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"The Goldman–Turaev Lie bialgebra and the mapping class group"

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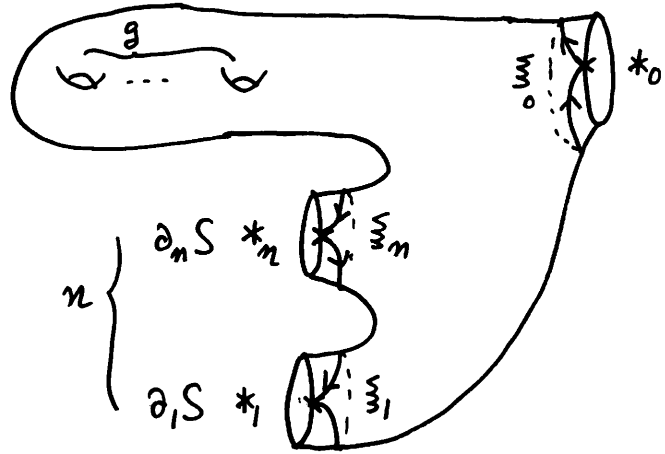
- arXiv : 1304.1885 (survey paper)
- http://www.ms.u-tokyo.ac.jp/~kawazumi/1309LosAngels.pdf

S : compact connected oriented surface with $\partial S \neq \phi$

$\Longrightarrow$ Classification Theorem $\quad \exists g, \exists n \geq 0 \quad S \cong \Sigma_{g,n+1} =$



$$\partial S = \coprod_{j=0}^{n} \partial_j S$$

$*_j \in \partial_j S, \quad 0 \leq j \leq n$

$E := \{*_j\}_{j=0}^{n} \subset \partial S$

$\xi_j \in \pi_1(S, *_j)$ boundary loop, $0 \leq j \leq n$

$\pi_1(S)$: free group of rank $2g+n$

$$\mathcal{M}(S) := \pi_0 \mathrm{Diff}(S, \mathrm{id} \text{ on } \partial S)$$
$$= \{\varphi : S \to S : \text{ori. pres. diffeo}, \ \varphi|_{\partial S} = \mathrm{id}_{\partial S}\} \Big/ \text{isotopy fixing } \partial S \ \underline{\text{pointwise}}$$

the mapping class group of S

$$\mathcal{I}^L(S) := \{\varphi \in \mathcal{M}(S) ; \ \varphi_* = \mathrm{id} \text{ on } \left(H_1(S:\mathbb{Z}) \Big/ \sum_{j=0}^{n} \mathbb{Z}[\xi_j] \right) \}$$

the largest Torelli group of S in the sense of Putman.

Goldman Lie algebra

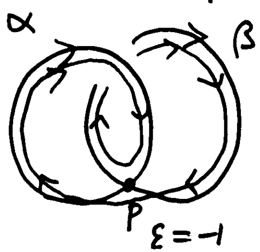
$\hat{\pi} = \hat{\pi}(S) := [S^1, S]$: the free homotopy set of free loops on S

$= \pi_1(S, p)/\text{conj.}$ $(p \in S)$ $(\because S$: connected $)$

$|\ | : \pi_1(S,p) \to \hat{\pi}(S)$, $\gamma \mapsto |\gamma|$, forgetting the basepoint p

$\alpha, \beta \in \hat{\pi}$ (Choose their representatives) in general position

$$[\alpha, \beta] \overset{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon(p : \alpha, \beta) \, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}(S)$$



$\varepsilon(p : \alpha, \beta) \in \{\pm 1\}$: local intersection number

α_p (resp. β_p) $\in \pi_1(S, p)$: based loop along α (resp. β) with basepoint p

Theorem (Goldman)

(1) $[\ ,\ ]$: well-defined (i.e., independent of the choice of representatives)

(2) $(\mathbb{Z}\hat{\pi}(S), [\ ,\ ])$: Lie algebra ($\longrightarrow$ Goldman Lie algebra)

Background : • Wolpert's study on the Weil-Petersson geometry

• Poisson structure on the moduli space of flat G-bundles on S

G : reductive Lie group (e.g., GL_n, SL_n, Sp_g, SO_n, $\cdots$)

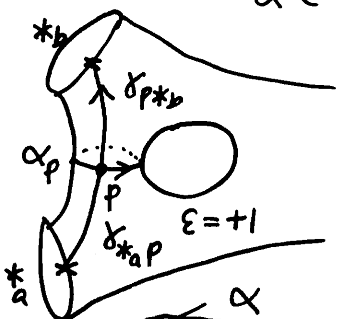
Our approach : (pro-)nilpotent nature of (a subalgebra of) the Goldman Lie algebra.

Key observation: $\mathbb{Z}\hat{\pi}(S)$ acts on $\mathbb{Z}\Pi S(*_a, *_b)$, $0 \le a, b \le n$,

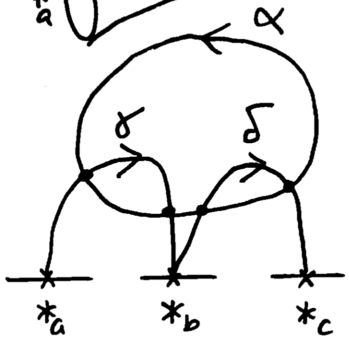
$$\Pi S(*_a, *_b) = \pi_1(S, *_a, *_b) = [([0,1], 0, 1), (S, *_a, *_b)]$$

the based homotopy set of paths from $*_a$ to $*_b$ on S

$\alpha \in \hat{\pi}(S)$, $\gamma \in \Pi S(*_a, *_b)$ in general position



$$\sigma(\alpha)(\gamma) \overset{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon(p : \alpha, \gamma)\, \gamma_{*_a p}\, \alpha_p\, \gamma_{p *_b} \in \mathbb{Z}\Pi S(*_a, *_b)$$



Theorem (Kuno–K.)

(1) σ : well-defined

(2) $\forall \alpha, \forall \beta \in \hat{\pi}(S)$, $\forall \gamma \in \Pi S(*_a, *_b)$, $\forall \delta \in \Pi S(*_b, *_c)$, $0 \le a, b, c \le n$

$$\sigma(\alpha)(\gamma\delta) = \sigma(\alpha)(\gamma)\delta + \gamma\sigma(\alpha)(\delta)$$

$$\sigma([\alpha, \beta])(\gamma) = \sigma(\alpha)(\sigma(\beta)(\gamma)) - \sigma(\beta)(\sigma(\alpha)(\gamma))$$

i.e., $\sigma : \mathbb{Z}\hat{\pi}(S) \to \mathrm{Der}(\mathbb{Z}\Pi S|_E)$ derivation Lie algebra.

Lie algebra homomorphism

where $\mathbb{Z}\Pi S|_E$: small additive category given by

objects : $\mathrm{Ob}(\mathbb{Z}\Pi S|_E) := E = \{*_a\}_{a=0}^{n}$

morphisms : $\mathbb{Z}\Pi S(*_a, *_b)$, the free $\mathbb{Z}$-module over the set $\Pi S(*_a, *_b)$.

<u>Consider</u> $\mathbb{Q}$ (or a field of characteristic 0) instead of $\mathbb{Z}$

($\because$) Later we will use log and exp)

$\sigma : \mathbb{Q}\hat{\pi}(S) \to \mathrm{Der}(\mathbb{Q}\Pi S|_E)$ Lie algebra homomorphism

<u>Some observations</u>

(1) $\mathrm{Image}\ \sigma \subset \mathrm{Der}_\partial(\mathbb{Q}\Pi S|_E) := \{D \in \mathrm{Der}(\mathbb{Q}\Pi S|_E) ;\ D({}^\forall\text{boundary loops}) = 0\}$

($\because$) may choose ${}^\forall\alpha \subset \mathrm{Int}\, S$)

(2) $1 \in \hat{\pi}(S)$ constant loop $\Rightarrow \sigma(1) = 0$

($\because$) may choose ${}^\forall\gamma \cap 1 = \phi$)

(3) $\sigma : \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}1 \to \mathrm{Der}_\partial(\mathbb{Q}\Pi S|_E)$ (injective) <u>not</u> surjective

($\because$) $\alpha \in \hat{\pi}(S)$

$(\gamma \in \Pi S(*_a, *_b) \mapsto (\alpha\cdot\gamma)\gamma) \in \mathrm{Der}_\partial(\mathbb{Q}\Pi S|_E)$

($\because$) $((\alpha\cdot\gamma) + (\alpha\cdot\delta))(\gamma\delta) = \{(\alpha\cdot\gamma)\gamma\}\delta + \gamma\{(\alpha\cdot\delta)\delta\}$)

$= \sigma(\log\alpha) \notin \sigma(\mathbb{Q}\hat{\pi}(S))$ if $[\alpha] \neq 0 \in H_1(S;\mathbb{Q})$

"$\log\alpha$" can be justified by taking a completion of $\mathbb{Q}\hat{\pi}(S)$.

Completion

G : group

$\widehat{\mathbb{Q}G} := \varprojlim_{m\to\infty} \mathbb{Q}G/(IG)^m$: the completed group ring

where $IG := \mathrm{Ker}(\varepsilon : \mathbb{Q}G \to \mathbb{Q})$ augmentation ideal
$$\sum_{x\in G} a_x x \mapsto \sum a_x$$

Classical result (Magnus, Witt, ⋯, Quillen, ⋯)

G : free group of finite rank (e.g., $\pi_1(S)$)

$\Longrightarrow$ $\widehat{\mathbb{Q}G} \cong \prod_{m=0}^{\infty} (G^{abel} \otimes_{\mathbb{Z}} \mathbb{Q})^{\otimes m}$ $\left(=: \widehat{T}(G^{abel} \otimes_{\mathbb{Z}} \mathbb{Q})\right)$

(non-canonical) isomorphism of complete Hopf algebras

$\text{Lie-like}(\widehat{\mathbb{Q}G}) \cong \widehat{\text{Free Lie}}(G^{abel} \otimes_{\mathbb{Z}} \mathbb{Q})$ pro-nilpotent completion of G

$\widehat{\mathbb{Q}\Pi S}|E$:
$\widehat{\mathbb{Q}\hat{\pi}}(S)$: completions with respect to the augmentation ideal $I\pi_1(S, p)$

$m \geq 0$, $0 \leq a, b \leq n$

$F_m \mathbb{Q}\Pi S(*_a, *_b) := \gamma(I\pi_1(S, p))^m \gamma'$

$\mathbb{Q}\hat{\pi}(S)(m) := \mathbb{Q}1 + |(I\pi_1(S, p))^m|$

indep. of $p \in S$, $\gamma \in \Pi S(*, p)$ and $\gamma' \in \Pi S(p, *')$

<u>Lemma</u> $\forall m_1, \forall m_2 \geq 1$

$\sigma(\mathbb{Q}\hat{\pi}(S)(m_1))(F_{m_2}\mathbb{Q}\Pi S(*_a, *_b)) \subset F_{m_1+m_2-2}\mathbb{Q}\Pi S(*_a, *_b)$

$[\mathbb{Q}\hat{\pi}(S)(m_1), \mathbb{Q}\hat{\pi}(S)(m_2)] \subset \mathbb{Q}\hat{\pi}(S)(m_1+m_2-2)$

$$\widehat{\mathbb{Q}\Pi S}(*_a, *_b) := \varprojlim_{m\to\infty} \mathbb{Q}\Pi S(*_a, *_b) / F_m\mathbb{Q}\Pi S(*_a, *_b)$$

$\widehat{\mathbb{Q}\Pi S}|_E$: small $\mathbb{Q}$-linear category given by

<u>objects</u>: $Ob(\widehat{\mathbb{Q}\Pi S}|_E) := E$

<u>morphisms</u>: $\widehat{\mathbb{Q}\Pi S}(*_a, *_b)$, $0 \leq a, b \leq n$,

$\mathrm{Der}(\widehat{\mathbb{Q}\Pi S}|_E)$: the <u>continuous</u> derivation Lie algebra of $\widehat{\mathbb{Q}\Pi S}|_E$

w.r.to $\{F_m\mathbb{Q}\Pi S(*_a, *_b)\}_{m\geq 0}$

$\cup$

$\mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E) := \{D \in \mathrm{Der}(\widehat{\mathbb{Q}\Pi S}|_E) ;\ D(\forall \text{boundary loop}) = 0\}$

$\widehat{\mathbb{Q}\hat{\pi}}(S) \overset{\text{def}}{=} \varprojlim_{m\to\infty} \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\hat{\pi}(S)(m)$ completed Goldman Lie algebra

$m \geq 2$ $\widehat{\mathbb{Q}\hat{\pi}}(S)(m) := \mathrm{Ker}(\widehat{\mathbb{Q}\hat{\pi}}(S) \longrightarrow \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\hat{\pi}(S)(m))$ Lie subalgebra

$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \longrightarrow \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E)$ continuous homomorphism of Lie algebras examples $\alpha \in \hat{\pi}(S)$ $(\overset{1:1}{\longleftrightarrow} {}^{\exists}\gamma \in \pi_1(S, {}^{\exists}p))$

(1) $\log \alpha \overset{\mathrm{def}}{=} |\log \gamma| = \left|\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(\gamma-1)^m\right| \in \widehat{\mathbb{Q}\hat{\pi}}(S)$

(remark $\forall m \geq 1$ $(\gamma-1)^m \in (I\pi_1(S,p))^m$)

(2) $(\log \alpha)^2 \overset{\mathrm{def}}{=} |(\log \gamma)^2| \in \widehat{\mathbb{Q}\hat{\pi}}(S)$

$(\log(\alpha^{-1}))^2 = (\log \alpha)^2 \in \widehat{\mathbb{Q}\hat{\pi}}(S)$: homotopy invariant of unoriented loop $\alpha^{\pm 1}$

Theorem (Kuno-K.)

$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} \mathrm{Der}_\omega(\widehat{\mathbb{Q}\pi S}|E)$

topological isomorphism of complete Lie algebras

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(tensorial description of $\widehat{\mathbb{Q}\hat{\pi}}(S) \cong \prod_{m=1}^{\infty} (H_1(S;\mathbb{Q})^{\otimes m})^{\mathbb{Z}/m}$

due to Kuno-K. and Massuyeau-Turaev, independently.

This theorem is a by-product of KK's proof.)

<u>mapping class group</u> $\mathcal{M}(S) = \pi_0(\mathrm{Diff}(S, \mathrm{id} \text{ on } \partial S))$

$\widehat{DN} : \mathcal{M}(S) \longrightarrow \mathrm{Aut}(\widehat{\mathbb{Q}\pi S}|_E)$, $\varphi \mapsto \varphi_*$

<u>injective</u> homomorphism (essentially due to Dehn-Nielsen)

$\mathcal{M}(S)^0 := \{ \varphi \in \mathcal{M}(S) ;\ \log \widehat{DN}(\varphi) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (\widehat{DN}(\varphi) - 1)^m \text{ converges} \}$

$\subsetneqq \mathcal{M}(S)$ subset (<u>not</u> subgroup)

- $\forall$ Dehn twist $\in \mathcal{M}(S)^0$
- $\mathcal{I}^L(S) \subset \mathcal{M}(S)^0$ largest Torelli group (in the sense of Putman)

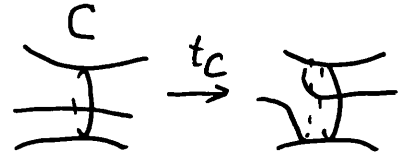
$$\begin{tikzcd}
\log \circ \widehat{DN} : \mathcal{M}(S)^0 \arrow[r, hook] \arrow[dr, dashed, "\exists ! \ \tau"'] & \mathrm{Der}(\widehat{\mathbb{Q}\pi S}|_E) \\
& \widehat{\mathbb{Q}\hat{\pi}}(S) \arrow[u, "\sigma"', "\cong"] \arrow[ul, phantom, "\circlearrowleft"]
\end{tikzcd}$$

$\tau : \mathcal{M}(S)^0 \hookrightarrow \widehat{\mathbb{Q}\hat{\pi}}(S)$ geometric Johnson homomorphism

<u>injective</u> map

<u>Dehn twists</u> $C \subset \operatorname{Int} S \;(= S \setminus \partial S)$ simple closed curve

$t_C \in \mathcal{M}(S)$ right-handed Dehn twist

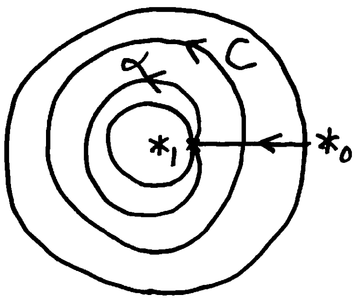


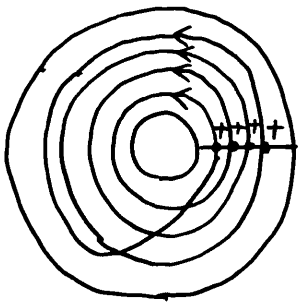
<u>Theorem</u> (<u>$\Sigma_{g,1}$</u> : Kuno-K., <u>general</u> : Kuno-K., Massuyeau-Turaev, independently)

$$\tau(t_C) = \frac{1}{2}(\log C)^2 \in \mathbb{Q}\widehat{\pi}(S)$$

i.e., $\widehat{DN}(t_C) = \exp(\sigma(\frac{1}{2}(\log C)^2))$ on $\mathbb{Q}\widehat{\Pi S}|E$.

<u>outline of proof</u> (I) S : annulus, $C = |\alpha|$





$$\forall n \geq 1 \begin{cases} \sigma(C^n)(\alpha) = 0 \\ \sigma(C^n)(\gamma) = n\gamma\alpha^n \end{cases}$$

$f(x)$: formal power series in $(x-1)$

$$\begin{cases} \sigma(f(C))(\alpha) = 0 \\ \sigma(f(C))(\gamma) = \gamma\alpha f'(\alpha) \end{cases}$$

$$\begin{cases} \log \widehat{DN}(t_C)(\alpha) = 0 \\ \log \widehat{DN}(t_C)(\gamma) = \gamma \log \alpha \end{cases} \longrightarrow \quad x f'(x) = \log x$$

$$f(x) = \int_1 \frac{1}{x} \log x \, dx = \frac{1}{2}(\log x)^2$$

$$\log \widehat{DN}(t_C) = \sigma(\tfrac{1}{2}(\log C)^2)$$

(II) general case $\Leftarrow$ van Kampen theorem //

Johnson image $\tau(\mathcal{I}^L(S)) \subset \mathbb{Q}\hat{\pi}(S)$

obvious estimate

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- trivial action on the homology of S
- coproduct $\Delta : \widehat{\mathbb{Q}\Pi S}(*_a, *_b) \to \widehat{\mathbb{Q}\Pi S}(*_a, *_b)^{\hat{\otimes} 2}$

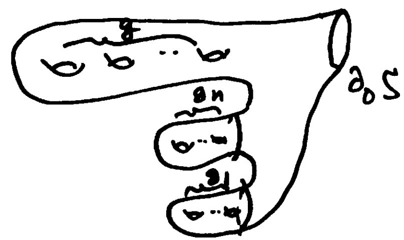
$$\gamma \in \Pi S(*_a, *_b) \mapsto \gamma \hat{\otimes} \gamma$$

$$L^+(S) := \left\{ u \in \mathbb{Q}\hat{\pi}(S)(2) ;\ \begin{array}{l} (\sigma(u) \hat{\otimes} 1 + 1 \hat{\otimes} \sigma(u))\Delta = \Delta\sigma(u) \\ i(u) \in \mathbb{Q}\hat{\pi}(\hat{S})(3) \end{array} \right\}$$

independent of the choice of $\partial_0 S$ and $g_j \geqq 1$!

where $\hat{S} := S \cup_{\partial S \setminus \partial_0 S} \left(\coprod_{j=1}^{m} \Sigma_{g_j, 1} \right) \quad (g_j \geqq 1)$

$i : S \hookrightarrow \hat{S} =$ (inclusion)



$L^+(S)$: pro-nilpotent Lie subalgebra $\subset \mathbb{Q}\hat{\pi}(S)$

$\underset{\text{exp, and log}}{\Longrightarrow}$ $L^+(S)$: pro-nilpotent Lie group

$\tau : \mathcal{I}^L(S) \hookrightarrow L^+(S)$ injective group homomorphism

(Other approaches : Putman , Church)

<u>genus 0 case</u>: $S = \Sigma_{0,n+1}$, $n \geq 0$.

$$0 \to \mathbb{Z}^n \to \mathcal{G}^L(\Sigma_{0,n+1}) \to PB_n \to 1 \quad (\text{exact})$$

$$\underset{1\leq j\leq n}{(0,\cdots 0,\overset{j}{1},0\cdots 0)} \mapsto t_{\partial_j S} \quad \text{Dehn twist}$$

Artin pure braid group

$$L^+(\Sigma_{0,n+1}) / \sum_{j=1}^{n} \mathbb{Q}\tau(t_{\partial_j S}) \cong \mathrm{sder}_n$$, the special derivation Lie algebra of $\widehat{\text{Free Lie}_n}$ (Ihara, ...)

<u>$n=2$</u>

- $\mathrm{sder}_2 \supset$ Grothendieck-Teichmüller Lie algebra
- $\underset{3\text{-dim}}{\tau(\mathcal{G}^L(\Sigma_{0,3}))} \subsetneqq \underset{\infty\text{-dim}}{L^+(\Sigma_{0,3})}$

<u>connected boundary case</u>: $S = \Sigma_{g,1}$, $g \geq 1$.

τ : equivalent to Massuyeau's total Johnson map

$\Longrightarrow$

$$\mathrm{gr}(\tau) : \underset{\text{w.r.to the Johnson filtration}}{\mathrm{gr}(\mathcal{G}^L(\Sigma_{g,1}))} \xrightarrow{\text{injective}} \underset{\text{w.r.to } \{\mathbb{Q}\hat{\pi}(m)\}_{m\geq 2}}{\mathrm{gr}(L^+(\Sigma_{g,1}))} = \mathfrak{h}^+_{g,1}$$

Morita's Lie algebra = positive part of Kontsevich's "Lie"

$\mathrm{gr}(\tau)$ = the classical Johnson homomorphism

[<u>Morita</u>

$\mathrm{gr}(\tau)$ is <u>not</u> surjective

The Morita traces

$\mathrm{Tr} : \mathfrak{h}_{g,1}^{+} \twoheadrightarrow \bigoplus_{m=1}^{\infty} \mathrm{Sym}^{2m+1}(H_1(S;\mathbb{Q}))$ surjective map

constructed in an algebraic manner

$\mathrm{Tr} \circ \mathrm{gr}(\tau) = 0$ (Morita)

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Turaev cobracket

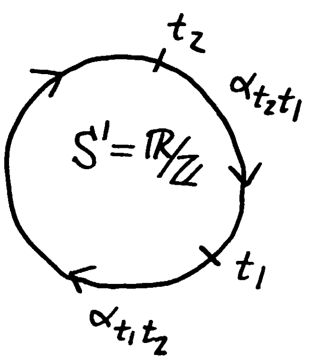
S : as above

$1 \in \hat{\pi}(S)$ constant loop $\quad \mathbb{Z}1 \subset \mathrm{Center}(\mathbb{Z}\hat{\pi}(S))$

$\mathbb{Z}\hat{\pi}'(S) := \mathbb{Z}\hat{\pi}(S)/\mathbb{Z}1$ Lie algebra

$|\ |' : \mathbb{Z}\pi_1(S) \xrightarrow{|\ |} \mathbb{Z}\hat{\pi}(S) \overset{\text{quotient}}{\twoheadrightarrow} \mathbb{Z}\hat{\pi}'(S)$

$\alpha \in \hat{\pi}(S)$ in general position



$D_\alpha := \{(t_1, t_2) \in S^1 = \mathbb{R}/\mathbb{Z} \ : \ t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$

parametrizing the double points

$$\delta(\alpha) \overset{\text{def}}{=} \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1t_2}|' \otimes |\alpha_{t_2t_1}|' \in \mathbb{Z}\hat{\pi}'(S) \otimes \mathbb{Z}\hat{\pi}'(S)$$

<u>Theorem</u> (Turaev)

(1) δ : well-defined

(2) $(\mathbb{Z}\hat{\pi}'(S), [\,,\,], \delta)$: Lie bialgebra

<u>Chas</u> : $\longrightarrow\!\!\!\!\!\!\!/\;\;$ ' involutive i.e., $[\,,\,] \circ \delta = 0$

- $\operatorname{Ker}\delta \subset \mathbb{Z}\hat{\pi}'(S)$: Lie subalgebra ($\because$ compatibility axiom)
- δ induces $(\widehat{\mathbb{Q}\hat{\pi}}(S), [\,,\,], \delta)$: complete Lie bialgebra

<u>Theorem</u> (Kuno-K.)

$\delta \circ \tau = 0 : \mathcal{M}(S)^0 \xrightarrow{\tau} \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\delta} \widehat{\mathbb{Q}\hat{\pi}}(S) \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}(S)$

($\Uparrow$ $\widehat{\mathbb{Q}\Pi S}(*_a, *_b)$: an $\mathcal{M}(S)$-equivariant $\widehat{\mathbb{Q}\hat{\pi}}(S)$-bimodule (Kuno-K.))

$\operatorname{gr}(\delta) : \operatorname{gr}(\widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{g,1})) \longrightarrow \operatorname{gr}(\widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{g,1}))^{\otimes 2}$

$\operatorname{gr}(\delta) \circ \operatorname{gr}(\tau) = 0$

<u>Theorem</u> (Kuno-K.)

The Morita traces are extracted from $\operatorname{gr}(\delta)$

$\Uparrow$ Massuyeau-Turaev's tensorial description of the Papakyriakopoulos-Turaev intersection form