第9回代数・解析・幾何学セミナー
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"The Goldman-Turaev Lie bialgebra and the mapping class group"

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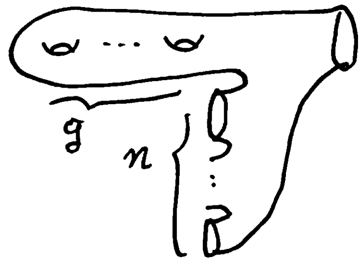
久野雄介氏（津田塾大・学芸）との共同研究
joint work with Yusuke Kuno (Tsuda College)

survey paper: arXiv: 1304.1885

S : compact connected oriented surface with $\partial S \neq \phi$

$\underset{\text{Classification Theorem}}{\Longrightarrow}$ $S \cong \Sigma_{g,n+1} =$ 

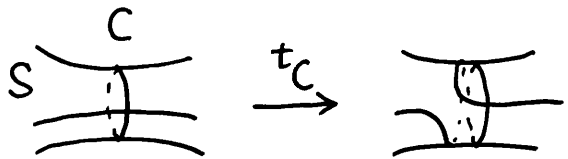
$\mathcal{M}(S) \overset{\text{def}}{=} \mathrm{Diff}(S, \mathrm{id} \text{ on } \partial S) / \text{isotopy fixing } \partial S$ <u>pointwise</u>

the mapping class group.

◎ <u>Beginning</u> — <u>Kuno's observation</u>

$C \subset S \setminus \partial S$ simple closed curve

$t_C \in \mathcal{M}(S)$ right-handed Dehn twist



<u>Kuno</u>: 「 t_C seems to be related to $\frac{1}{2}(\log C)^2$ 」

cf). <u>Picard - Letschetz formula</u>

$t_{C*} : H_1(S) \to H_1(S)$, $u \mapsto u - (u \cdot [C])[C]$

<u>Kuno</u>: an explicit formula for the extended 1^{st} Johnson homom. of t_C, where $\frac{1}{2}(\log C)^2$ appears

◎ <u>Generalization</u> (Kuno-K.)

$\Uparrow$

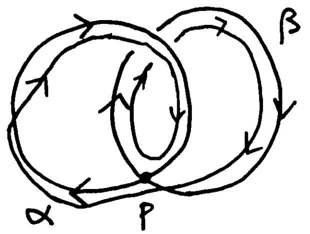
<u>Goldman Lie algebra</u>

$\hat{\pi} = \hat{\pi}(S) \overset{\text{def}}{=} [S^1, S] = \{\ell : S^1 \to S \text{ conti. map}\}/\text{homotopy} = \pi_1(S)/\text{conj}$

free homotopy set of free loops on S

$|\ | : \pi_1(S) \to \hat{\pi}(S)$ quotient map $=$ the forgetful map of base point

$\alpha, \beta \in \hat{\pi}$ in general position



$\varepsilon = -1$

$$[\alpha, \beta] \overset{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon(p : \alpha, \beta) \, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}$$

$\varepsilon(p : \alpha, \beta) \in \{\pm 1\}$ local intersection number

$\alpha_p, \beta_p \in \pi_1(S, p)$

[Goldman (1) $[\ ,\ ]$: well-defined
(2) $(\mathbb{Z}\hat{\pi}, [\ ,\ ])$: Lie algebra $\Rightarrow$ Goldman Lie algebra

$1 \in \hat{\pi}$ constant loop. $1 \in \mathrm{Center}(\mathbb{Z}\hat{\pi})$ $(\because\ 1 \cap {}^\forall \alpha = \phi)$

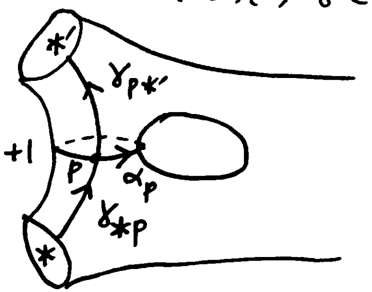
$\mathbb{Z}\hat{\pi} / \mathbb{Z}1$: Lie algebra

$I = [0, 1] \subset \mathbb{R}$, $\ast, \ast' \in \partial S$

$$\Pi S(\ast, \ast') \overset{\text{def}}{=} [(I, 0, 1), (S, \ast, \ast')] = \{\ell : I \to S \text{ conti. map} \,;\, \ell(0) = \ast, \ell(1) = \ast'\} / \text{homotopy rel } \partial$$

fundamental groupoid

$\alpha \in \hat{\pi}$, $\gamma \in \Pi S(\ast, \ast')$



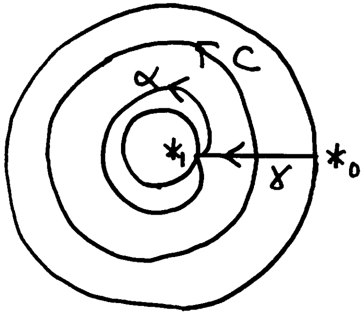
$\varepsilon = +1$

$$\sigma(\alpha)(\gamma) \overset{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon(p : \alpha, \gamma) \, \gamma_{\ast p} \alpha_p \gamma_{p \ast'} \in \mathbb{Z}\Pi S(\ast, \ast')$$

[Kuno-K. (1) σ : well-defined
(2) $\sigma : \mathbb{Z}\hat{\pi} \to \mathrm{Der}(\mathbb{Z}\Pi S|_{\partial S})$ Lie algebra homomorphism

$\sigma(1) = 0$, $\sigma : \mathbb{Z}\hat{\pi}/\mathbb{Z}1 \to \mathrm{Der}(\mathbb{Z}\Pi S|_{\partial S})$

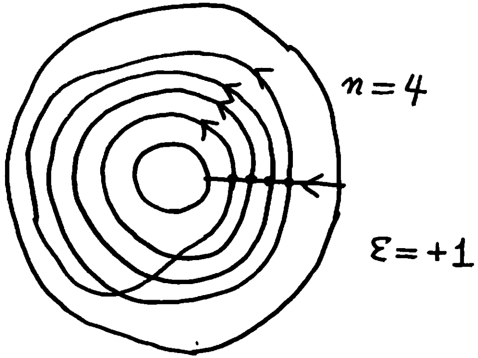
Example $S = \Sigma_{0,2}$



$C = |\alpha| \in \hat{\pi}$

$n \geqq 0$

$$\begin{cases} \sigma(C^n)(\alpha) = 0 & (\because\ \alpha \cap C = \phi) \\ \sigma(C^n)(\gamma) = n\gamma\alpha^n \end{cases}$$



$f(x)$: "function" in x

$$\Rightarrow \begin{cases} \sigma(f(C))(\alpha) = 0 \\ \sigma(f(C))(\gamma) = \gamma\alpha f'(\alpha) \end{cases}$$

compare

Dehn twist

$$\begin{cases} t_C(\alpha) = \alpha \\ t_C(\gamma) = \gamma\alpha \end{cases} \qquad \begin{cases} (\log t_C)(\alpha) = 0 \\ (\log t_C)(\gamma) = \gamma \log\alpha \end{cases}$$

$x f'(x) = \log x$

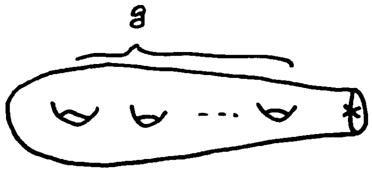
$f(x) = \int \frac{1}{x} \log x \, dx = \frac{1}{2}(\log x)^2$

Theorem ($\Sigma_{g,1}$ Kuno-K., general Kuno-K., Massuyeau-Turaev)

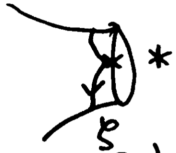
$$(t_C)_* = e^{\sigma(\frac{1}{2}(\log C)^2)} \in \mathrm{Aut}(\widehat{\mathbb{Q}\pi S|_{\partial S}})$$

completion w.r.to $I\pi_1(S)$ the augmentation ideal.

$g \geq 1$

$S = \Sigma = \Sigma_{g,1} =$  $* \in \partial\Sigma$

$\pi := \pi_1(\Sigma, *)$, free group of rank $2g$



$\zeta \in \pi$ negative ∂-loop

$\widehat{\mathbb{Q}\pi} := \varprojlim_{m\to\infty} \mathbb{Q}\pi / (I\pi)^m$, $I\pi := \mathrm{Ker}(\varepsilon : \mathbb{Q}\pi \to \mathbb{Q})$

completed group ring $\qquad \sum a_x x \mapsto \sum a_x$

$\Delta : \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi} \hat{\otimes} \widehat{\mathbb{Q}\pi}$ coproduct, $(\Delta x = x \hat{\otimes} x, \ \forall x \in \pi)$

Tensorial description

$H := H_1(\Sigma; \mathbb{Q}) = (\pi / [\pi, \pi]) \otimes_{\mathbb{Z}} \mathbb{Q}$

$x \in \pi \longmapsto [x] := (x \bmod [\pi, \pi]) \otimes 1 \in H$

$\hat{T} = \hat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$ complete tensor algebra

$\Delta : \hat{T} \to \hat{T} \hat{\otimes} \hat{T}$, coproduct, $(\Delta X = X \hat{\otimes} 1 + 1 \hat{\otimes} X \quad \forall X \in H)$

$\omega := \sum_{i=1}^{g} A_i B_i - B_i A_i \in H^{\otimes 2} \subset \hat{T}$ symplectic form

indep. of the choice of symplectic basis $\{A_i, B_i\}_{i=1}^{g} \subset H$

Definition (Massuyeau)

$\theta : \pi \to \hat{T}$ symplectic expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 0) $\theta : \pi \to \hat{T}$ map

1) $\forall x \in \pi \quad \theta(x) = 1 + [x] +$ higher terms

2) $\forall x, \forall y \in \pi \quad \theta(xy) = \theta(x)\theta(y)$

3) (group-like) $\forall x \in \pi \quad \Delta\theta(x) = \theta(x) \hat{\otimes} \theta(x) \in \hat{T} \hat{\otimes} \hat{T}$

4) (symplectic) $\theta(\zeta) = \exp(\omega) \left(= \sum_{k=0}^{\infty} \frac{1}{k!} \omega^k\right) \in \hat{T}$

examples (1) (K.) harmonic Magnus expansion $/\mathbb{R}$

(2) (Massuyeau) LMO functor

(3) (Kuno) combinatorial

$\Longrightarrow \quad \theta : (\widehat{\mathbb{Q}\pi}, \widehat{\mathbb{Q}\langle\zeta\rangle}) \xrightarrow{\cong} (\hat{T}, \mathbb{Q}[[\omega]])$

isomorphism of pairs of complete Hopf algebra

$N : \hat{T} \to \hat{T}$ linear map (cyclic symmetrizer)

$N|_{H^{\otimes 0}} := 0, \quad N(X_1 X_2 \cdots X_m) := \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1} \quad (X_j \in H)$

$$\begin{array}{ccccc}
\mathrm{Der}(\hat{T}) & \cong & H^* \otimes \hat{T} & \overset{\text{P.d.}}{=} & H \otimes \hat{T} \\
\cup & & \circlearrowleft & & \cup \\
\mathrm{Der}_\omega(\hat{T}) & & = & & N(\hat{T})
\end{array}$$

$\widehat{\mathbb{Q}\hat{\pi}} := \varprojlim_{m\to\infty} \mathbb{Q}\hat{\pi}/(\mathbb{Q}1 + |I\pi^m|)$ completed Goldman Lie algebra

Theorem (Kuno–K.) $\theta : \pi \to \hat{T}$ symplectic expansion

(1) $-N\theta : \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} N(\hat{T}) = \mathrm{Der}_\omega(\hat{T})$, $|x| \longmapsto -N\theta(x)$,

isomorphism of Lie algebras

(2)
$$\begin{array}{ccc} \widehat{\mathbb{Q}\hat{\pi}} \otimes \widehat{\mathbb{Q}\pi} & \xrightarrow{\sigma} & \widehat{\mathbb{Q}\pi} \\ {\scriptstyle -N\theta\otimes\theta}\downarrow\wr & \circlearrowright & \wr\downarrow{\scriptstyle\theta} \\ \mathrm{Der}_\omega(\hat{T}) \otimes \hat{T} & \xrightarrow[\text{derivation}]{} & \hat{T} \end{array}$$

$\mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi}) := \{ D : \text{continuous derivation of } \widehat{\mathbb{Q}\pi} \; ; \; D(\zeta) = 0 \}$

Corollary $\sigma : \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi})$

isomorphism of Lie algebras

Johnson homomorphism

$$L^+(\Sigma) := \left\{ u \in \mathrm{Ker}\left(\widehat{\mathbb{Q}\hat{\pi}} \to \mathbb{Q}\hat{\pi} / |\mathbb{Q}1 + [I\pi]^3|\right) ; (\sigma(u) \hat{\otimes} 1 + 1 \hat{\otimes} \sigma(u))\Delta = \Delta\sigma(u) \right\}$$

pro-nilpotent Lie subalgebra.

$\xRightarrow[\text{exp-log}]{}$ pro-nilpotent Lie group

$\mathrm{gr}(L^+(\Sigma)) = \mathfrak{h}^+_{g,1}$ (Morita's Lie algebra) = positive part of Kontsevich's "Lie"

$\widehat{DN} : \mathcal{M}(\Sigma) \to \mathrm{Aut}(\widehat{\mathbb{Q}\pi})$ injective (Dehn-Nielsen)

$\sigma : \mathbb{Q}\hat{\pi} \xrightarrow{\cong} \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi})$

(ex) $(\sigma^{-1} \circ \log \circ \widehat{DN})(t_C) = \frac{1}{2}(\log C)^2 \in \widehat{\mathbb{Q}\hat{\pi}}$)

$\mathcal{I}(\Sigma) := \mathrm{Ker}(\mathcal{M}(\Sigma) \to \mathrm{Aut}(H_1(\Sigma;\mathbb{Q})))$ Torelli group

$$\begin{tikzcd}
\mathcal{I}(\Sigma) \arrow[r, "\sigma^{-1}\circ\log\circ\widehat{DN}"] \arrow[dr, dashed, "\exists!\,\tau"'] & \widehat{\mathbb{Q}\hat{\pi}} \\
\arrow[r, phantom, "\circlearrowleft"] & L^+(\Sigma) \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$

$L^+(\Sigma)$ ← pro-nilpotent Lie group

$\tau : \mathcal{I}(\Sigma) \to L^+(\Sigma)$ injective group homomorphism

$\mathrm{gr}(\tau) : \mathrm{gr}(\mathcal{I}(\Sigma)) \to \mathrm{gr}(L^+(\Sigma))$ classical Johnson homomorphism

w.r.t. Johnson filtration

<u>injective</u>

Morita $gr(\tau) : gr(\mathcal{I}(\Sigma)) \to gr(L^+(\Sigma))$ is not surjective.

more precisely

$$Tr : gr(L^+(\Sigma)) \xrightarrow[\text{surjective}]{} \bigoplus_{m=1}^{\infty} \mathrm{Sym}^{2m+1}(H) \qquad \text{Morita trace}$$

(constructed in an algebraic way.)

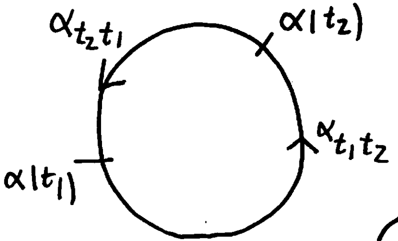
$$Tr \circ gr(\tau) = 0$$

Turaev cobracket

S : (connected) oriented surface , $1 \in \hat{\pi}(S)$ constant loop (S : connected)

$| \ |' : \mathbb{Z}\pi_1(S) \to \mathbb{Z}\hat{\pi} \to \mathbb{Z}\hat{\pi}/\mathbb{Z}1 =: \mathbb{Z}\hat{\pi}'$ quotient map

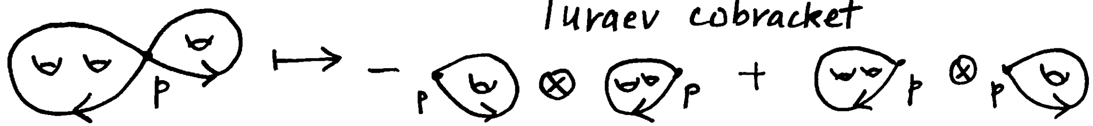
$\alpha \in \hat{\pi}$ in general position



$$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 \,;\, t_1 \neq t_2 ,\, \alpha(t_1) = \alpha(t_2)\}$$

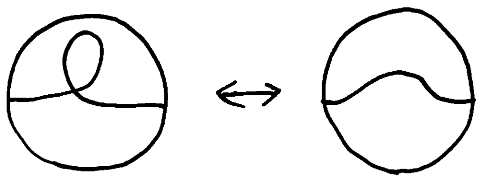
$$\delta(\alpha) \overset{\text{def}}{=} \sum_{(t_1,t_2) \in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \, |\alpha_{t_1 t_2}|' \otimes |\alpha_{t_2 t_1}|' \in \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$$

Turaev cobracket



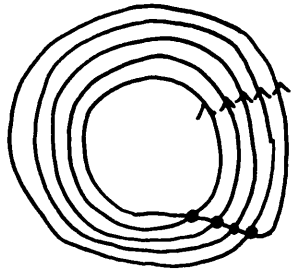
◎ The reason why we take the quotient $\mathbb{Z}\hat{\pi}' = \mathbb{Z}\hat{\pi}/\mathbb{Z}1$:

birth - death move of a monogon



[<u>Turaev</u> (1) δ ; well-defined
(2) $(\mathbb{Z}\hat{\pi}', [\ ,\ ], \delta)$: Lie bialgebra

(<u>Chas</u> : involutive : $[\ ,\ ] \circ \delta = 0$)



(*) $\alpha \in \hat{\pi}$: simple loop , $n \in \mathbb{Z}$
$\Rightarrow \delta(\alpha^n) = 0$

($\because$) $n \geq 0$, $\delta(\alpha^n) = \sum_{k=1}^{n-1} \alpha^k \otimes \alpha^{n-k} - \sum_{k=1}^{n-1} \alpha^{n-k} \otimes \alpha^k = 0$

$\delta(\alpha^{-n}) = \delta((\alpha^{-1})^n) = 0$ ($\because$) α^{-1} : simple) //

[<u>Theorem</u> (Kuno-K.)
$\delta \circ \tau = 0 : \mathcal{I}(\Sigma) \to L^+(\Sigma) \to \widehat{\mathbb{Q}\hat{\pi}} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}$

($\because$) <u>Johnson</u> $\mathcal{I}(\Sigma)$ is generated by

<u>BP-maps</u>

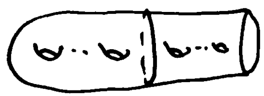
C_1

C_2

$t_{C_1} \circ t_{C_2}^{-1}$

and

<u>BSCC-maps</u>



C_0

t_{C_0}

$(\delta \circ \tau)(t_{C_1} \circ t_{C_2}^{-1})$
$= \delta(\frac{1}{2}(\log C_1)^2) - \delta(\frac{1}{2}(\log C_2)^2) \overset{(*)}{=} 0$

$(\delta \circ \tau)(t_{C_0}) = \delta(\frac{1}{2}(\log C_0)^2) \overset{(*)}{=} 0$

//

$\mathcal{K}(\Sigma)$:= the subgroup generated by BSCC-maps.
Johnson kernel.

$S = \Sigma = \Sigma_{g,1}$

$\theta : \pi \to \widehat{T}$ symplectic expansion

$\delta^\theta := ((-N\theta)\hat{\otimes}(-N\theta)) \circ \delta \circ (-N\theta)^{-1} : \mathrm{Der}_\omega(\widehat{T}) \to \mathrm{Der}_\omega(\widehat{T}) \hat{\otimes} \mathrm{Der}_\omega(\widehat{T})$

Theorem (Massuyeau–Turaev, Kuno–K., independently)

$\forall X_1, \cdots \forall X_k \in H$

$$\delta^\theta(N(X_1 \cdots X_k)) = \underbrace{\delta^{alg}(N(X_1 \cdots X_k))}_{\text{degree } k-2} + \underbrace{\text{higher terms}}_{\text{degree } \geq k+1}$$

where

$$\delta^{alg}(N(X_1 \cdots X_k)) = \sum_{i<j} (X_i \cdot X_j) \left\{ \begin{array}{l} N(X_{i+1} \cdots X_{j-1}) \hat{\otimes} N(X_{j+1} \cdots X_k X_1 \cdots X_{i-1}) \\ - N(X_{j+1} \cdots X_k X_1 \cdots X_{i-1}) \hat{\otimes} N(X_{i+1} \cdots X_{j-1}) \end{array} \right\}$$

Schedler's cobracket ($\Leftarrow$ quiver theory)

$\Uparrow$ Massuyeau–Turaev's tensorial description of the homotopy intersection form

Theorem (Kuno–K.)

The Morita trace is included in δ^{alg}

Enomoto-Satoh

$$\widehat{\mathrm{Tr}} = \sum_{m=1}^{\infty} \widehat{\mathrm{Tr}}_m : \mathrm{gr}(L^+(\Sigma)) \longrightarrow \bigoplus_{m=1}^{\infty} (H^{\otimes m})^{\mathbb{Z}/m} \quad \left(= \bigoplus_{m=1}^{\infty} N(H^{\otimes m}) = \mathrm{gr}(\widehat{\mathbb{Q}\pi})\right)$$

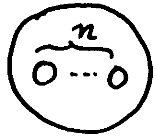
Enomoto-Satoh trace (← refinement of the Morita trace)

$$\widehat{\mathrm{Tr}}_{\geq 2} \circ \mathrm{gr}(\tau) = 0 : \mathrm{gr}(\mathfrak{g}(\Sigma)) \longrightarrow \bigoplus_{m=2}^{\infty} (H^{\otimes m})^{\mathbb{Z}/m}$$

constructed in a purely algebraic way

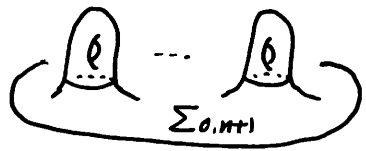
(**Remark** $\widehat{\mathrm{Tr}}_1 \circ \mathrm{gr}(\tau) \neq 0$ on $\mathfrak{g}(\Sigma)$
$= 0$ on $\mathcal{K}(\Sigma)$)

divergence cocycle in the Kashiwara-Vergne problem [Alekseev-Torossian]

$S = \Sigma_{0,n+1} =$ 

$$\mathrm{div} : \mathrm{gr}(L^+(\Sigma_{0,n+1})) \longrightarrow \bigoplus_{m=1}^{\infty} (H_1(\Sigma_{0,n+1};\mathbb{Q})^{\otimes m})^{\mathbb{Z}/m} \quad \left(= \mathrm{gr}(\widehat{\mathbb{Q}\pi}(\Sigma_{0,n+1}))\right)$$

$\iota : S = \Sigma_{0,n+1} \hookrightarrow \Sigma_{n,1}$ capping map

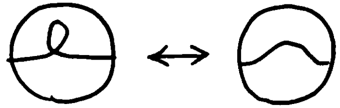


$\Big\} \; \Sigma_{n,1}$

$\iota^* \widehat{\mathrm{Tr}} = \mathrm{div} + ($a low degree term $H_1(\Sigma_{0,n+1})^{\otimes 2} \longrightarrow H_1(\Sigma_{0,n+1})$)

$(\; \iota^* \widehat{\mathrm{Tr}} \text{ "} \doteqdot \text{" } \mathrm{div} \;)$

<u>Enomoto</u> The Enomoto-Satoh trace is <u>not</u> included in δ^{alg}

($\Leftarrow$ monogon )

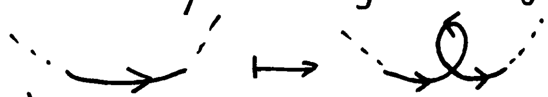
<u>Topological re-construction of the Enomoto-Satoh trace ?</u>

$\Nearrow$

<u>regular/immersed version of the Turaev cobracket (K.)</u>

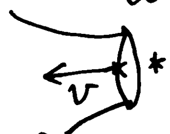
$\hat{\pi}^+ := \{ \ell : S^1 \to \Sigma : C^\infty \text{immersion} \} / \text{regular homotopy}$

$\langle r \rangle (\cong \mathbb{Z}) \curvearrowright \hat{\pi}^+$ free action. by inserting a monogon into an immersed loop



- $\hat{\pi}^+ / \langle r \rangle = \hat{\pi} \; (= [S^1, \Sigma])$
- $1 :=$  null-homotopic loop with rotation number 0
- f : framing of $T\Sigma \Rightarrow \hat{\pi}^+ \cong \hat{\pi} \times \underset{\sim}{\mathbb{Z}}$ $\leftarrow$ rotation number w.r.to f.

$0 \neq v \in T_* \Sigma$ inward vector



$\pi^+ := \{ \ell : (I, \partial I) \to (\Sigma, *) : C^\infty \text{immersion } \dot{\ell}(0) = -\dot{\ell}(1) = v \} / \text{regular homotopy fixing } \dot{\ell}(0) \text{ and } \dot{\ell}(1)$

$\pi^+ / \langle r \rangle = \pi = \pi_1(\Sigma, *)$

$\pi^+ \cong \pi \times \mathbb{Z} \Leftarrow f$: framing of $T\Sigma$

$[\ ,\ ]^+ : \mathbb{Q}\hat{\pi}^+ \otimes_{\mathbb{Q}\langle r\rangle} \mathbb{Q}\hat{\pi}^+ \longrightarrow \mathbb{Q}\hat{\pi}^+$ the regular/immersed Goldman bracket

$$\begin{tikzcd}
\delta^+ : \mathbb{Q}\hat{\pi}^+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Q}\hat{\pi}^+ \otimes_{\mathbb{Q}\langle r\rangle} \mathbb{Q}\hat{\pi}^+ \\
\mathbb{Q}\hat{\pi}^+ / \mathbb{Q}\langle r\rangle 1 \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd}$$

the regular/immersed Turaev cobracket

(rmk monogon $\neq 0 \in \mathbb{Q}\hat{\pi}^+$)

$\sigma^+ : \mathbb{Q}\hat{\pi}^+ \otimes_{\mathbb{Q}\langle r\rangle} \mathbb{Q}\pi^+ \longrightarrow \mathbb{Q}\pi^+$ the regular/immersed action σ

$\widehat{\mathbb{Q}\hat{\pi}^+}$, $\widehat{\mathbb{Q}\pi^+}$: completions w.r.to the augmentation ideal $I\pi^+$

$L^+(\Sigma)^+ < \widehat{\mathbb{Q}\hat{\pi}^+} / \widehat{\mathbb{Q}\langle r\rangle} 1$: inverse image of $L^+(\Sigma)$

$\Rightarrow \sigma^+ : \widehat{\mathbb{Q}\hat{\pi}^+} / \widehat{\mathbb{Q}\langle r\rangle} 1 \xrightarrow{\cong} \mathrm{Der}_\partial^{\widehat{\mathbb{Q}\langle r\rangle}} (\widehat{\mathbb{Q}\pi^+})$ isomorphism of Lie algebras

$\Rightarrow \tau^+ : \mathcal{g}(\Sigma) \longrightarrow L^+(\Sigma)^+$

$\delta^+ \circ \tau^+ = 0 : \mathcal{g}(\Sigma) \to L^+(\Sigma)^+ \longrightarrow \widehat{\mathbb{Q}\hat{\pi}^+} \otimes_{\widehat{\mathbb{Q}\langle r\rangle}} \widehat{\mathbb{Q}\hat{\pi}^+}$

f : framing of TS

$\Longrightarrow$ $\varepsilon_f : \widehat{\mathbb{Q}\hat{\pi}^+} \longrightarrow \widehat{\mathbb{Q}\langle r\rangle}$, augmentation map , $\alpha \in \hat{\pi}^+ \longmapsto r^{\mathrm{rot}_f(\alpha)}$

$s_f : \widehat{\mathbb{Q}\hat{\pi}} \longrightarrow \widehat{\mathbb{Q}\hat{\pi}^+}/\widehat{\mathbb{Q}\langle r\rangle}1$ embedding of Lie algebra

$$
\begin{tikzcd}
\mathfrak{g}(\Sigma) \arrow[r, hook, "\tau^+"] & L^+(\Sigma)^+ \arrow[r, "\delta^+"] & \widehat{\mathbb{Q}\hat{\pi}^+} \hat{\otimes}_{\widehat{\mathbb{Q}\langle r\rangle}} \widehat{\mathbb{Q}\hat{\pi}^+} \arrow[d, "\varepsilon_f \hat{\otimes} 1"] \\
\mathcal{K}(\Sigma) \arrow[u, hook] \arrow[r, hook, "\tau"'] \arrow[ur, phantom, "\circlearrowright"] & L^+(\Sigma) \arrow[u, "s_f"'] \arrow[ur, phantom, "\circlearrowright"] \arrow[dr, "ES_f"'] & \widehat{\mathbb{Q}\hat{\pi}^+}/\widehat{\mathbb{Q}\langle r\rangle}1 \arrow[d, "\text{forgetting } C^\infty \text{ structure}"] \\
& & \widehat{\mathbb{Q}\hat{\pi}}
\end{tikzcd}
$$

Theorem (K.)

$\mathrm{gr}(ES_f)$ = the Enomoto-Satoh traces

Remark ◎ $\mathrm{gr}_1(ES_f)$ = Furuta's cocycle associated with f

◎ Similar consideration for $S = \Sigma_{0,n+1}$

$\Longrightarrow$ $\mathrm{gr}(ES_f)$ = the divergence cocycle in the Kashiwara-Vergne problem :

$$
\begin{array}{ccc}
\mathrm{gr}(L^+(\Sigma_{0,n+1})) & \longrightarrow & \mathrm{gr}(\widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{0,n+1})) \\
\| & & \| \\
\text{positive part of } \mathrm{sder}_n & & \mathrm{tr}_n
\end{array}
$$