル表示について.

$S = \Sigma_{g,n+1}$ (一般), $\ast \in \partial S$

$\mathbb{Z}\hat{\pi}' := \mathbb{Z}\hat{\pi}/\mathbb{Z}\mathbb{1}$, $|\ |' : \mathbb{Z}\pi_1(S) \xrightarrow{|\ |} \mathbb{Z}\hat{\pi} \xrightarrow{\text{quotient}} \mathbb{Z}\hat{\pi}/\mathbb{Z}\mathbb{1}$.

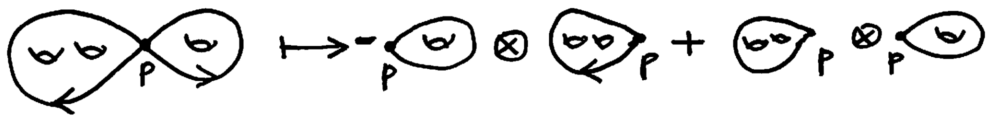


<u>Turaev cobracket</u>

$\alpha \in \hat{\pi}$ in general position

$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 \ ; \ t_1 \neq t_2, \ \alpha(t_1) = \alpha(t_2)\}$

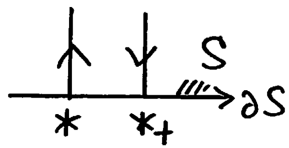
$$\delta(\alpha) \overset{\text{def}}{=} \sum_{(t_1,t_2) \in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \, |\alpha_{t_1 t_2}|' \otimes |\alpha_{t_2 t_1}|' \in \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$$



<u>Turaev</u> (1) δ : well-defined

(2) $(\mathbb{Z}\hat{\pi}', [\ ,\ ], \delta)$: Lie bialgebra (in the sense of Drinfel'd)

$\overset{\text{completion}}{\Longrightarrow}$ $(\widehat{\mathbb{Q}\hat{\pi}}, [\ ,\ ], \delta)$: Lie bialgebra (in the sense of Drinfel'd)

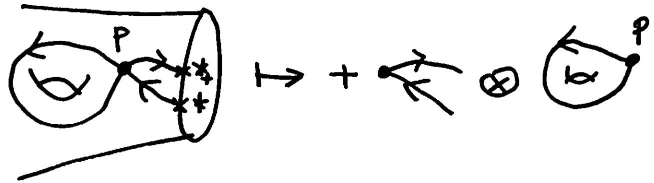


$\gamma \in \pi = \pi_1(S, \ast) \cong \pi_1(S, \ast, \ast_+)$ in general position

$\Gamma_\gamma := \{\text{double points of } \gamma\} \not\ni \ast, \ast_+$

$\forall p$. $0 < t_1^p \lneq t_2^p < 1$ s.t. $\gamma^{-1}(p) = \{t_1^p, t_2^p\}$

$$\mu(\gamma) \overset{\text{def}}{=} -\sum_{p\in\Gamma_\gamma} \varepsilon(\dot\gamma(t_1^p), \dot\gamma(t_2^p))\, \gamma_{0t_1^p}\gamma_{t_2^p 1} \otimes |\gamma_{t_1^p t_2^p}|' \in \mathbb{Z}\pi \otimes \mathbb{Z}\hat\pi'$$



Kuno-K. (inspired by Turaev)

(1) μ: well-defined

(2) $\mathbb{Z}\pi$: $\mathbb{Z}\hat\pi'$-bimodule through μ $\overset{\text{completion}}{\Longrightarrow}$ $\widehat{\mathbb{Q}\pi}$: $\widehat{\mathbb{Q}\hat\pi}$-bimodule through μ

$S = \Sigma_{g,1}$ の場合 $\theta : \pi \to \hat{T}$ symplectic expansion, $-N\theta : \widehat{\mathbb{Q}\hat\pi} \overset{\cong}{\Rightarrow} N(\hat T) = \mathrm{Der}_\omega(\hat T)$ Lie algebra isomorphism

$$\delta^\theta := ((-N\theta)\otimes(-N\theta))\circ\delta\circ(-N\theta)^{-1} : N(\hat T) \to N(\hat T)\hat\otimes N(\hat T)$$

$$\delta^\theta = \sum_{p=-\infty}^{+\infty} \delta^\theta_{(p)}, \quad \delta_{(p)}(N(H^{\otimes m})) \subset \bigoplus_{k+l=m} N(H^{\otimes k})\hat\otimes N(H^{\otimes l}), \ (\forall m \geq 0)$$

Laurent expansion of δ^θ

Theorem (Massuyeau-Turaev, Kuno-K.)

$$\delta^\theta_{(p)} = \begin{cases} 0, & \text{if } p \leqq -3,\ p=-1. \\ \text{Schedler's cobracket } \delta^{alg} & \text{if } p=-2 \end{cases}$$

$$\delta^{alg}(N(X_1\cdots X_m)) = \sum_{j-i\geqq 2}(X_i\cdot X_j)\{N(X_{i+1}\cdots X_{j-1})\hat\otimes N(X_{j+1}\cdots X_m X_1\cdots X_{i-1}) - N(X_{j+1}\cdots X_m X_1\cdots X_{i-1})\hat\otimes N(X_{i+1}\cdots X_{j-1})\}$$

$N(1)=0$ $\qquad (X_k \in H)$

Proposition (Kuno-K.) δ^{alg} includes the Morita traces (except the 1^{st} term) ___

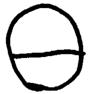
Enomoto–Satoh trace ES ---- refinement of the Morita traces

$$ES(N(X_1 \cdots X_m)) := \sum_{i=1}^{m} (X_i \cdot X_{i+1}) N(X_{i+2} \cdots X_m X_1 \cdots X_{i-1})$$

$j-i=1$

Proposition (Enomoto) δ^{alg} does <u>not</u> include the Enomoto–Satoh trace. —

Theorem (K.). A <u>regular homotopy</u> version of the Turaev cobracket includes the ES trace. —

-  $\underset{reg}{\not\simeq}$  not regular homotopic
- Furuta's construction of the 1^{st} Morita = the 1^{st} ES trace.

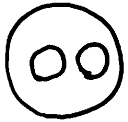
Question 1 $\overset{?}{\exists}\,\theta$: symplectic expansion s.t. $\delta^{\theta} \overset{?}{=} \delta^{alg}$. —

Observation (Kuno)

(1) δ^{θ} depends on the choice of a symplectic expansion θ

(2) For $\Sigma_{1,1} =$ , there exists such an expansion <u>modulo</u> degree $\geqq 11$.

$S = \Sigma_{0,3}$ (a pair of pants) の場合



- a central extension of the Grothendieck-Teichmüller Lie algebra $\subset \mathbb{Q}\hat{\pi}(\Sigma_{0,3})$
-

$\Sigma_{g,1}$	symplectic expansion	the Enomoto-Satoh trace ES
$\Sigma_{0,n+1}$ $n \geq 2$	special expansion	the divergence cocycle in the Kashiwara-Vergne problem div

regular homotopy version of the Turaev cobracket $[\kappa]$

Kashiwara-Vergne problem (in a formulation by Alekseev-Torossian)

Find a special expansion for $\Sigma_{0,3}$ compatible with div

Theorem (Alekseev-Meinrenken)

$\exists$ a solution to the Kashiwara-Vergne problem

Question 2 θ : a solution of the Kashiwara-Vergne problem

$\Longrightarrow \quad \delta^{\theta} \overset{?}{=} \delta^{\theta}_{[-1]}$

If Yes, Question 1 = "positive genus Kashiwara-Vergne problem"

If No, $\exists$ a new geometric constraint for the Johnson image

$\exists$ a new filtration on the Grothendieck-Teichmüller Lie algebra.