東工大複素解析セミナー 2014年11月20日 15:15-16:45 東京工業大学本館2階 H224B 教室

「Goldman-Turaev Lie 双代数のテンソル表示について」

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§1. 自由群の群環のテンソル表示 (古典的)

→ Johnson 準同型の構成

§2. Goldman Lie 代数のテンソル表示

→ Dehn twist の表示公式, Johnson 準同型の幾何的再構成

§3. Turaev 余括弧積のテンソル表示について

→ Johnson 準同型像の外からの評価, 柏原 Vergne 問題の発散 cocycle

§1. 自由群の群環のテンソル表示.

π : free group of finite rank

$\mathbb{Q}\pi := \{ \sum_{x\in\pi} a_x x \; ; \; a_x \in \mathbb{Q}, \; a_x = 0$ for all but finitely many $x \in \pi \}$

rational group ring

$H := (\pi/[\pi,\pi]) \otimes_{\mathbb{Z}} \mathbb{Q} = H_1(\pi;\mathbb{Q})$

$x \in \pi \longmapsto [x] := (x \bmod [\pi,\pi]) \otimes 1$

$\hat{T} = \hat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$. completed tensor algebra.

topology coming from the degree filtration $\hat{T}_{\geq p} := \prod_{m\geq p} H^{\otimes m}$, $p \geq 1$

$1 + \hat{T}_{\geq 1} < \hat{T}$ multiplicative group

<u>Definition</u> $\theta : \pi \to \hat{T}$ (generalized) Magnus expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\forall x, \forall y \in \pi$. $\theta(xy) = \theta(x)\theta(y)$

2) $\forall x \in \pi$. $\theta(x) \equiv 1 + [x] \mod \hat{T}_{\geq 2}$

$\Rightarrow$ $\theta : \mathbb{Q}\pi \to \hat{T}$, $\sum a_x x \mapsto \sum a_x \theta(x)$, algebra homomorphism

$\widehat{\mathbb{Q}\pi} := \varprojlim_{p\to\infty} \mathbb{Q}\pi/(I\pi)^p$ completed group ring

where $I\pi := \mathrm{Ker}(\varepsilon : \mathbb{Q}\pi \to \mathbb{Q})$ augmentation ideal
$\qquad\qquad \sum a_x x \mapsto \sum a_x$

$\Rightarrow$ $\theta : \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \widehat{T}$ algebra isomorphism

$T^\theta : \mathrm{Aut}(\pi) \longrightarrow \mathrm{Aut}(\widehat{T}) := \{ U : \widehat{T} \to \widehat{T} \text{ ; topological algebra automorphism} \}$

$\varphi \longmapsto T^\theta(\varphi) := \theta \circ \varphi \circ \theta^{-1}$ group embedding

total Johnson map.

$$\mathrm{Aut}(\widehat{T}) \cong \mathrm{Hom}(H, \widehat{T}_{\geq 1})_0 := \{ u \in \mathrm{Hom}(H, \widehat{T}_{\geq 1}) \ ; \ |u| := (H \xrightarrow{u} \widehat{T}_{\geq 1} \twoheadrightarrow \widehat{T}_{\geq 1}/\widehat{T}_{\geq 2} = H) \in GL(H) \}$$

$$U \longmapsto U|_H$$

$$\begin{tikzcd}
\mathrm{Aut}(\widehat{T}) \arrow[r, phantom, "\cong"] & \mathrm{Hom}(H, \widehat{T}_{\geq 1})_0 \arrow[d, phantom, "\cong"] & u \arrow[d, mapsto] \\
& \mathrm{Hom}(H, \widehat{T}_{\geq 2}) \times GL(H) & (u \circ |u|^{-1}, |u|) \\
\mathrm{Aut}(\pi) \arrow[uu] \arrow[ur] & \varphi \arrow[r, mapsto] & (\tau^\theta(\varphi), |\varphi|)
\end{tikzcd}$$

$$\tau^\theta(\varphi) = \sum_{k=1}^{\infty} \tau_k^\theta(\varphi), \quad \tau_k^{\varphi}(\varphi) \in \mathrm{Hom}(H, H^{\otimes(k+1)}), \ k \geq 1$$

$\tau_k^\theta : \mathrm{Aut}(\pi) \longrightarrow \mathrm{Hom}(H, H^{\otimes(k+1)})$

an extension of the k^{th} Johnson homomorphism

(a generalization of Kitano's constuction)

$\tau_1^\theta \in Z^1(\mathrm{Aut}(\pi) ; \mathrm{Hom}(H, H^{\otimes 2}))$ 1-cocycle

$\rightsquigarrow$ all the (twisted) Morita–Mumford classes on the mapping class group (Morita + Kawazumi)

$IA(\hat{T}) := \mathrm{Ker}(|\ | : \mathrm{Aut}(\hat{T}) \to GL(H))$

$\mathrm{Der}^+(\hat{T}) := \left\{ D : \hat{T} \to \hat{T} \ \vdots \ \text{continuous linear map.} \ \begin{array}{l} \forall u, \forall v \in \hat{T} \quad D(uv) = (Du)v + u(Dv) \\ \qquad\qquad \text{derivation} \\ D(H) < \hat{T}_{\geq 2} \end{array} \right\}$

Lie algebra

$$\mathrm{Der}^+(\hat{T}) \underset{\log}{\overset{\exp}{\rightleftarrows}} IA(\hat{T}) \quad (\cong)$$

$$D \longmapsto \exp(D) := \sum_{m=0}^{\infty} \frac{1}{m!} D^m$$

$$\log U := \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (U-1)^m \longleftarrow U$$

$IA(\pi) := \mathrm{Ker}(\mathrm{Aut}(\pi) \to GL(H))$, $T^\theta(IA(\pi)) < IA(\hat{T})$

$\tau^{\theta, M} := \log \circ T^\theta : IA(\pi) \to IA(\hat{T}) \longrightarrow \mathrm{Der}^+(\hat{T})$

Massuyeau's total Johnson map

<u>Hopf algebra structure</u>

$\Delta : \mathbb{Q}\pi \to \mathbb{Q}\pi \otimes \mathbb{Q}\pi$. coproduct. $x \in \pi \mapsto \Delta x := x \otimes x$, algebra homom

$\mathbb{Q}\pi$: Hopf algebra

$\overset{\Longrightarrow}{\text{induces}}$ $\Delta : \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi} \hat{\otimes} \widehat{\mathbb{Q}\pi}$ coproduct

$\widehat{\mathbb{Q}\pi}$: complete Hopf algebra

$\Delta : \hat{T} \to \hat{T}\hat{\otimes}\hat{T}$ coproduct, $X \in H \mapsto \Delta(X) := X\hat{\otimes}1 + 1\hat{\otimes}X$, continuous algebra homom

$\hat{T}$: complete Hopf algebra.

$\hat{\mathcal{L}} := \mathrm{Lie}(\hat{T}) = \{u \in \hat{T} : \Delta u = u\hat{\otimes}1 + 1\hat{\otimes}u\}$ Lie-like elements

complete free Lie algebra over H

$\exp(\hat{\mathcal{L}}) = \{g \in \hat{T} ;\ g \neq 0,\ \Delta g = g\hat{\otimes}g\}$ group-like elements

<u>Definition</u> $\theta : \pi \to \hat{T}$ group-like expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\theta : \pi \to \hat{T}$ (generalized) Magnus expansion

2) $\forall x \in \pi,\ \Delta\theta(x) = \theta(x)\hat{\otimes}\theta(x)$, i.e., $\theta(\pi) \subset \exp(\hat{\mathcal{L}})$

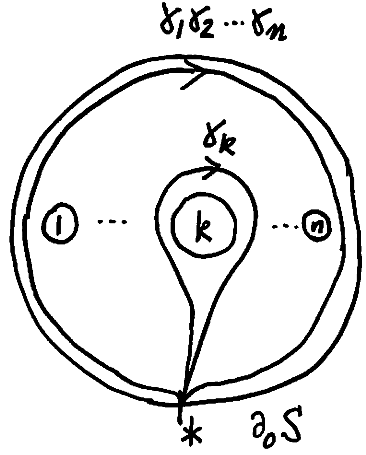
$\Longrightarrow$ $\theta : \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \hat{T}$ isomorphism of complete Hopf algebras

$\tau^{\theta,M}(IA(\pi)) \subset \mathrm{Der}^+(\hat{\mathcal{L}}) = \{D \in \mathrm{Der}^+(\hat{T}) : (D\hat{\otimes}1 + 1\hat{\otimes}D)\circ\Delta = \Delta\circ D\}$

<u>Special expansions</u> and <u>symplectic expansions</u>

$g, n \geq 0$

$\Sigma_{g,n+1} :=$



<u>special expansion</u> $\Sigma_{0,n+1}$

$\pi = \pi_1(\Sigma_{0,n+1}, *)$, $* \in \partial_0 S$

$= \langle \gamma_k ; 1 \le k \le n \rangle$ free group of rank n

$x_k := [\gamma_k] \in H = H_1(\Sigma_{0,n+1} ; \mathbb{Q})$, $1 \le k \le n$

$x_0 := -\sum_{k=1}^{n} x_k \in H$

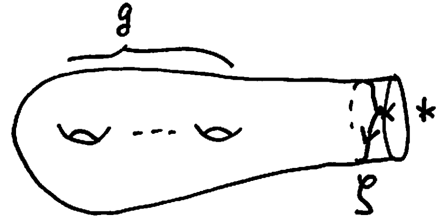
<u>Definition</u> $\theta : \pi \longrightarrow \hat{T}$ special expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\theta : \pi \longrightarrow \hat{T}$ group-like expansion

2) (tangential condition) $1 \le {}^{\forall} k \le n$, ${}^{\exists} g_k \in \exp(\hat{\mathcal{L}})$ $\theta(\gamma_k) = g_k^{-1} e^{x_k} g_k$

3) (special condition) $\theta(\gamma_1\gamma_2 \cdots \gamma_n) = e^{\sum_{k=1}^{n} x_k} \left(= e^{-x_0}\right)$

ex) (Habegger–Masbaum) Kontsevich integral



<u>symplectic expansion</u> $\Sigma_{g,1}$

$\pi = \pi_1(\Sigma_{g,1}, *)$, $* \in \partial S$, free of rank $2g$

$H = H_1(\Sigma_{g,1} ; \mathbb{Q})$

$\cdot : H \times H \longrightarrow \mathbb{Q}$, $(X, Y) \longmapsto X \cdot Y$, intersection number (non-degenerate)

$H \cong H^*$, $X \longmapsto (Y \mapsto Y \cdot X)$, Poincaré duality

$\omega \in H \otimes H \overset{\text{P.d.}}{\cong} H^* \otimes H = \mathrm{Hom}(H, H)$ symplectic form

$\omega \leftarrow\!\!\shortmid -1_H \in \mathrm{Hom}(H,H)$

$\omega = \sum_{i=1}^{g} A_i B_i - B_i A_i \in H^{\otimes 2} \subset \hat{T}$, $\{A_i, B_i\} \subset H$ symplectic basis

Definition (Massuyeau) $\theta : \pi \to \hat{T}$ symplectic expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\theta : \pi \to \hat{T}$ group-like expansion

2) (symplectic condition) $\theta(\zeta) = e^{\omega} \in \exp(\hat{\mathcal{L}})$

ex) 1) (K.) harmonic Magnus expansion $/\mathbb{R}$ (後述)

2) (Massuyeau) the LMO functor

3) (Kuno) combinatorial construction coming from a free generator system of π

4) (Bene-K.-Kuno-Penner) fatgraph symplectic expansion

Observation (Massuyeau) $\theta : \pi \to \hat{T}$ symplectic expansion

$\Longrightarrow$ $\tau^{\theta, M}$ (Torelli group) $\subset \mathrm{Der}^{+}_{\omega}(\hat{\mathcal{L}}) \cong \mathrm{Ker}([\,,\,] : H \otimes \hat{\mathcal{L}}_{\geq 2} \to \hat{\mathcal{L}})$

Morita's Lie algebra = the positive part of Kontsevich's "Lie"

(an extension of Morita's result)

harmonic Magnus expansion

Generalities X : C^∞ manifold, A : $\mathbb{R}$-algebra

$\nabla = d + \omega : C^\infty(X:A) \to \Omega^1(X:A)$ connection on $X \times A \to X$

$\omega \in \Omega^1(X:A)$ connection 1-form ($\neq$ symplectic form)

$\ell : [0,1] \to X$: C^∞ path

$\{\ell(0)\} \times A \to \{\ell(1)\} \times A$, $(\ell(0), a) \mapsto (\ell(1), T_\ell a)$, holonomy along ℓ

$$T_\ell a = \Big(1 + \sum_{m=1}^{\infty} \int_\ell \overbrace{\omega \cdots \omega}^{m}\Big) a$$

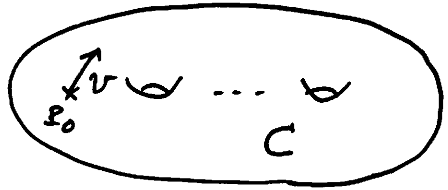
$$\int_\ell \overbrace{\omega \cdots \omega}^{m} = \int_0^1 \overbrace{(\ell^*\omega) \cdots (\ell^*\omega)}^{m} \quad \text{Chen's iterated integral}$$

$$\int_0^1 (f_1(t)dt) \cdots (f_m(t)dt) := \int \cdots \int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_m \geq 0} f_1(t_1) \cdots f_m(t_m)\, dt_1 \cdots dt_m$$

∇ : flat $\Leftrightarrow$ [integrable condition: $d\omega = \omega \wedge \omega \in \Omega^2(X:A)$]

$\overset{\text{homotopy invariance}}{\Longrightarrow}$ $\pi_1(X) \to \begin{pmatrix}\text{multiplicative}\\ \text{group of } A\end{pmatrix}$ group homomorphism

$$[\ell] \mapsto T_\ell = 1 + \sum_{m=1}^{\infty} \int_\ell \overbrace{\omega \cdots \omega}^{m}$$



C : (closed) compact Riemann surface of genus g (≥ 1)

$p_0 \in C$, $v \in T_{p_0}C \setminus \{0\}$, $H_{\mathbb{R}} = H \underset{\mathbb{Q}}{\otimes} \mathbb{R} = H_1(C;\mathbb{R})$

$$\pi_1(C \setminus \{p_0\}, v) := \left\{ \ell : [0,1] \to C \; ; \; \begin{array}{l} \text{piecewise } C^\infty,\ \ell(]0,1[) \subset C \setminus \{p_0\} \\ \ell(0) = \ell(1) = p_0,\ \dot{\ell}(0) = -\dot{\ell}(1) = v \end{array} \right\} \Big/ \text{homotopy}$$

free group of rank $2g$

<u>Flat connection on</u> $(C \setminus \{p_0\}) \times \widehat{T}(H_{\mathbb{R}})$?

$$\omega = \sum_{m=1}^{\infty} \omega_{(m)}, \quad \omega_{(m)} \in \Omega^1(C \setminus \{p_0\} ; H_{\mathbb{R}}^{\otimes m})$$

$$\omega_{(1)} \in \Omega^1(C ; H_{\mathbb{R}}), \quad {}^\forall [\ell] \in \pi_1(C \setminus \{p_0\}, v), \quad \int_\ell \omega_{(1)} = [\ell] \in H_{\mathbb{R}} \quad (\because\ \theta(x) = 1 + [x] + \cdots)$$

$\Leftarrow$ complex structure of C ($\Rightarrow$ harmonic 1-forms)

$$\omega_{(1)} := \left(\begin{array}{l} \text{harmonic} \\ \text{representatives} \end{array} : H_{\mathbb{R}} \to \Omega^1(C) \right) \in H_{\mathbb{R}}^* \otimes \Omega^1(C) \overset{P.D.}{=} H_{\mathbb{R}} \otimes \Omega^1(C) = \Omega^1(C ; H_{\mathbb{R}})$$

$$\int_C \omega_{(1)} \wedge \omega_{(1)} = \langle 1_H \cup 1_H, [C] \rangle = \sum_{i=1}^{g} A_i B_i - B_i A_i \left(\overset{\text{symplectic form}}{=:} I \right) \neq 0 \in H_{\mathbb{R}}^{\otimes 2}$$

$\nexists\, \omega_{(2)} \in \Omega^2(C ; H_{\mathbb{R}}^{\otimes 2})$ $\quad d\omega_{(2)} = \omega_{(1)} \wedge \omega_{(1)}$ $\quad$ The integrable condition <u>fails</u>!

$\delta_0 : C^\infty(C) \to \mathbb{R}$ delta current at p_0, 2-current on C

$f \mapsto f(p_0)$ $\qquad \left(\int_C \delta_0 = 1 \right)$

$$\boxed{\underline{\text{modified integrable condition}} \qquad d\omega = \omega\wedge\omega - I\,\delta_0}$$

Solve the equations

$$\begin{cases} d\omega_{(2)} = \omega_{(1)}\wedge\omega_{(1)} - I\,\delta_0 \\ d\omega_{(m)} = \sum_{p+q=m} \omega_{(p)}\wedge\omega_{(q)}\ , \quad m\geq 3 \end{cases}$$

inductively

$\Uparrow$ Green operator $\quad A^p(C) := \{\mathbb{R}\text{-valued } p\text{-currents on } C\}$, $0\leq p\leq 2$.

$$0 \to \mathbb{R} \xrightarrow[\text{fct}]{\text{const}} A^0(C) \xrightarrow{d*d} A^2(C) \xrightarrow{\int_C} \mathbb{R} \to 0 \quad (\text{exact})$$

($*: A^1(C) \circlearrowleft$ Hodge-$*$, depends only on the conformal str. of C.)

$\exists!\ \Phi : A^2(C) \to A^0(C)/\mathbb{R}$ Green operator

s.t. $\forall \Omega \in A^2(C)$, $\quad d*d\,\Phi(\Omega) = \Omega - (\int_C \Omega)\,\delta_0$

$$\omega_{(m)} := *d\Phi\Big(\sum_{\substack{p+q=m\\ p,q\geq 1}} \omega_{(p)}\wedge\omega_{(q)}\Big) \in A^1(C)\otimes H_{\mathbb{R}}^{\otimes m}, \quad m\geq 2$$

- C^∞ on $C\setminus\{P_0\}$
- $\omega_{(2)} - \frac{1}{2\pi} I\,d\theta$: C^∞ near P_0 (where $z = re^{\sqrt{-1}\theta}$: complex coordinate of C centered at P_0)

- $^{\forall} m \geqq 3\ ^{\exists} C_m > 0$

  $$\left|\omega_{(m)}\left(\frac{d}{dz}\right)\right| = \left|\omega_{(m)}\left(\frac{d}{d\bar z}\right)\right| \leqq C_m \left|\log|z|\right|^{[(m-1)/2]} \quad \text{near } P_0$$

- $^{\forall} m \geqq 1.\quad \omega_{(m)} \in A^1(C) \otimes \left(\hat{\mathcal{L}}(H_{\mathbb{R}}) \cap H_{\mathbb{R}}^{\otimes m}\right)$

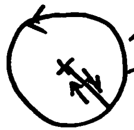
$$\omega := \sum_{m=1}^{\infty} \omega_{(m)} \in A^1(C ; \hat{\mathcal{L}}(H_{\mathbb{R}}))$$

$$\theta^{(C,P_0,v)} : [\ell] \in \pi_1(C \setminus \{P_0\}, v) \longmapsto \theta^{(C,P_0,v)}(\ell) := 1 + \sum_{m=1}^{\infty} \int_{\ell} \overbrace{\omega \cdots \omega}^{m} \in \hat{T}(H_1(C;\mathbb{R}))$$

group-like expansion

$\theta^{(C,P_0,v)}$: symplectic expansion

$\because$) $\varepsilon > 0$



ℓ_ε $\quad z^{-1}(\varepsilon e^{\pi\sqrt{-1}\theta}),\ 0 \leq \theta \leq 2\pi,$

$$\theta^{(C,P_0,v)}(\ell_\varepsilon) = \lim_{\varepsilon \downarrow 0} \theta^{(C,P_0,v)}(\ell_\varepsilon) \quad (\because \text{ homotopy invariance})$$

$$= \lim_{\varepsilon\downarrow 0}\left(1 + \sum_{n=1}^{\infty} \int_{\ell_\varepsilon} \underbrace{\left(\frac{1}{2\pi} I\, d\theta\right) \cdots \left(\frac{1}{2\pi} I\, d\theta\right)}_{n}\right)$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2\pi} I\right)^n \frac{1}{n!} (2\pi)^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} I^n = e^{I} \quad //$$

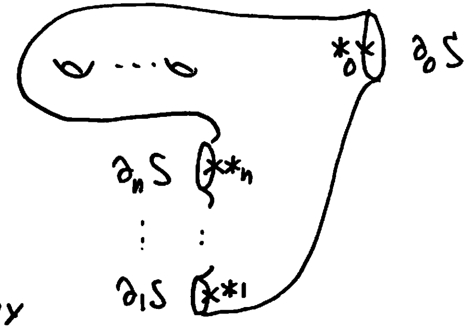
§2. Goldman Lie 代数のテンソル表示

S : compact connected oriented surface with $\partial S \neq \phi$

$\xRightarrow[\text{Theorem}]{\text{Classification}}$ $\exists g, \exists n \geq 0$, $S \cong \Sigma_{g,n+1}$

$\partial S = \coprod_{j=0}^{n} \partial_j S$, Choose $*_j \in \partial_j S$.

$E := \{*_j\}_{j=0}^{n} \subset \partial S$



$\mathcal{M}(S) := \{\varphi : S \to S$: ori. pres. diffeo. $\varphi|_{\partial S} = 1_{\partial S}\}$ / isotopy fixing ∂S pointwise

mapping class group

Dehn-Nielsen embedding

$$DN : \mathcal{M}(S) \xhookrightarrow[\text{injective}]{} \mathrm{Aut}(\Pi S|_E)$$

where $\Pi S|_E$: restriction of the fundamental groupoid ΠS to the objectset E

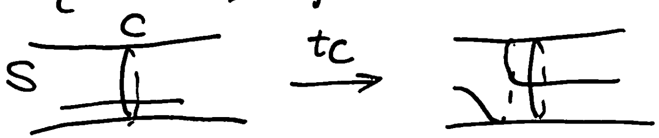
$$Ob(\Pi S|_E) = E$$

$$(\Pi S|_E)(*_a, *_b) = (\Pi S)(*_a, *_b) = \pi_1(S, *_a, *_b) = [([0,1], 0, 1), (S, *_a, *_b)]$$

$$(0 \leq a, b \leq n)$$

Dehn twist $C \subset S \setminus \partial S$ simple closed curve

$t_C \in \mathcal{M}(S)$ right handed Dehn twist



Picard-Lefschetz formula

$(t_C)_*(u) = u - (u \cdot [C])[C]$ $(\forall u \in H_1(S;\mathbb{Z}))$

Kuno

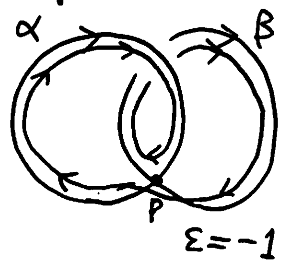
explicit formula for $\tau_1^\theta(t_C)$

$$t_C \underset{?}{=} \frac{1}{2}(\log C)^2$$

§ 2.1. Completed Goldman Lie algebra

$\hat{\pi}(S) := [S^1, S] = \pi_1(S)/_{\text{conj.}}$ the free homotopy set of free loops on S

$|\ | : \pi_1(S) \to \hat{\pi}(S)$ quotient map = forgetful map of a basepoint



<u>Goldman bracket</u> $\alpha, \beta \in \hat{\pi}(S)$ in general position.

$$[\alpha, \beta] \overset{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta)\, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}(S)$$

where $\varepsilon_p(\alpha, \beta) \in \{\pm 1\}$ local intersection number

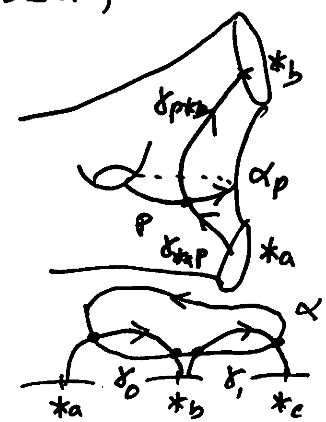
$\alpha_p, \beta_p \in \pi_1(S, p)$ based loops along α, β

<u>Goldman</u> (1) $[\ ,\ ]$: well-defined

(2) $(\mathbb{Z}\hat{\pi}(S), [\ ,\ ])$: Lie algebra

$\mathbb{Z}\hat{\pi}(S)$ <u>-module structure on</u> $\mathbb{Z}\Pi S|_E$, $\alpha \in \hat{\pi}(S)$, $\gamma \in \Pi S(*_a, *_b)$ in general position $(0 \leq a, b \leq n)$

$$\sigma(\alpha)(\gamma) \overset{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon_p(\alpha, \gamma)\, \gamma_{*_a p}\, \alpha_p\, \gamma_{p *_b} \in \mathbb{Z}\Pi S(*_a, *_b)$$



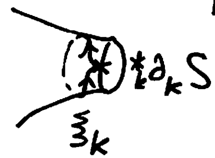
<u>Kuno-K,</u> (1) σ : well-defined

(2) $\sigma : \mathbb{Z}\hat{\pi}(S) \to \mathrm{Der}(\mathbb{Z}(\Pi S|_E))$ Lie algebra homomorphism

i.e., $\forall \alpha, \forall \beta \in \hat{\pi}(S)$, $\forall \gamma_0 \in \Pi S(*_a, *_b)$, $\forall \gamma_1 \in \Pi S(*_b, *_c)$

$$\sigma(\alpha)(\gamma_0 \gamma_1) = (\sigma(\alpha)\gamma_0)\gamma_1 + \gamma_0(\sigma(\alpha)\gamma_1)$$

$$\sigma(\alpha)(\sigma(\beta)(\gamma_0)) - \sigma(\beta)(\sigma(\alpha)(\gamma_0)) = \sigma([\alpha, \beta])(\gamma_0)$$



$\xi_k \in \Pi S(*_k, *_k) = \pi_1(S, *_k)$, $0 \le k \le n$, positive boundary loop

$\mathrm{Der}_\partial(\mathbb{Z}\Pi S|_E) := \{ D \in \mathrm{Der}(\mathbb{Z}\Pi S|_E)\ ;\ 0 \le {}^\forall k \le n,\ D(\xi_k) = 0 \}$

$\cup$

$\sigma(\mathbb{Z}\hat{\pi}(S))$ ($\because$ ${}^\forall \alpha \in \hat{\pi}(S)$ we may choose $\alpha \subset S \setminus \partial S$)

$\sigma(\mathbb{1}) = 0$, $\mathbb{1}$: trivial loop $\in \mathrm{Center}(\mathbb{Z}\hat{\pi}(S))$

$\sigma : \mathbb{Z}\hat{\pi}'(S) := \mathbb{Z}\hat{\pi}(S)/\mathbb{Z}\mathbb{1} \to \mathrm{Der}_\partial(\mathbb{Z}\Pi S|_E)$ Lie algebra homomorphism

$\otimes \mathbb{Q} \Downarrow$

$\sigma : \mathbb{Q}\hat{\pi}'(S) := \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\mathbb{1} \to \mathrm{Der}_\partial(\mathbb{Q}\Pi S|_E)$ Lie algebra homomorphism

injective but <u>not</u> surjective

<u>Completion</u> $q \in S$, $\gamma_k \in \Pi S(*_k, q)$, $0 \le k \le n$.

$\Pi S(*_a, *_b) = \gamma_a \pi_1(S, q) \gamma_b^{-1}$, $0 \le a, b \le n$

$\mathbb{Q}\Pi S(*_a, *_b) = \gamma_a(\mathbb{Q}\pi_1(S, q))\gamma_b^{-1}$

$\widehat{\mathbb{Q}\Pi S}(*_a, *_b) := \varprojlim_{p \to \infty} \mathbb{Q}\Pi S(*_a, *_b) / \gamma_a (I\pi_1(S, q))^p \gamma_b^{-1}$ (indep of the choice of q and γ_k)

"completed groupoid ring"

$\widehat{\mathbb{Q}\Pi S}|_E$: $\mathbb{Q}$-linear small category

<u>object</u> $*_k \in E$, $0 \le k \le n$

<u>morphism</u> $\widehat{\mathbb{Q}\Pi S}(*_a, *_b)$, $0 \le a, b \le n$

$|\ | : \mathbb{Q}\pi_1(S,*) \to \mathbb{Q}\hat{\pi}(S)$ forgetful map of the basepoint $* \in S$, $1 \in \pi_1(S,*)$ identity

$$\widehat{\mathbb{Q}\hat{\pi}}(S) := \varprojlim_{p\to\infty} \mathbb{Q}\hat{\pi}(S) / |\mathbb{Q}1 + (I\pi_1(S,*))^p| , \quad \left(\begin{array}{c}\text{indep. of the choice} \\ \text{of } *\end{array}\right)$$

the completed Goldman (-Turaev) Lie (bi)algebra

- $\widehat{\mathbb{Q}\hat{\pi}}(S)$: Lie algebra
- σ extends to $\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \to \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E)$ Lie algebra homomorphism

Theorem 1 (Kuno-K.) (Infinitesimal Dehn-Nielsen Theorem)

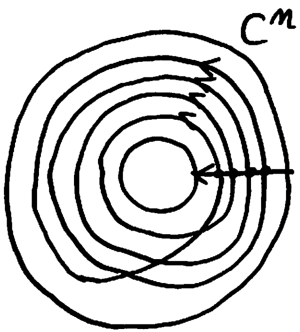
$$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E) \quad \text{Lie algebra isomorphism}$$

($\Leftarrow$ tensorial description of $\widehat{\mathbb{Q}\hat{\pi}}(S)$ and σ)

$DN : \mathcal{M}(S) \to \mathrm{Aut}(\widehat{\mathbb{Q}\Pi S}|_E) = \{\text{topological automorphisms}\}$

Dehn twist $S = \Sigma_{0,2}$ annulus, $\gamma \in \Pi S(*_0, *_1)$, $\alpha \in \pi_1(S, *_1)$

$C = |\alpha|$.



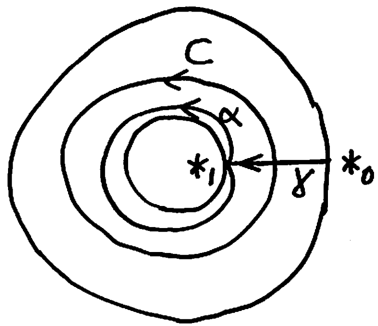
$n=4$

$$\begin{cases} t_C(\gamma) = \gamma\alpha \\ t_C(\alpha) = \alpha \end{cases} \quad \log t_C \in \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E)$$

$$\begin{cases} (\log t_C)(\gamma) = \gamma \log \alpha \\ (\log t_C)(\alpha) = 0 \end{cases}$$

$n \geq 0$

$$\begin{cases} \sigma(C^n)(\gamma) = +n\gamma\alpha^n \\ \sigma(C^n)(\alpha) = 0 \end{cases}$$



$f(t) \in \mathbb{Q}[[t-1]]$, $f(C) \in \mathbb{Q}\hat{\pi}(S)$

$$\begin{cases} \sigma(f(C))(\gamma) = \gamma\alpha f'(\alpha) \\ \sigma(f(C))(\alpha) = 0 \end{cases}$$

$t f'(t) = \log t$, $\quad f(t) = \int_1 \frac{1}{t} \log t \, dt = \frac{1}{2}(\log t)^2$

<u>Theorem 2</u> ($\Sigma_{g,1}$/ Kuno–K., <u>general</u>/ Kuno–K., Massuyeau–Turaev)

S : compact connected oriented surface with $\partial S \neq \phi$

$C = |x| \subset S \setminus \partial S$ simple closed curve, $x \in \pi_1(S)$

$\frac{1}{2}(\log C)^2 := \left| \frac{1}{2}(\log x)^2 \right| \in \widehat{\mathbb{Q}\hat{\pi}}(S)$

$\Rightarrow \quad DN(t_C) = \exp \sigma\left(\frac{1}{2}(\log C)^2\right) \in \mathrm{Aut}(\widehat{\mathbb{Q}\pi S}|_E)$

<u>quantization of Theorem 2</u> (S. Tsuji)

$$(t_C)_* = \exp\left(\frac{(\cosh^{-1}(-C/2))^2}{4\log(-A)}\right)$$ on the completed Kauffman skein module

<u>mapping class group</u>

$DN : \mathcal{M}(S) \hookrightarrow \mathrm{Aut}(\mathbb{Q}\widehat{\Pi S}|_E)$

$\mathcal{M}(S)^\circ := \{ \varphi \in \mathcal{M}(S) ;\ {}^\exists \log DN(\varphi) := \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (DN(\varphi) - 1)^m \}$ sub<u>set</u>

- ${}^\forall$ Dehn twist $t_C \in \mathcal{M}(S)^\circ$
- the largest Torelli group (in the sense of Putman) $\subset \mathcal{M}(S)^\circ$
- ${}^\forall \varphi \in \mathcal{M}(S)^\circ$, $\log DN(\varphi) \in \mathrm{Der}(\mathbb{Q}\widehat{\Pi S}|_E) \underset{\sigma}{\cong} \mathbb{Q}\hat{\hat{\pi}}(S)$

$$\Delta \circ (\log DN(\varphi)) = ((\log DN(\varphi)) \hat{\otimes} 1 + 1 \hat{\otimes} (\log DN(\varphi))) \circ \Delta$$

$\tau := \sigma^{-1} \circ \log \circ DN : \mathcal{M}(S)^\circ \longrightarrow \mathbb{Q}\hat{\hat{\pi}}(S)$ geometric Johnson homomorphism

- $\tau(t_C) = \frac{1}{2}(\log C)^2$ (Thm 2)
- $S = \Sigma_{g,1}$, $\tau|_{\text{Torelli gp}} \cong$ Massuyeau's total Johnson map

§ 2.2. Tensorial description of the completed Goldman Lie algebra

$H = H_1(S;\mathbb{Q})$

$N : \hat{T}(H) \to \hat{T}(H)$ cyclic symmetrizer (cyclicizer)

$N|_{H^{\otimes 0}} := 0$

$N(X_1X_2 \cdots X_m) := \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1} \quad (X_j \in H)$

- $N(uv) = N(vu) \quad (\forall u, \forall v \in \hat{T})$
- $N(\hat{T}) = N(\hat{T}_{\geq 1})$

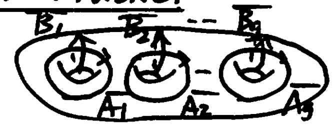
Observation (Kuno-K.) $\forall \theta : \pi_1(S) \to \hat{T} = \hat{T}(H_1(S;\mathbb{Q}))$ Magnus expansion

$N\theta : \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} N(\hat{T}_{\geq 1}), \quad |x| \in \hat{\pi}(S) \mapsto N(\theta(x))$

isomorphism of filtered $\mathbb{Q}$-vector spaces

Goldman bracket ?

Additional datum for describing the Goldman bracket ---- $s \in \mathrm{Sect}\, i_*$

$\overline{S} := S \cup_{\partial S} ((n+1)\text{-discs}) \cong \Sigma_g =$ 

$i : S \hookrightarrow \overline{S}$ inclusion map

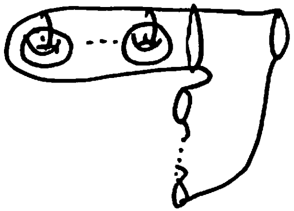
$H_2(\overline{S}, S;\mathbb{Q}) \xrightarrow{\partial_*} H_1(S;\mathbb{Q}) \xrightarrow{i_*} H_1(\overline{S};\mathbb{Q}) \to 0$ (exact)

$\mathrm{Sect}\, i_* := \{ \text{section of } i_* : H_1(S;\mathbb{Q}) \to H_1(\overline{S};\mathbb{Q}) \}$

<u>examples</u>

(0) $\# \mathrm{Sect}\, i_* = 1 \iff S = \Sigma_{g,1}$ or $\Sigma_{0,n+1}$

(1) $/\mathbb{Z}$



$\Rightarrow s_0 \in \mathrm{Sect}\, i_*$

(2) $/\mathbb{R}$ Regard $\mathrm{Int}\, S$ as a punctured Riemann surface $C \setminus \{P_0, P_1, \cdots, P_n\}$
$P_a \neq P_b$ if $a \neq b$

<u>normalized Abelian differential of the third kind</u>

C : compact Riemann surface, $P \neq Q \in C$

$\exists !\ \omega = \omega(C; P, Q)$: meromorphic 1-form on C

s.t. (i) holomorphic on $C \setminus \{P, Q\}$

(ii) $\mathrm{ord}_P\, \omega = \mathrm{ord}_Q\, \omega = -1$

(iii) $\mathrm{Res}_P\, \omega = -\mathrm{Res}_Q\, \omega = \dfrac{1}{2\pi\sqrt{-1}}$

(iv) $\forall \varphi$: holo. 1-form on C $\quad \int_C \omega \wedge \overline{\varphi} = 0$

$n \geq 1$. $\mathrm{Int}\, S = C \setminus \{P_0, P_1, \cdots, P_n\}$, $P_a \neq P_b$ if $a \neq b$.

$$\bigcap_{k=1}^{n} \mathrm{Ker}\left(\int \mathrm{Re}\, \omega(C, P_0, P_k) : H_1(S) = H_1(\mathrm{Int}\, S) \to \mathbb{R} \right)$$

$\underset{\text{induces}}{\Longrightarrow}$ canonical section $/\mathbb{R}$ $\quad s_{(C, P_0, \cdots, P_n)} \in \mathrm{Sect}\, i_*$

<u>Remark</u> $n=1$

$\mathbb{C}_g \to \mathbb{M}_g$ universal Riemann surface over the moduli of compact Riemann surfaces of genus g, $\mathbb{M}_g$

$k_0 :=$ (the 1^{st} g.c. variation of $s_{(C,\ell_0,\ell_1)}$) extends to $\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g$

$\int_{fiber} k_0^3 =$ (the 1^{st} g.c. variation of pointed harmonic volumes)

(a twisted 1-form on $\mathbb{C}_g$)

$\Longrightarrow$ differential forms representing the Morita-Mumford classes on $\mathbb{M}_g$

$e_n = (-1)^{n+1}\kappa_n$

$\Longrightarrow$ different two (1, 1)-forms on $\mathbb{M}_g$ representing $e_1 = \kappa_1$

$\underset{(K.)}{\Longrightarrow}$ a real-valued function $a_g \doteq \varphi : \mathbb{M}_g \to \mathbb{R}$

" Kawazumi - Zhang invariant " (Robin de Jong)

<u>Lie bracket on</u> $N(\hat{T}_{\geq 1}) = N(\hat{T}(H_1(S:\mathbb{Q}))_{\geq 1})$

$\mathrm{Ker}(i_*: H_1(S) \twoheadrightarrow H_1(\bar{S})) = \sum_{k=0}^{n} \mathbb{Q} C_k = \bigoplus_{k=1}^{n} \mathbb{Q} C_k, \quad C_k := [\zeta_k] \overset{= x_k}{\in} H_1(S)$ boundary loop

Fix a section $s \in \mathrm{Sect}\, i_*$ and $\{\bar{A}_i, \bar{B}_i\}_{i=1}^{g} \subset H_1(\bar{S}:\mathbb{Q})$ symplectic basis

$A_i^s := s(\bar{A}_i), \quad B_i^s := s(\bar{B}_i) \in H_1(S:\mathbb{Q})$

$u = \sum_{i=1}^{g} A_i^s u_i' + \sum_{i=1}^{g} B_i^s u_i'' + \sum_{k=1}^{n} C_k u_k^0, \quad v = \sum_{i=1}^{g} A_i^s v_i' + \sum_{i=1}^{g} B_i^s v_i'' + \sum_{k=1}^{n} C_k v_k^0 \in N(\hat{T}_{\geq 1}) \subset H \otimes \hat{T}$

$[u, v]_s \overset{\mathrm{def}}{=} N\left(\sum_{i=1}^{g} u_i' v_i'' - u_i'' v_i' + \sum_{k=1}^{n} C_k (u_k^0 v_k^0 - v_k^0 u_k^0) \right) \in N(\hat{T}_{\geq 1})$

$\Longrightarrow$ Lie algebra $N(\hat{T}_{\geq 1})_s := (N(\hat{T}_{\geq 1}), [\ ,\ ]_s)$

$N(\hat{T}_{\geq 1})_s$-<u>module structure on</u> $\hat{T}_E$

$\hat{T}_E$: $\mathbb{Q}$-linear small category

<u>object</u> $*_k \in E, \quad 0 \leq k \leq n$

<u>morphism</u> $\hat{T}_E(*_a, *_b) := \hat{T}, \quad 0 \leq a, b \leq n$

$$\mathrm{Der}_\partial(\hat{T}_E) := \left\{ D : \begin{array}{l}\text{continuous}\\ \text{derivation of } \hat{T}_E\end{array} ; \begin{array}{l} D(C_k) = 0 \text{ for } C_k \in \hat{T}_E(*_k, *_k),\ \forall\, \underset{\sim}{1} \leq k \leq n \\ D(C_0 - \sum_{i=1}^{g} (A_i^s B_i^s - B_i^s A_i^s)) = 0 \text{ at } *_0 \end{array} \right\}$$

$\sigma_s : N(\hat{T}_{\geq 1})_s \to \mathrm{Der}_\partial(\hat{T}_E)$ Lie algebra homom.

$\sigma_s^0 : N(\hat{T}_{\geq 1})_s \to \mathrm{Der}(\hat{T}) = \mathrm{Der}(\hat{T}_E(*_0, *_0))$ Lie algebra homom

$\sigma_s^0(u)(A_i^s) := u_i'', \quad \sigma_s^0(u)(B_i^s) := -u_i', \quad \sigma_s^0(u)(C_j) := u_j^0 C_j - C_j u_j^0$

$\sigma_s(u)(v) := -u_a^0 v + \sigma_s^0(u)(v) + v u_b^0, \quad v \in (\hat{T}_E)(*_a, *_b), \quad 0 \leq a, b \leq n, \quad (u_0^0 := 0)$

$(1+\hat{T}_{\geq 1})_E$: groupoid over E, $\quad (1+\hat{T}_{\geq 1})_E(*_a, *_b) := 1+\hat{T}_{\geq 1}$, $\quad 0 \leq a, b \leq n$

<u>Magnus expansion of</u> $\Pi S|_E$

θ : Magnus expansion of $\Pi S|_E$

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\theta : \Pi S|_E \to (1+\hat{T}_{\geq 1})_E$: homomorphism of groupoids over E

2) $0 \leq {}^\forall k \leq n$, $\quad \theta : \pi_1(S, *_k) \to (1+\hat{T}_{\geq 1})_E(*_k, *_k) = 1+\hat{T}_{\geq 1}$ Magnus expansion

$s \in \mathrm{Sect}\, i_*$

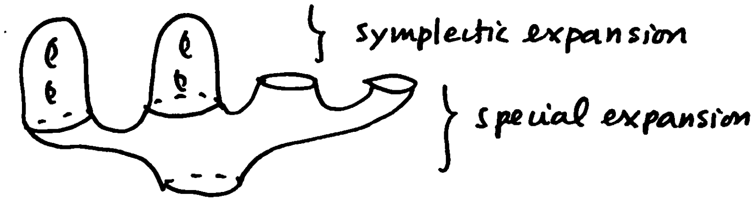
<u>boundary condition</u> $(\#_s)$

$$\theta(\zeta_k^s) = \begin{cases} \exp(C_k) & \text{for } 1 \leq {}^\forall k \leq n \\ \exp\left(C_0 - \sum_{i=1}^{g} (A_i^s B_i^s - B_i^s A_i^s)\right) & \text{for } k = 0 \end{cases}$$

ex) (1) $S = \Sigma_{g,1}$, symplectic expansion

(2) $S = \Sigma_{0,n+1}$, special expansion

(3)



(4) $/\mathbb{R}$ harmonic Magnus expansion for $(C, P_0, P_1, \cdots, P_n)$

Theorem 3 ($\Sigma_{g,1}$ | Kuno–K., general | Massuyeau–Turaev, Kuno–K.)

$s \in \mathrm{Sect}\, i_*$, θ : Magnus expansion of $\Pi S|_E$ satisfying $(\#_s)$

$\Longrightarrow$ (1) $-N\theta : \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} N(\hat{T})_s$ Lie algebra isomorphism

(2)

$$\begin{tikzcd}
\widehat{\mathbb{Q}\hat{\pi}}(S) \arrow[r, "\sigma"] \arrow[d, "-N\theta"', "\wr"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E) \arrow[d, "\theta", "\wr"'] \\
N(\hat{T}_{\geq 1})_s \arrow[r, "\sigma_s"'] & \mathrm{Der}_\partial(\hat{T}_E)
\end{tikzcd}$$

Massuyeau–Turaev

- tensorial description of the Papakyriakopoulos–Turaev homotopy intersection form
- quiver theory
- double Poisson bracket

Thm 3 $\Rightarrow$ Thm 1 :

$\sigma_s : N(\hat{T}_{\geq 1})_s \xrightarrow{\cong} \mathrm{Der}_\partial(\hat{T}_E)$ isom $\Leftarrow$ straightforward computation

§3. Turaev 余括弧積のテンソル表示について

§3.1. Turaev cobracket

$\hat{\pi} = \hat{\pi}(S)$, $\mathbb{1} = |1| \in \hat{\pi}$ constant loop, $\in \mathrm{Center}(\mathbb{Z}\hat{\pi})$

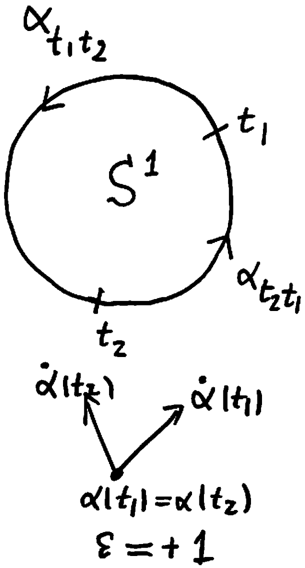
$\mathbb{Z}\hat{\pi}' := \mathbb{Z}\hat{\pi}/\mathbb{Z}\mathbb{1}$ Lie algebra, $|\ |' : \mathbb{Z}\pi_1(S) \xrightarrow{|\ |} \mathbb{Z}\hat{\pi} \xrightarrow{\text{quotient}} \mathbb{Z}\hat{\pi}'$

$\delta : \mathbb{Z}\hat{\pi}' \to \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$ Turaev cobracket

$\alpha \in \hat{\pi}(S)$ in general position

$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 ; t_1 \neq t_2, \alpha(t_1) = \alpha(t_2)\}$ double points

$$\delta(\alpha) := \sum_{(t_1,t_2) \in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1 t_2}|' \otimes |\alpha_{t_2 t_1}|' \in \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$$



$\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \in \{\pm 1\}$ local intersection number

$\left[\begin{array}{l} \underline{\text{Turaev}}\ (1)\ \delta : \mathbb{Z}\hat{\pi}' \to \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}' \text{ well-defined} \\ \quad (2)\ (\mathbb{Z}\hat{\pi}', [\ ,\ ], \delta) : \text{Lie bialgebra (in the sense of Drinfel'd)} \end{array}\right.$

($\Rightarrow \mathrm{Ker}\,\delta' < \mathbb{Z}\hat{\pi}'$ Lie subalgebra)

the Goldman-Turaev Lie bialgebra

$\Longrightarrow (\mathbb{Q}\hat{\pi}, [\,,\,], \delta)$: complete Lie bialgebra.
the completed Goldman-Turaev Lie algebra

Rmk) 商 $\mathbb{Z}\hat{\pi}' = \mathbb{Z}\pi/\mathbb{Z}\mathbb{1}$ をとらなければならない理由 : monogon の生成消滅

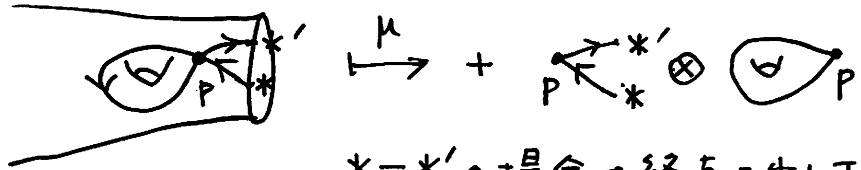
homotopic (しかし, 正則 homotopy ではない)

$\mathbb{Z}\hat{\pi}'$-comodule structure on $\mathbb{Z}\Pi S(*,*')$

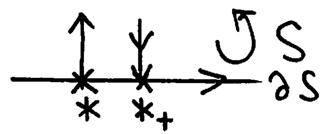
$*, *' \in \partial S$, $* \neq *'$, $\gamma \in \Pi S(*,*')$ in general position

$\Gamma_\gamma := \{\text{double points of } \gamma\} \subset S$, $p \in \Gamma_\gamma$, $\gamma^{-1}(p) = \{t_1^p, t_2^p\} \subset [0,1]$, $t_1^p < t_2^p$

$$\mu(\gamma) := -\sum_{p \in \Gamma_\gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) (\gamma_{0t_1^p}\gamma_{t_2^p 1}) \otimes |\gamma_{t_1^p t_2^p}|' \in \mathbb{Z}\Pi S(*,*') \otimes \mathbb{Z}\hat{\pi}'(S)$$



$* = *'$ の場合は終点を少し正の方向にズラす



Kuno-K. (inspired by Turaev)

(1) $\mu : \mathbb{Z}\Pi S(*,*') \to \mathbb{Z}\Pi S(*,*') \otimes \mathbb{Z}\hat{\pi}'(S)$ well-defined

(2) $(\mathbb{Z}\Pi S(*,*'), \sigma, \mu)$: $\mathbb{Z}\hat{\pi}'(S)$-bimodule

$\Longrightarrow$ $\widehat{\mathbb{Q}\Pi S}(*_a, *_b)$: complete $\mathbb{Q}\hat{\pi}(S)$-bimodule $(0 \le a, b \le n)$

<u>Theorem 4</u> (Kuno-K.)

$$\delta \circ \tau = 0 : \mathcal{M}(S)^\circ \xrightarrow{\tau} \mathbb{Q}\hat{\pi}(S) \xrightarrow{\delta} \mathbb{Q}\hat{\pi}(S) \hat{\otimes} \mathbb{Q}\hat{\pi}(S)$$

<u>proof</u> $\forall \varphi \in \mathcal{M}(S)^\circ$ preserves μ

$\forall n \in \mathbb{Z}$ $\quad e^{n\sigma\tau(\varphi)}$ preserves μ

i.e., $\forall v \in \widehat{\mathbb{Q}\Pi S}(*_a, *_b)$ $\quad \mu(e^{n\sigma\tau(\varphi)} v) = (e^{n\sigma\tau(\varphi)} \hat{\otimes} e^{n\sigma\tau(\varphi)}) \mu(v)$

$\Longrightarrow$ linear term in n $\quad \mu(\sigma\tau(\varphi) v) = (\sigma\tau(\varphi) \hat{\otimes} 1 + 1 \hat{\otimes} \operatorname{ad} \tau(\varphi)) \mu(v)$

$\Longrightarrow$ compatibility axiom of bimodules $\quad (\overline{\sigma} \otimes 1_{\mathbb{Q}\hat{\pi}})(v \otimes \delta\sigma(\varphi)) = 0$

(where $\overline{\sigma} : \widehat{\mathbb{Q}\Pi S} \otimes \mathbb{Q}\hat{\pi} \to \widehat{\mathbb{Q}\Pi S}$, $v \otimes u \mapsto -\sigma(u) v$)

$\Longrightarrow$ $\sigma : \mathbb{Q}\hat{\pi} \to \operatorname{Der}(\widehat{\mathbb{Q}\Pi S})$ injective ($\because$ Thm 1) $\quad \delta\sigma(\varphi) = 0$ //

$\tau(\mathcal{M}(S)^\circ) \subset \operatorname{Ker} \delta$ subset

$\operatorname{Ker} \delta \subset \mathbb{Q}\hat{\pi}(S)$ Lie subalgebra

§3.2. On a tensorial description of the Turaev cobracket

S : compact connected oriented surface with $\partial S \neq \phi$

θ : Magnus expansion of $\pi_1(S)$

$-N\theta : \mathbb{Q}\widehat{\hat{\pi}}(S) \xrightarrow{\widehat{\cong}} N(\hat{T}_{\geq 1})$ isom

$\delta^\theta := ((-N\theta)\hat{\otimes}(-N\theta)) \circ \delta \circ (-N\theta)^{-1} : N(\hat{T}_{\geq 1}) \to N(\hat{T}_{\geq 1}) \hat{\otimes} N(\hat{T}_{\geq 1})$

tensorial description of the Turaev cobracket

$\delta^\theta = \sum_{p=-\infty}^{+\infty} \delta^\theta_{(p)}$, $\forall m \geq 0$ $\delta^\theta_{(p)}(N(H^{\otimes m})) \subset N(H^{\otimes m+p})$, "Laurent expansion of δ^θ"

$S = \Sigma_{g,1}$, $N(\hat{T}_{\geq 1}) = \mathrm{Der}_\omega(\hat{T}) \supset \mathrm{Der}^+_\omega(\hat{\mathcal{L}})$

<u>Theorem 5</u> (Kuno-K., Massuyeau-Turaev)

(based on Massuyeau-Turaev's description of the homotopy intersection form)

$$\delta^\theta_{(p)} = \begin{cases} 0 & \text{if } p \leq -3 \\ \text{Schedler's cobracket } \delta^{alg} & \text{if } p = -2 \end{cases}$$

$$\delta^{alg}(N(X_1 \cdots X_m)) = \sum_{j-i \geq 2} (X_i \cdot X_j) \left\{ \begin{array}{l} N(X_{i+1} \cdots X_{j-1}) \hat{\otimes} N(X_{j+1} \cdots X_m X_1 \cdots X_{i-1}) \\ - N(X_{j+1} \cdots X_m X_1 \cdots X_{i-1}) \otimes N(X_{i+1} \cdots X_{j-1}) \end{array} \right\}$$

$N1 = 0$ $\qquad (X_i \in H)$

Proposition (Kuno–K.)

$\delta^{alg}|_{\mathrm{Der}^+_\omega(\hat{\mathcal{L}})}$ includes the Morita traces (except the 1st term)

Corollary

The Turaev cobracket includes the Morita traces (except the 1st term)

Enomoto–Satoh trace : a refinement of the Morita traces

$ES : N(\hat{T}_{\geq 1}) \to N(\hat{T}_{\geq 1})$

$ES(N(X_1 \cdots X_m)) := \sum_{i=1}^{m} (X_i \cdot X_{i+1})\, N(X_{i+2} \cdots X_m X_1 \cdots X_{i-1})$

Theorem (Enomoto–Satoh)

ES (except the 1st term) vanishes on $gr(\tau(\text{Torelli group}))$

Observation (Enomoto)

Schedler's cobracket δ^{alg} does not include the Enomoto–Satoh traces ES

Rmk The 1st Morita trace = The 1st Enomoto–Satoh trace

: Furuta's description in terms of a framing of $\Sigma_{g,1}$

An alternative cyclic symmetrizer N^+

$H = H_1(S;\mathbb{Q})$, $\quad \hat{T} = \hat{T}(H)$.

$N^+ : \hat{T} \to \hat{T}$ continuous $\mathbb{Q}$-linear map

$$N^+|_{H^{\otimes m}} := \begin{cases} 1_{H^{\otimes 0}} & \text{if } m = 0 \\ N|_{H^{\otimes m}} & \text{if } m \geq 1 \end{cases}$$

$\Uparrow$

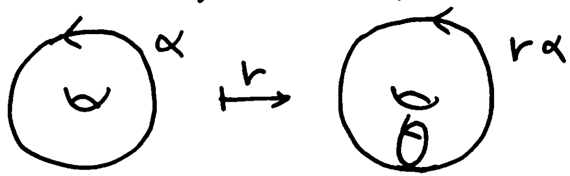
a regular homotopy version of the Turaev cobracket

S : compact connected oriented surface with $\partial S \neq \emptyset$

$\hat{\pi}^+ = \hat{\pi}^+(S) := \{\ell : S^1 \to S \ C^\infty \text{ immersion}\} / \text{regular homotopy}$

$\langle r \rangle$ $(\cong \mathbb{Z})$ infinite cyclic group generated by a formal letter r

$\langle r \rangle \curvearrowright \hat{\pi}^+$ free action

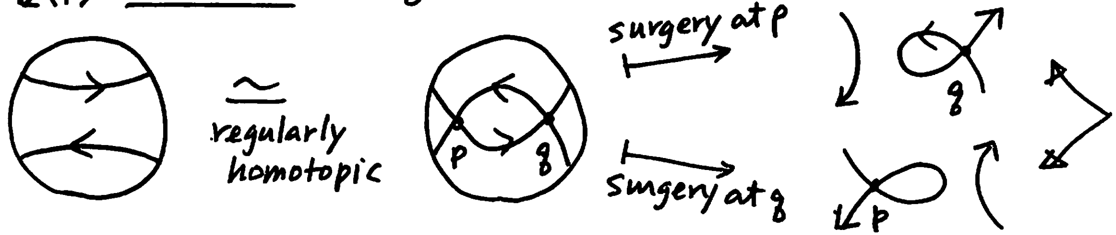


inserting a monogon into α

$\hat{\pi}^+(S)/\langle r \rangle = \hat{\pi}(S)$

$\delta^+ : \mathbb{Z}\hat{\pi}^+(S) \to \mathbb{Z}\hat{\pi}^+(S) \otimes_{\mathbb{Z}\langle r \rangle} \mathbb{Z}\hat{\pi}^+(S)$ $\quad$ the "regular" Turaev cobracket

$\otimes_{\mathbb{Z}\langle r\rangle}$ する理由: bigon の生成消滅



これらは $\mathbb{Z}(\hat\pi^+ \times_{\langle r\rangle} \hat\pi^+) = \mathbb{Z}\hat\pi^+ \otimes_{\mathbb{Z}\langle r\rangle} \mathbb{Z}\hat\pi^+$ においてはじめて一致する.

$\varpi : TS \to S$ tangent bundle

$TS \overset{\exists}{\cong} S \times \mathbb{R}^2$ global trivialization. $(\because \partial S \neq \emptyset)$

$v \mapsto (\varpi(v), f(v))$ $\quad$ ($f : TS \to \mathbb{R}^2$ framing of S)

$\mathrm{rot}_f : \hat\pi^+(S) \to \mathbb{Z}$ rotation number w.r.to f

$$\alpha \mapsto \mathrm{rot}_f \alpha := \deg\left(S^1 \xrightarrow{\dot\alpha} (TS \setminus 0\text{-section}) \xrightarrow{f} \mathbb{R}^2 \setminus \{0\}\right)$$

$\Phi_f : \hat\pi^+(S) \cong \hat\pi(S) \times \langle r\rangle, \quad \alpha \mapsto (\alpha, r^{\mathrm{rot}_f \alpha}),$

$\widehat{\mathbb{Q}\langle r\rangle} = \mathbb{Q}[[\rho]], \quad \rho = \log r$

$\Longrightarrow -N^+\theta_f : \widehat{\mathbb{Q}\hat\pi^+} \xrightarrow{\cong} N^+(\hat T)\hat\otimes \mathbb{Q}[[\rho]]$, $\mathbb{Q}[[\rho]]$-isomorphism

$$\delta^{+,\theta_f} := ((-N^+\theta_f)\hat\otimes(-N^+\theta_f)) \circ \delta^+ \circ (-N^+\theta_f)^{-1} = \sum_{p=-\infty}^{+\infty} \delta^{+,\theta_f}_{(p)}$$

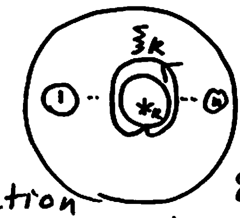
Theorem 5 (K.) $S=\Sigma_{g,1}$, θ : symplectic expansion, f : framing of $\Sigma_{g,1}$

$\Longrightarrow$ $\delta^{+,\theta_f}_{(p)} = 0$ if $p \leqq -3$

$$\delta^{+,\theta_f}_{(-2)} N^+(X_1\cdots X_m) = \sum_{j-i\geq 1} (X_i\cdot X_j)\left\{ \begin{array}{l} N^+(X_{i+1}\cdots X_{j-1})\,\hat\otimes\, N^+(X_{j+1}\cdots X_m X_1\cdots X_{i-1}) \\ \qquad - N^+(X_{j+1}\cdots X_m X_1\cdots X_{i-1})\,\hat\otimes\, N^+(X_{i+1}\cdots X_{j-1}) \end{array}\right\}$$

Corollary The regular Turaev cobracket includes the Enomoto-Satoh traces ___

$S=\Sigma_{0,n+1}$, $\hat T = \hat T(H_1(\Sigma_{0,n+1};\mathbb{Q}))$



$x_k := [\xi_k] = C_k \in H = H_1(\Sigma_{0,n+1};\mathbb{Q})$, $1\leq k\leq n$

$\dot{\rightsquigarrow} : \hat T_{\geqq 1}\times \hat T_{\geqq 1} \longrightarrow \hat T_{\geqq 1}$ associative multiplication

$\partial_0 S$

(Massuyeau-Turaev)

$$x_{i_1}\cdots x_{i_{l-1}} x_{i_l} \,\dot{\rightsquigarrow}\, x_{j_1} x_{j_2}\cdots x_{j_m} := -\delta_{i_l j_1}\, x_{i_1}\cdots x_{i_{l-1}} x_{j_1} x_{j_2}\cdots x_{j_m}$$

$x_0 = -\sum_{k=1}^{n} x_k$: unit of $\dot{\rightsquigarrow}$

Theorem 6 (K.) $S=\Sigma_{0,n+1}$, θ : special expansion, f : framing coming from  $\hookrightarrow \mathbb{R}^2$

$\partial_0 S$

$\Longrightarrow$ $\delta^{+,\theta_f}_{(p)} = 0$ if $p \leqq -2$

$$\begin{aligned}\delta^{+,\theta_f}_{(-1)}(N^+(X_1\cdots X_m)) = \sum_{j-i\geq 1} & N^+(X_i \dot{\rightsquigarrow} X_j X_{j+1}\cdots X_m X_1\cdots X_{i-1})\,\hat\otimes\, N^+(X_{i+1}\cdots X_{j-1}) \\ & + N^+(X_j \dot{\rightsquigarrow} X_i X_{i+1}\cdots X_{j-1})\,\hat\otimes\, N^+(X_{j+1}\cdots X_m X_1\cdots X_{i-1}) \\ & - N^+(X_{i+1}\cdots X_{j-1})\,\hat\otimes\, N^+(X_i \dot{\rightsquigarrow} X_j X_{j+1}\cdots X_m X_1\cdots X_{i-1}) \\ & - N^+(X_{j+1}\cdots X_m X_1\cdots X_{i-1})\,\hat\otimes\, N^+(X_j \dot{\rightsquigarrow} X_i X_{i+1}\cdots X_{j-1})\end{aligned}$$

<u>divergence cocycle in the Kashiwara–Vergne problem</u>

$\mathrm{div}: \hat{T}_{\geq 1} \to N(\hat{T}_{\geq 1})$, $\quad X_1 \cdots X_m \mapsto \mathrm{div}(X_1 \cdots X_m) := -N(X_1 \cdots X_{m-1} \rightsquigarrow X_m)$

$$\mathrm{div}\, N(X_1 \cdots X_m) = \sum_{i=1}^{m} N(X_i \rightsquigarrow X_{i+1} \cdots X_m X_1 \cdots X_{i-1})$$
$$= (\mathbb{Q}\hat{\pi}^{+} \otimes \mathbb{Q}1\text{-component of } \delta_{(-1)}^{+,\theta_f})$$

$\Sigma_{0,3}$, $(n=2)$

<u>Kashiwara–Vergne problem</u> (in a formulation by Alekseev–Torossian)

Find a special expansion of $\Sigma_{0,3}$ compatible with div

<u>Theorem</u> (Alekseev–Meinrenken) $\exists$ a solution of the KV problem —

<u>Speculation</u> θ: a solution of the KV problem

$$\Longrightarrow \quad \delta^{\theta} = \delta^{\theta}_{(-1)}$$

If Yes,

$\rightsquigarrow$ "<u>positive genus KV problem</u>"

Find a symplectic expansion θ satisfying $\delta^{\theta} = \delta^{alg}$ —