JSPS-CNRS Special Day on "Mapping class groups of surfaces and automorphism groups of free groups" IRMA, University of Strasbourg,

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"The Turaev cobracket, the Enomoto-Satoh traces and the divergence cocycle in the Kashiwara-Vergne problem"

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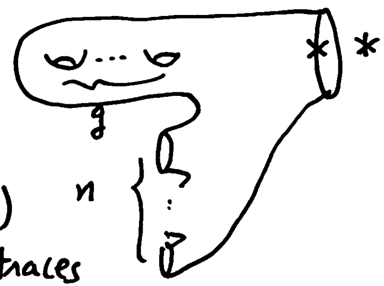
S : compact connected oriented surface with $\partial S \neq \phi$

$\underset{\text{Classification Theorem}}{\Longrightarrow}$ $\exists g, \exists n \geqq 0$

$S \cong \Sigma_{g,n+1}$

$=$ 

Abstract

regular homotopy version of the Turaev cobracket on S

$\downarrow$ take the lowest degree term

$S = \Sigma_{g,1}$ $\swarrow$ $\searrow$ $S = \Sigma_{0,3}$

the Enomoto-Satoh traces = the Enomoto-Satoh obstructions for the surjectivity of the Johnson homomorphism

the divergence cocycle in the Kashiwara-Vergne problem

Contents

§1. The Turaev cobracket (for general S)

S : compact connected oriented surface with $\partial S \neq \phi$

$\pi := \pi_1(S)$ based loops ---> free group ($\because$ $\partial S \neq \phi$)

$\hat{\pi} := [S^1, S]$ free loops

$\quad = \pi/\text{conj.}$

$|\ | : \pi \to \hat{\pi}$ quotient map = forgetful map of basepoint

$\hat{\pi}^+ := \pi_0(\text{Immersion}(S^1, S))$

$\quad = \{\alpha : S^1 \to S,\ C^\infty \text{immersion}\}/\text{regular homotopy}$

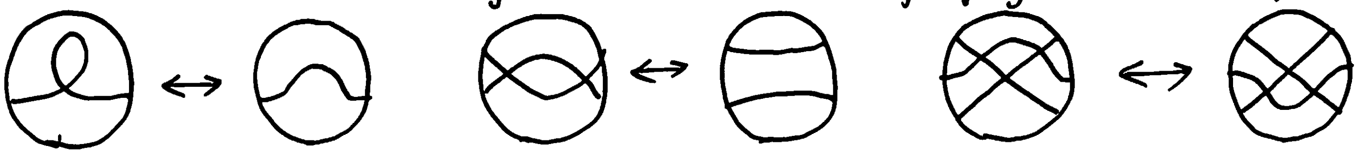
$\Phi : \hat{\pi}^+ \twoheadrightarrow \hat{\pi}$ forgetful map of smooth structure

$\hat{\pi}^+ = \{\text{generic immersions } S^1 \to S\}/(\text{isotopy, move(II), move(III)})$

move (I) monogon

move (II) bigon

move (III) jumping over a double point



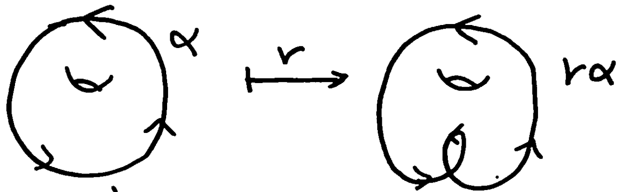
$\hat{\pi} = \hat{\pi}^+/\text{move (I)}$

<u>cyclic group action</u>

$\langle r\rangle (\cong \mathbb{Z})$ infinite cyclic group generated by a formal letter r

$\langle r\rangle \curvearrowright \hat{\pi}^+$ free action

$\hat{\pi}^+/\langle r\rangle = \hat{\pi}$



inserting a monogon.

<u>framing</u> $TS \cong S\times\mathbb{R}^2$ $(\because\ \partial S \neq \phi)$

f : framing of S

$\overset{\text{def}}{\Longleftrightarrow}$ $f: TS \to \mathbb{R}^2$ C^∞ map. s.t. $\forall p \in S$, $f|_{T_pS} : T_pS \xrightarrow{\cong} \mathbb{R}^2$ orientation-pres. isom.

<u>Remark</u> : Our construction is a generalization of Furuta's cocycle $\in Z^1(\mathcal{M}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Z}))$ defined by a framing of $\Sigma_{g,1}$. Its cohomology class equals the first term of the Enomoto-Satoh traces = the first term of the Morita traces.

<u>rotation number</u> f : framing of S

$\mathrm{rot}_f : \hat{\pi}^+ \to \mathbb{Z}$, $\alpha \mapsto \deg(S^1 \to \mathbb{R}^2\setminus\{0\},\ t \mapsto f(\dot{\alpha}(t)))$ rotation number

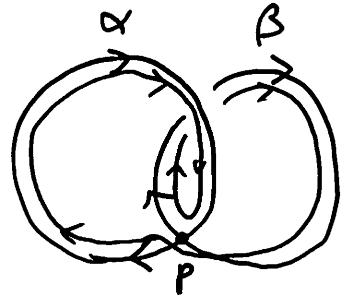
$\Phi_f : \hat{\pi}^+ \xrightarrow{\cong} \hat{\pi} \times \langle r\rangle$ $\langle r\rangle$-equivariant bijection

$\alpha \mapsto (\Phi\alpha,\ r^{\mathrm{rot}_f \alpha})$

Goldman bracket $[\ ,\ ] : \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi} \to \mathbb{Z}\hat{\pi}$

$\alpha, \beta \in \hat{\pi}$ in general position (i.e., $\alpha \amalg \beta$: generic immersion)

$$[\alpha, \beta] \overset{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta)\, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}$$



$\varepsilon = -1$

where $\varepsilon(p : \alpha, \beta) \in \{\pm 1\}$ local intersection number

$\alpha_p, \beta_p \in \pi_1(S, p)$ based loop along α, β with basepoint p

$\alpha_p \beta_p \in \pi_1(S, p) \overset{|\ |}{\longmapsto} |\alpha_p \beta_p| \in \hat{\pi}$ forgetting the basepoint p

product in π_1

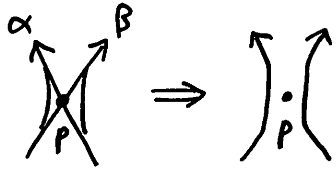
Theorem (Goldman)

- $[\ ,\ ]$: well-defined (i.e., invariant under the moves (I)(II) and (III))
- $(\mathbb{Z}\hat{\pi}, [\ ,\ ])$; Lie algebra $\rightsquigarrow$ Goldman Lie algebra of S

regular homotopy version of the Goldman bracket

$$[\ ,\ ]^+ : \mathbb{Z}\hat{\pi}^+ \otimes_{\mathbb{Z}\langle r \rangle} \mathbb{Z}\hat{\pi}^+ \to \mathbb{Z}\hat{\pi}^+$$

by smoothing $\alpha_p \beta_p$ at each $p \in \alpha \cap \beta$



$$\mathrm{rot}_f |\alpha_p \beta_p| = \mathrm{rot}_f \alpha + \mathrm{rot}_f \beta \quad (\forall f : \text{framing of } S)$$

$$\Phi_f : (\mathbb{Z}\hat{\pi}^+, [\ ,\ ]^+) \xrightarrow{\cong} (\mathbb{Z}\hat{\pi}, [\ ,\ ]) \otimes \mathbb{Z}\langle r \rangle$$

isom. of $\mathbb{Z}\langle r \rangle$ - Lie algebras

<u>Remarks</u> (1) Similarly we can consider a regular homotopy version κ^+ of

$$\kappa : \mathbb{Z}\pi \otimes \mathbb{Z}\pi \to \mathbb{Z}\pi \otimes \mathbb{Z}\pi$$

$\left\{\begin{array}{l}\text{a derived operation of the coaction } \mu \\ \text{a modification of the Papakyriakopoulos–Turaev homotopy intersection form}\end{array}\right.$

and have

$$\kappa^+ \cong \kappa \otimes \mathbb{Z}\langle r\rangle,$$

which enables us to use the tensorial description of κ by Massuyeau–Turaev

(2). $\mathbb{1} :=$  $\in \hat{\pi}^+$ trivial loop with rotation number 0

$\downarrow \Phi$

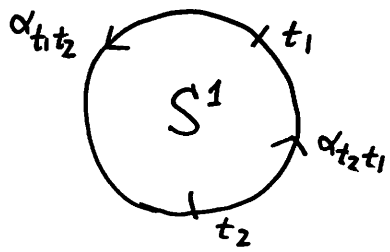
$\mathbb{1} = |1| \in \hat{\pi}$

both of $\mathbb{1}$'s $\in$ Center

$\Rightarrow$ One can consider the quotient Lie algebras

$$\mathbb{Z}\hat{\pi}' := \mathbb{Z}\hat{\pi}/\mathbb{Z}\mathbb{1} \quad \text{and} \quad \mathbb{Z}\hat{\pi}^+/\mathbb{Z}\langle r\rangle\mathbb{1}$$

$$|\ |' : \mathbb{Z}\pi \xrightarrow{|\ |} \mathbb{Z}\hat{\pi} \xrightarrow{\text{quotient}} \mathbb{Z}\hat{\pi}'$$



<u>Turaev cobracket</u> $\delta : \mathbb{Z}\hat{\pi}' \longrightarrow \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$

$\alpha \in \hat{\pi}$ in general position.

$D_\alpha := \{ (t_1, t_2) \in S^1 \times S^1 \ ; \ t_1 \neq t_2, \ \alpha(t_1) = \alpha(t_2) \}$

$$\delta(\alpha) := \sum_{(t_1,t_2) \in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \, |\alpha_{t_1t_2}|' \otimes |\alpha_{t_2t_1}|' \in \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$$

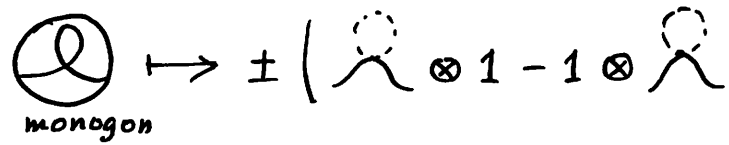
<u>Theorem</u> (Turaev)

δ : well-defined

$(\mathbb{Z}\hat{\pi}', [\ ,\ ], \delta)$: Lie bialgebra (in the sense of Drinfel'd)

need to take the quotient $\mathbb{Z}\hat{\pi}' = \mathbb{Z}\hat{\pi} / \mathbb{Z}1$

($\because$)



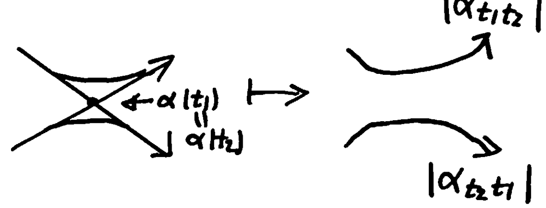
$\neq 0$ in $(\mathbb{Z}\hat{\pi})^{\otimes 2}$)

<u>regular homotopy version</u> of the Turaev cobracket

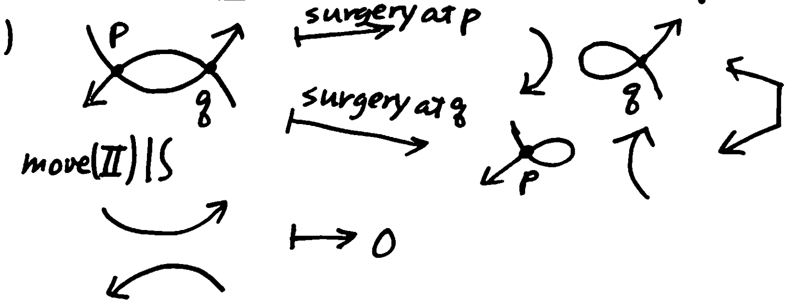
$$\delta^+ : \mathbb{Z}\hat{\pi}^+ \longrightarrow \mathbb{Z}\hat{\pi}^+ \underset{\mathbb{Z}\langle r \rangle}{\otimes} \mathbb{Z}\hat{\pi}^+$$

by smoothing $\alpha_{t_1t_2}$ and $\alpha_{t_2t_1}$ at $\alpha(t_1) = \alpha(t_2)$

$\mathrm{rot}_f(\alpha_{t_1t_2}) + \mathrm{rot}_f(\alpha_{t_2t_1}) = \mathrm{rot}_f \alpha$



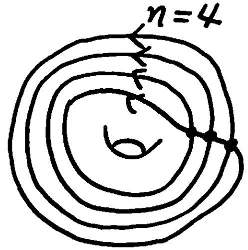
<u>need to define</u> $\delta^+(\alpha)$ <u>as an element of</u> $\mathbb{Z}\hat{\pi}^+ \otimes_{\mathbb{Z}\langle r\rangle} \mathbb{Z}\hat{\pi}^+$

(∵) 

These coincide with each other in $\mathbb{Z}(\hat{\pi}^+ \times_{\langle r\rangle} \hat{\pi}^+) = \mathbb{Z}\hat{\pi}^+ \otimes_{\mathbb{Z}\langle r\rangle} \mathbb{Z}\hat{\pi}^+$

- $\forall n \in \mathbb{Z}$, $\delta^+(r^n\alpha) = r^n\delta^+\alpha + nr^n(\alpha\otimes 1 - 1\otimes\alpha)$

  (in particular, δ^+ is <u>not</u> $\mathbb{Z}\langle r\rangle$-linear)



- α : simple closed curve

$\Longrightarrow \forall n \in \mathbb{Z}$ $\delta^+(\alpha^n) = 0$

(∵ $\delta^+(\alpha^n) = \sum_{k=1}^{n-1} \alpha^k \otimes \alpha^{n-k} - \sum_{k=1}^{n-1} \alpha^{n-k} \otimes \alpha^k = 0$)

Some tensor algebra

Choose a basepoint $* \in \partial S$

$\pi := \pi_1(S, *)$, free group of rank $2g+n$ for $S \cong \Sigma_{g,n+1}$

$H := H_1(S; \mathbb{Q}) = (\pi/[\pi,\pi]) \otimes_{\mathbb{Z}} \mathbb{Q}$

$\gamma \in \pi \longmapsto [\gamma] := (\gamma \bmod [\pi,\pi]) \otimes 1 \in H$

$T = T(H) := \bigoplus_{m=0}^{\infty} H^{\otimes m}$, the tensor algebra over H

Δ : coproduct, $\quad \Delta : T \to T \otimes T$, $\quad X \in H \mapsto X \otimes 1 + 1 \otimes X$

$\mathcal{L} :=$ Lie-like$(T) = \{u \in T ; \Delta u = u \otimes 1 + 1 \otimes u\}$

$= \bigoplus_{m=1}^{\infty} \mathcal{L}_m$, $\quad \mathcal{L}_m := \mathcal{L} \cap H^{\otimes m}$ $\quad$ free Lie algebra over H

$T = U(\mathcal{L})$ the universal enveloping algebra of $\mathcal{L}$

$\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$ the <u>completed</u> tensor algebra over H

$\widehat{T}_{\geq p} := \prod_{m=p}^{\infty} H^{\otimes m} \subset \widehat{T}$ two-sided ideal $\rightsquigarrow$ $\widehat{T}_{\geq 1}$-adic topology on $\widehat{T}$

$\widehat{\mathcal{L}} := \prod_{m=1}^{\infty} \mathcal{L}_m =$ Lie-like$(\widehat{T})$ The <u>completed</u> free Lie algebra over H

$\widehat{\mathcal{L}}_{\geq p} := \prod_{m=p}^{\infty} \mathcal{L}_m$

$\mathrm{Der}(\hat{T}) := \{D : \hat{T} \to \hat{T} \text{ continuous derivation}\}$

$\mathrm{Der}(\hat{\mathcal{L}}) := \{D : \hat{\mathcal{L}} \to \hat{\mathcal{L}} \text{ continuous derivation}\}$

can be identified with the subalgebra of $\mathrm{Der}(\hat{T})$

$$\{D \in \mathrm{Der}(\hat{T}) \,;\, (D \hat{\otimes} 1 + 1 \hat{\otimes} D)\Delta = \Delta D\}$$

$\mathrm{Der}^+(\hat{\mathcal{L}}) := \{D \in \mathrm{Der}(\hat{\mathcal{L}}) \,;\, D(H) \subset \mathcal{L}_{\geq 2}\}$

ι_H : restriction to $H \subset \hat{\mathcal{L}} \subset \hat{T}$

$$\begin{array}{ccccccc} \mathrm{Der}(\hat{T}) & \supset & \mathrm{Der}(\hat{\mathcal{L}}) & \supset & \mathrm{Der}^+(\hat{\mathcal{L}}) \\ {\scriptstyle \iota_H}\ \wr\| & \circlearrowleft & \wr\| & \circlearrowleft & \wr\| \\ H^* \otimes \hat{T} & \supset & H^* \otimes \hat{\mathcal{L}} & \supset & H^* \otimes \hat{\mathcal{L}}_{\geq 2} \end{array}$$

<u>group-like expansion of the free group</u> π

Definition

$\theta : \pi \to \hat{T}$ group-like expansion of π

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\forall \gamma \in \pi \quad \theta(\gamma) \equiv 1 + [\gamma] \mod \hat{T}_{\geq 2}$

2) $\forall \gamma, \forall \delta \in \pi \quad \theta(\gamma\delta) = \theta(\gamma)\theta(\delta)$

3) (group-like condition)

$\forall \gamma \in \pi \quad \Delta\theta(\gamma) = \theta(\gamma) \hat{\otimes} \theta(\gamma) \quad$ i.e., $\theta(\gamma) \in \exp(\hat{\mathcal{L}})$

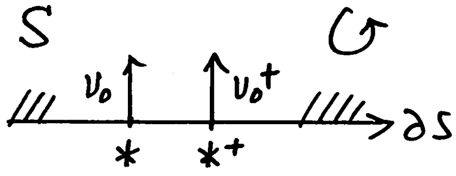
$\Longrightarrow \quad \theta : \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \widehat{T}$ isom. of complete Hopf algebras

where $\quad \widehat{\mathbb{Q}\pi} := \varprojlim_{m\to\infty} \mathbb{Q}\pi / (I\pi)^m$

$I\pi := \mathrm{Ker}\,(\varepsilon : \mathbb{Q}\pi \to \mathbb{Q})$ augmentation ideal.
$\qquad\qquad \sum_{x\in\pi} a_x x \mapsto \sum a_x$

<u>regular homotopy version of the fundamental group</u> π

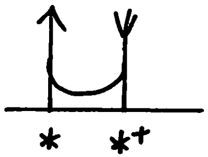
Choose tangent vectors v_0, v_0^+ as follows



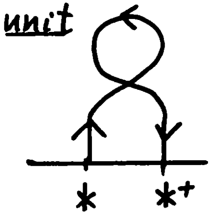
$\pi^+ := \{ \ell : [0,1] \to S : \begin{array}{l} C^\infty \text{ immersion} \\ \dot{\ell}(0) = v_0,\ \dot{\ell}(1) = -v_0^+ \end{array} \} / \text{regular homotopy}$

<u>group-structure</u>

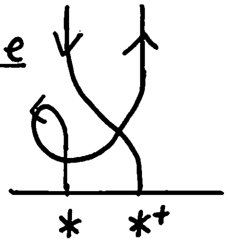
<u>product</u>



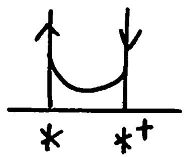
<u>unit</u>



<u>inverse</u>



<u>closing</u> $\quad |\ | : \pi^+ \twoheadrightarrow \hat{\pi}^+$



f : framing of S

$\Longrightarrow \quad \Phi_f : \pi^+ \xrightarrow{\cong} \pi \times \langle r \rangle, \quad [\ell] \mapsto (\overset{\text{forgetting smooth str}}{\overline{\Phi}\ell},\ r^{(\mathrm{rot}_f \ell - \frac{1}{2})})$

$\qquad$ isom of groups

$\Longrightarrow \quad \Phi_f : \widehat{\mathbb{Q}\pi^+} \xrightarrow{\cong} \widehat{\mathbb{Q}\pi} \hat{\otimes} \mathbb{Q}[[\rho]]$, <u>where</u> $\mathbb{Q}[[\rho]] = \widehat{\mathbb{Q}\langle r \rangle}$, $\rho = \log r$

θ : group-like expansion of π

$\Longrightarrow \quad \theta_f := (\theta \hat{\otimes} 1_{\mathbb{Q}[[\rho]]}) \circ \Phi_f : \widehat{\mathbb{Q}\pi^+} \xrightarrow{\cong} \widehat{T} \hat{\otimes} \mathbb{Q}[[\rho]]$ isom. of complete Hopf algebras

<u>cyclic symmetrizers</u> (cyclicizers)

$N, N^+ : \hat{T} \to \hat{T}$ $\mathbb{Q}$-linear maps

<u>$m \geq 1$</u> $\quad N(X_1 \cdots X_m) = N^+(X_1 \cdots X_m) \overset{\text{def}}{=} \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1} \quad (X_j \in H)$

<u>$m = 0$</u> $\quad N|_{H^{\otimes 0}} := 0, \quad N^+|_{H^{\otimes 0}} := 1_{H^{\otimes 0}}, \quad H^{\otimes 0} = \mathbb{Q} \subset \hat{T}$

$(\leftrightarrow \mathbb{Q}\hat{\pi}/\mathbb{Q}1) \qquad (\leftrightarrow \mathbb{Q}\hat{\pi}^+)$

$N^+(\hat{T}) = \mathbb{Q} \oplus N(\hat{T})$

$\widehat{\mathbb{Q}\hat{\pi}} := \varprojlim_{m\to\infty} \mathbb{Q}\hat{\pi} / \mathbb{Q}1 + |(I\pi)^m|$ completed Goldman-Turaev Lie bialgebra

<u>Observation</u> (Kuno-K.)

θ : group-like expansion

$\Longrightarrow -N\theta : \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} N(\hat{T})$ isom. of filtered $\mathbb{Q}$-vector spaces

$|x| \in \hat{\pi} \longmapsto -N\theta(x)$

$\widehat{\mathbb{Q}\hat{\pi}^+} := \varprojlim_{m\to\infty} \mathbb{Q}\hat{\pi}^+ / |(I\pi^+)^m|$ completion of the "regular" Goldman-Turaev Lie bialgebra

$-N^+ \circ \theta_f : \widehat{\mathbb{Q}\hat{\pi}^+} \xrightarrow{\cong} N^+(\hat{T}) \hat{\otimes} \mathbb{Q}[[\rho]] = (\mathbb{Q} \oplus N(\hat{T})) \hat{\otimes} \mathbb{Q}[[\rho]]$

isom. of filtered $\mathbb{Q}$-vector spaces

§ 2. The Enomoto-Satoh traces (for $\Sigma_{g,1}$)

$g \geq 1$

$S = \Sigma_{g,1} =$  $* \in \partial S$

$\pi = \pi_1(S, *)$ free group of rank $2g$

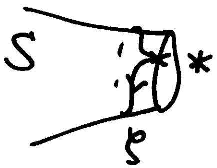
$H = H_1(S;\mathbb{Q}) \overset{\text{P.d.}}{\cong} H^* = H^1(S;\mathbb{Q}) \quad X \mapsto (Y \mapsto Y \cdot X)$ Poincaré duality (intersection number)

$\omega := \sum_{i=1}^{g} A_i B_i - B_i A_i \in \mathcal{L}_2 \subset H^{\otimes 2}$ symplectic form

(indep. of the choice of a symplectic basis $\{A_i, B_i\}_{i=1}^{g} \subset H$)

$$
\begin{array}{ccccccc}
\mathrm{Der}_\omega(\hat{T}) := \{D \in \mathrm{Der}(\hat{T})\,;\, D\omega = 0\} & & \cong & & N(\hat{T}) & & \text{Kontsevich's "associative"} \\
\cap & & & & \cap & & \\
\mathrm{Der}(\hat{T}) & \underset{1_H}{\cong} & H^* \otimes \hat{T} & \underset{\text{P.d.}}{\cong} & H \otimes \hat{T} = \hat{T}_{\geq 1} & & \\
\cup & & & & \cup & & \\
\mathrm{Der}^+_\omega(\hat{\mathcal{L}}) := \{D \in \mathrm{Der}^+(\hat{\mathcal{L}})\,;\, D\omega = 0\} & & \cong & & N(H \otimes \mathcal{L}_{\geq 2}) & & \text{Morita's Lie algebra } \hat{\mathfrak{h}}_{g,1} = \text{positive part of Kontsevich's "Lie"}
\end{array}
$$

$\zeta \in \pi = \pi_1(S, *)$ negative boundary loop



Definition (Massuyeau)

$\theta : \pi \to \hat{T}$ symplectic expansion

$\overset{\text{def}}{\iff}$ 0) $\theta : \pi \to \hat{T}$ group-like expansion

1) $\theta(\zeta) = \exp \omega \; \left(= \sum_{m=0}^{\infty} \frac{1}{m!} \omega^m\right) \in \hat{T}$

Theorem (Kuno-K.) θ: symplectic expansion

$\Rightarrow$ $-N\theta: \mathbb{Q}\hat{\pi} \xrightarrow{\cong} N(\hat{T}) \cong \mathrm{Der}_\omega(\hat{T})$ isom. of filtered Lie algebras

$|x| \in \hat{\pi} \longmapsto -N\theta(x)$

$\delta^\theta := ((-N\theta)\hat{\otimes}(-N\theta)) \circ \delta \circ (-N\theta)^{-1}$: Turaev cobracket on $N(\hat{T}) = \mathrm{Der}_\omega(\hat{T})$

Theorem (Kuno-K., Massuyeau-Turaev) based on a result of Massuyeau-Turaev

The lowest degree term of δ^θ equals Schedler's cobracket. i.e.,

$$\delta^\theta_{(-2)}(N(X_1\cdots X_m)) = \sum_{j-i\geq 2}(X_i\cdot X_j)\{N(X_{i+1}\cdots X_{j-1})\hat{\otimes}N(X_{j+1}\cdots X_m X_1\cdots X_{i-1}) - N(X_{j+1}\cdots X_m X_1\cdots X_{i-1})\hat{\otimes}N(X_{i+1}\cdots X_{j-1})\}$$

$N(1)=0$ $\quad (X_i \in H)$

Corollary (Kuno-K.)

$\delta^\theta_{(-2)}|_{\mathrm{Der}^+_\omega(\hat{\mathcal{L}})}$ includes the Morita traces (except the first one)

Johnson homomorphisms

$\Gamma_k = \Gamma_k(\pi)$, $k\geq 1$, lower central series, $\Gamma_1 := \pi$, $\Gamma_{k+1} := [\pi, \Gamma_k]$ $(k\geq 1)$

$\mathrm{gr}(\Gamma) := \bigoplus_{k=1}^{\infty} \Gamma_k/\Gamma_{k+1}$, $\quad \mathrm{gr}(\Gamma)\otimes_{\mathbb{Z}}\mathbb{Q} \cong \mathcal{L}$ (Magnus-Witt)

$\mathcal{M}(k) := \mathrm{Ker}(\mathrm{MCG}(\Sigma_{g,1}) \to \mathrm{Aut}(\pi/\Gamma_{k+1}))$ the Johnson filtration

$\mathcal{I}_{g,1} := \mathcal{M}(1)$ Torelli group

$\mathcal{K}_{g,1} := \mathcal{M}(2)$ Johnson kernel

$$gr(\mathfrak{g}) := \bigoplus_{k=1}^{\infty} m(k)/m(k+1) \quad , \quad gr(K) := \bigoplus_{k=2}^{\infty} m(k)/m(k+1)$$

<u>Johnson homomorphism</u>

$\tau : gr(\mathfrak{g}) \hookrightarrow \mathrm{Der}(\mathcal{L})$ injective homom. of Lie algebras

<u>Theorem</u> (Morita)

$\tau(gr(\mathfrak{g})) \subsetneqq \mathrm{Der}^+_\omega(\mathcal{L})$ (= Morita's Lie algebra $\mathfrak{h}_{g,1}$)

Morita traces

extension

<u>Enomoto-Satoh traces</u>

$$\begin{tikzcd}
\mathrm{ES}: \mathrm{Der}^+_\omega(\mathcal{L}) \arrow[r] \arrow[rr, bend left, "\times"] & N(T) \arrow[r, "\circlearrowleft"] & \mathrm{Sym}(H) \\
H^* \otimes \mathcal{L}_{\geq 2} \arrow[u, phantom, "\cap"] \arrow[r, phantom, "\circlearrowleft"] & & \\
H^* \otimes H \otimes T \arrow[u, phantom, "\cap"] \arrow[r, "\text{contraction}"'] & T \arrow[uu, "N"'] &
\end{tikzcd}$$

<u>Theorem</u> (Enomoto-Satoh)

- $\mathrm{ES}(gr(K)) = 0$
- $\mathrm{ES}(\mathrm{Ker}(\text{the Morita traces})) \neq 0$

<u>Theorem</u> (Enomoto)

$\delta^{\theta}_{(-2)}|_{\mathrm{Der}^+_\omega(\mathcal{L})}$ does <u>not</u> include the ES traces __regular homotopy version__ θ : symplectic expansion, f : framing of $\Sigma_{g,1}$

$$-N^+\circ\theta_f : \widehat{\mathbb{Q}\hat{\pi}^+} \xrightarrow{\cong} N^+(\hat{T})\hat{\otimes}\mathbb{Q}[[\rho]] = (\mathbb{Q}\tilde{\times}_e \mathrm{Der}_\omega(\hat{T}))\hat{\otimes}\mathbb{Q}[[\rho]]$$

isom. of Lie algebras

__where__ $e : \mathrm{Der}_\omega(\hat{T})\times\mathrm{Der}_\omega(\hat{T}) \xrightarrow[\text{deg 1 part}]{} H\times H \xrightarrow{\cdot} \mathbb{Q}$ 2-cocycle

$\mathbb{Q}\tilde{\times}_e \mathrm{Der}_\omega(\hat{T})$: central extension by the 2-cocycle e.

δ^{+,θ_f} : the Turaev cobracket on $N^+(\hat{T})\hat{\otimes}\mathbb{Q}[[\rho]]$ induced by $-N\circ\theta_f$

__Theorem (K.)__ θ : symplectic expansion, f : framing of $\Sigma_{g,1}$

$\Longrightarrow$ The lowest degree term of δ^{+,θ_f} is given by

$$\delta^{+,\theta_f}_{(-2)} N^+(X_1\cdots X_m)$$
$$= \sum_{j-i\geq 1}(X_i\cdot X_j)\{N^+(X_{i+1}\cdots X_{j-1})\otimes N^+(X_{j+1}\cdots X_m X_1\cdots X_{i-1}) - N^+(X_{j+1}\cdots X_m X_1\cdots X_{i-1})\otimes N^+(X_{i+1}\cdots X_{j-1})\}$$

$(X_i\in H)$

In particular, the $N^+(\hat{T})\otimes N^+(1)$-component equals the ES-traces

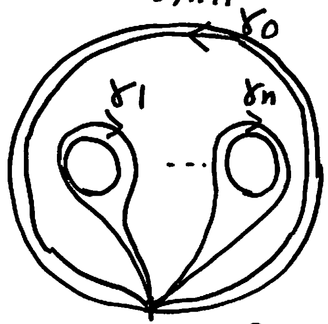
$\rightsquigarrow$ a geometric proof of the result of Enomoto-Satoh : $(ES)\circ\tau = 0$ on $\mathrm{gr}(\mathcal{K})$

§3. The divergence cocycle (for $\Sigma_{0,3}$)

$n \geqq 2$

$S = \Sigma_{0,n+1}$

$=$



$* \in \partial S$

$\pi = \pi_1(S, *)$ free group of rank n

$\gamma_k \in \pi$, $0 \leq k \leq n$, as on the left

$x_k := [\gamma_k] \in H = H_1(S : \mathbb{Q})$

$x_0 = -\sum_{k=1}^{n} x_k \in H$

[Alekseev-Torossian]

$\mathrm{tder}_n := \{ D \in \mathrm{Der}(\hat{\mathcal{L}}) ;\ 1 \leq {}^\forall k \leq n,\ {}^\exists u_k \in \hat{\mathcal{L}}\ \ D(x_k) = [x_k, u_k] \}$

tangential derivations

$D = (u_1, u_2, \cdots, u_n)$

$\mathrm{TAut}_n := \left\{ \varphi \in \mathrm{Aut}(\hat{\mathcal{L}}) \subset \mathrm{Aut}(\hat{T}) ;\ \begin{array}{l} 1 \leq {}^\forall k \leq n.\ {}^\exists g_k \in \exp(\hat{\mathcal{L}}) \\ \varphi(x_k) = g_k^{-1} x_k g_k \end{array} \right\}$

$\mathrm{sder}_n := \{ D \in \mathrm{tder}_n : D(x_0) = 0 \}$

(normalized) special derivations

<u>Observation</u> (Kuno)

$D = (u_1, u_2, \cdots, u_n) \in \mathrm{tder}_n$

$D \in \mathrm{sder}_n \iff \sum_{k=1}^{n} x_k \otimes u_k \in N(\hat{T})$

cyclic invariant

Definition

$\theta : \pi \to \hat{T}$ special expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) θ : group-like expansion

2) $1 \le {}^\forall k \le n$, ${}^\exists g_k \in \exp(\hat{\mathcal{L}})$ $\theta(\gamma_k) = g_k e^{x_k} g_k^{-1}$

3) $\theta(\gamma_0) = e^{x_0} = \exp\left(-\sum_{k=1}^{n} x_k\right) \in \hat{T}$

Theorem (Kuno-K., Massuyeau-Turaev)

$\theta : \pi \to \hat{T}$ special expansion

$\Longrightarrow$ Under the isomorphism $-N\theta : \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} N(\hat{T})$,

the Goldman bracket is given by

$$\left[\sum_{i=1}^{n} x_i \otimes u_i, \sum_{i=1}^{n} x_i \otimes v_i\right] = N\left(\sum_{i=1}^{n} u_i [x_i, v_i]\right)$$

for $\sum_{i=1}^{n} x_i \otimes u_i$, $\sum_{i=1}^{n} x_i \otimes v_i \in N(\hat{T}) \subset H \otimes \hat{T}$

Corollary

$$0 \to \hat{\bigoplus_{i=1}^{n}} \mathbb{Q}N(x_i^2) \to N(\hat{T}) \to \mathrm{sder}_n \to 0$$

$$\sum x_i \otimes u_i \mapsto (u_1, \cdots, u_n)$$

central extension of Lie algebras

(Lie bracket on sder_n $\leftarrow\!\!\!|$ Goldman bracket)

<u>divergence cocycle</u> ([Alekseev-Torossian])

⊙ $tr_n := \hat{T}_{\geq 1}/\overline{[\hat{T}, \hat{T}]}$ $(\cong N(\hat{T}) \cong \widehat{\mathbb{Q}\hat{\pi}})$

$tr : \hat{T}_{\geq 1} \to tr_n$: quotient map

$div : tder_n \to tr_n$ divergence cocycle

$$div(D) = div(u_1, u_2, \cdots, u_n) := tr\left(\sum_{k=1}^{n} x_k (\partial_k u_k)\right)$$

<u>where</u> $\partial_k (x_{k_1} \cdots x_{k_m}) := \delta_{k k_m} x_{k_1} \cdots x_{k_{m-1}}$

⊙ $\dot{\leadsto} : \hat{T}_{\geq 1} \times \hat{T}_{\geq 1} \to \hat{T}_{\geq 1}$ associative multiplication $\leadsto$ adaptation of Massuyeau-Turaev's $\dot{\leadsto}$ on $\Sigma_{g,1}$

defined by

$$x_{i_1} \cdots x_{i_l} \dot{\leadsto} x_{j_1} \cdots x_{j_m} \overset{\text{def}}{=} -\delta_{i_m j_1} x_{i_1} \cdots x_{i_l} x_{j_2} \cdots x_{j_m} \quad (i_\alpha, j_\beta \in \{1, 2, \cdots, n\})$$

$\left(x_0 = -\sum_{k=1}^{n} x_k\right.$: unit w.r.t. $\left.\dot{\leadsto}\right)$

$$div(u_1, u_2, \cdots, u_n) = -tr\left(\sum_{k=1}^{n} u_k \dot{\leadsto} x_k\right) = -N \circ (\dot{\leadsto}) \circ \left(\sum_{k=1}^{n} u_k \otimes x_k\right)$$

$$div|_{sder_n} : N(H \otimes \hat{\mathcal{L}}) \subset H \otimes \hat{\mathcal{L}} \xrightarrow{\dot{\leadsto}} \hat{T} \xrightarrow{-N} \hat{T}$$

⊙ div : 1-cocycle

$\Longrightarrow$ $\widehat{tr_n} := tr_n \oplus \mathbb{Q}c$ extension by div.

$F \in TAut_n \mapsto j(F) := F \cdot c - c \in tr_n$

$D \in tder_n \mapsto j(e^D) = \frac{e^D - 1}{D} \cdot (div(D))$

$\underline{n=2}$ $tr_1 = \mathbb{Q}[[x_1]]_{\geq 1} \ni f$

$\tilde{\delta} : tr_1 \to tr_2$ $(\tilde{\delta} f)(x_1, x_2) \overset{\text{def}}{=} f(x_1) + f(x_2) - f(ch(x_1, x_2))$

<u>where</u> $ch : \hat{\mathcal{L}} \times \hat{\mathcal{L}} \to \hat{\mathcal{L}}$ Baker-Campbell-Hausdorff map

$$ch(u, v) := \log(e^u e^v)$$

<u>Kashiwara-Vergne problem</u> (in a formulation by Alekseev-Torossian)

Find an automorphism $F \in TAut_2$ satisfying

(i) $F(x_1 + x_2) = ch(x_1, x_2)$

(ii) $j(F) = \tilde{\delta}\left(\frac{1}{2} \sum_{k=1}^{\infty} \frac{B_{2k}\, x^{2k}}{2k \cdot (2k)!}\right)$. <u>where</u> B_{2k} : Bernoulli number

<u>Condition (i) $\longleftrightarrow$ Goldman bracket</u>

(<u>$n \geq 1$</u> standard expansion

$\theta_{std} : \pi \to \hat{T}$, $\gamma_k \mapsto e^{x_k}$ $(1 \leq {}^\forall k \leq n)$

Condition (i) $\Longleftrightarrow$ $F^{-1} \circ \theta_{std}$: special expansion.

<u>Condition (ii) $\overset{?}{\longleftrightarrow}$ Turaev cobracket</u> --- 2 evidences

<u>1st evidence</u>

$\mu : \mathbb{Q}\pi \to \mathbb{Q}\pi \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1})$ coaction (ess. due to Turaev) $\overset{\text{alt. part}}{\rightsquigarrow}$ Turaev cobracket δ

$\gamma \in \pi = \pi_1(S, *)$ in general position

$\Gamma_\gamma := \{\text{double points of } \gamma\} \ni p$. $0 \leq \overset{\exists!}{t_1^p} \lneqq \overset{\exists!}{t_2^p} \leq 1$ $\gamma(t_1^p) = \gamma(t_2^p) = p$

$$\mu(\gamma) \overset{\text{def}}{=} -\sum_{p\in\Gamma_\gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))\ \gamma_{0t_1^p}\gamma_{t_2^p 1} \otimes |\gamma_{t_1^p t_2^p}|' \in \mathbb{Z}\pi \otimes \mathbb{Z}\hat{\pi}'$$

$\mu^{\theta_{std}}$: coaction on $\hat{T}$ induced by $\theta_{std} : \widehat{\mathbb{Q}\pi} \cong \hat{T}$.

<u>Proposition</u> (K.) $\forall k_1, \cdots, \forall k_m \in \{1,2\}$

$$\mu^{\theta_{std}}(x_{k_1}\cdots x_{k_m}) = -(1\otimes N)\sum_{1\le i<j\le m}(x_{k_1}\cdots x_{k_{i-1}} \hat{\otimes} 1)\left[(1\otimes\iota)\Delta\left(x_{k_i} \overset{\circ}{\rightsquigarrow} \left(-x_1-x_2+\frac{x_1^2}{e^{-x_1}-1}+\frac{x_2^2}{e^{-x_2}-1}\right) \overset{\circ}{\rightsquigarrow} x_{k_j}\right)\right](x_{k_{j+1}}\cdots x_{k_m}\otimes x_{k_{i+1}}\cdots x_{k_{j-1}})$$
$$-\frac{1}{2}\sum_{i=1}^{m} x_{k_1}\cdots x_{k_{i-1}} x_{k_{i+1}}\cdots x_{k_m}\otimes N x_{k_i}$$
$$+\sum_{i=1}^{m}\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}\sum_{p=0}^{2n-1}(-1)^p\binom{2n}{p} x_{k_1}\cdots x_{k_{i-1}} x_{k_i}^{p} x_{k_{i+1}}\cdots x_{k_m}\otimes N x_{k_i}^{2n-p}$$

where $\iota : \hat{T} \to \hat{T}$ antipode.

<u>2nd evidence</u> --- "regular" Turaev cobracket

<u>Theorem</u> (K.) $n \geqq 2$, f : framing coming from $\Sigma_{0,n+1} \overset{\text{standard embedding}}{\subset} \mathbb{R}^2$

$\Longrightarrow$ The value at $p=0$ of the lowest degree term of δ^{+,θ_f} is

$$\delta_{(-1)}^{+,\theta_f}(N^+(X_1\cdots X_m))\big|_{p=0}$$
$$= \sum_{1\le i<j\le m} N^+(X_i \rightsquigarrow X_j X_{j+1}\cdots X_{i-1}) \hat{\otimes} N^+(X_{i+1}\cdots X_{j-1}) - N^+(X_{j+1}\cdots X_{i-1}) \hat{\otimes} N^+(X_j \rightsquigarrow X_i X_{i+1}\cdots X_{j-1})$$

(the $N^+(\hat{T})\otimes N^+(1)$ - component) $= -\operatorname{div} N^+(X_1\cdots X_m)$ divergence cocycle

$(X_i \in H)$ —