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"The Goldman-Turaev Lie bialgebra and the largest Torelli group"

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I. The Goldman-Turaev Lie bialgebra and the mapping class group

II. A tensorial description of the completed Goldman-Turaev Lie bialgebra

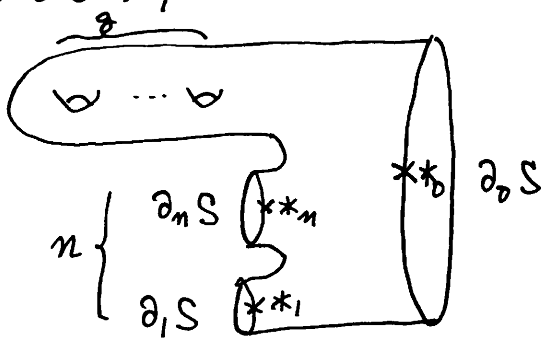
$\boxed{\text{I.}}$

S : a compact connected oriented surface with $\partial S \neq \phi$

$\cong \Sigma_{g,n+1}$, $\exists g, \exists n \geqq 0$

$\partial S = \coprod_{j=0}^{n} \partial_j S$

$E := \{*_j\}_{j=0}^{n}$, $*_j \in \partial_j S$



would like to discuss some relation among

(i) free loops on S

(ii) based paths / loops on (S, E)

(iii) the mapping class group (the Teichmüller modular group) of S

$$\mathcal{M}(S) := \{\varphi : S \to S : \text{ori. pres. diffeo. } \varphi|_{\partial S} = 1_{\partial S}\} / \text{isotopy fixing } \partial S \text{ pointwise}$$

(ii)-(iii) classical.

Dehn-Nielsen

$$DN : \mathcal{M}(\Sigma_{g,1}) \xrightarrow{\cong} \{\bar{\varphi} \in \mathrm{Aut}(\pi_1(\Sigma_{g,1}, *_0)) \,;\, \bar{\varphi} \text{ fixes any boundary loops}\}$$

can prove (similarly)

$$DN : \mathcal{M}(S) \underset{\text{injective}}{\hookrightarrow} \mathrm{Aut}(\Pi S|_E)$$

where $\Pi S|_E$: restriction of the fundamental groupoid ΠS to the object set E.

$$\mathrm{Ob}(\Pi S|_E) = E$$

$$(\Pi S|_E)(*_a, *_b) = \underbrace{\Pi S(*_a, *_b) = [([0,1], 0, 1), (S, *_a, *_b)]}_{\text{the homotopy set of paths from } *_a \text{ to } *_b} \quad (0 \leq a, b \leq n)$$

(i) --- the Goldman-Turaev Lie bialgebra.

$\hat{\pi}(S) := [S^1, S]$ the homotopy set of free loops on S

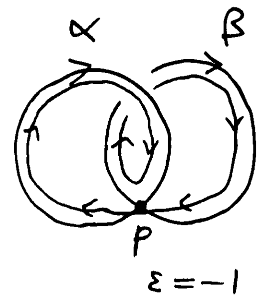
$|\ | : \pi_1(S) \to \hat{\pi}(S)$ forgetful map of a base point.

<u>Goldman bracket</u> $\alpha, \beta \in \hat{\pi}(S)$ in general position.

$$[\alpha, \beta] \overset{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta)\, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}(S)$$

where $\varepsilon_p(\alpha, \beta) \in \{\pm 1\}$ local intersection number

$\alpha_p, \beta_p \in \pi_1(S, p)$ based loops along α, β



<u>Goldman</u> (1) $[\ ,\ ]$: well-defined

(2) $(\mathbb{Z}\hat{\pi}(S), [\ ,\ ])$: Lie algebra ---▷ the Goldman Lie algebra

In general,

<u>Lie algebra : a language for an infinitesimal symmetry</u>

Whose symmetry does $\mathbb{Z}\hat{\pi}(S)$ describe ?

<u>Goldman</u> : the moduli space of flat bundles over S

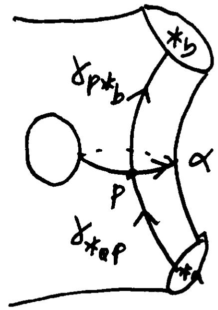
<u>Kuno-K.</u> : the "groupoid ring" $\mathbb{Z}(\Pi S|_E)$

⇝ connect (i) and (ii)+(iii)

Action of $\mathbb{Z}\hat{\pi}(S)$ on $\mathbb{Z}(\Pi S|_E)$

$\alpha \in \hat{\pi}(S)$, $\gamma \in \Pi S(*_a, *_b)$, $(0 \leq a, b \leq n)$ in general position

$$\sigma(\alpha)(\gamma) \overset{\mathrm{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon_p(\alpha, \gamma)\, \gamma_{*_a p}\, \alpha_p\, \gamma_{p *_b} \in \mathbb{Z}\Pi S(*_a, *_b)$$



Kuno-K. (1) σ: well-defined

(2) $\sigma: \mathbb{Z}\hat{\pi}(S) \to \mathrm{Der}(\mathbb{Z}(\Pi S|_E))$ Lie algebra homomorphism

$\xi_j \in \Pi S(*_j, *_j) = \pi_1(S, *_j)$, $0 \leq j \leq n$

boundary loop with positive direction

$\mathrm{Der}_\xi(\mathbb{Z}(\Pi S|_E)) \overset{\mathrm{def}}{=}$ the stabilizer of $\{\xi_j\}_{j=0}^{n}$ in $\mathrm{Der}(\mathbb{Z}\Pi S|_E)$

$\cup$

$\sigma(\mathbb{Z}\hat{\pi}(S))$ $\quad$ ($\because$ $\forall \alpha \in \hat{\pi}(S)$ we may choose $\alpha \cap \xi_j = \phi$)

<u>completion</u>

G : a group

$$\sum_{x\in G} a_x x \mapsto \sum a_x$$

$\mathbb{Q}G$: $\mathbb{Q}$-group ring $\quad IG := \mathrm{Ker}(\varepsilon : \mathbb{Q}G \to \mathbb{Q})$ augmentation ideal

$\widehat{\mathbb{Q}G} := \varprojlim_{m\to\infty} \mathbb{Q}G/(IG)^m$ the completed group ring

$$\mathrm{Aut}(\widehat{\mathbb{Q}G}) \supset \boxed{} \underset{\exp}{\overset{\log}{\rightleftarrows}} \boxed{} \subset \mathrm{Der}(\widehat{\mathbb{Q}G})$$

$$v \longmapsto \log v = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (v-1)^k$$

$$\exp u = \sum_{k=0}^{\infty} \frac{1}{k!} u^k \longleftarrow u$$

$\Delta : \widehat{\mathbb{Q}G} \to \widehat{\mathbb{Q}G} \hat{\otimes} \widehat{\mathbb{Q}G}$ coproduct $\quad \Delta x = x \hat{\otimes} x \ (\forall x \in G)$

$\mathcal{G}$: a groupoid with $\mathrm{Ob}\,\mathcal{G} = E$

$\mathbb{Q}\mathcal{G}$: $\mathbb{Q}$-linear small category with $\mathrm{Ob}(\mathbb{Q}\mathcal{G}) = E$, $(\mathbb{Q}\mathcal{G})(*_a, *_b) := \mathbb{Q}(\mathcal{G}(*_a, *_b))$ $\quad {}^{0\le a,b\le n}$

"groupoid ring"

$\widehat{\mathbb{Q}\mathcal{G}}$: completion with respect to $I(\mathcal{G}(*_a, *_a))$ $(0\le a\le n)$.

$$\mathrm{Aut}(\widehat{\mathbb{Q}\mathcal{G}}) \supset \boxed{} \underset{\exp}{\overset{\log}{\rightleftarrows}} \boxed{} \subset \mathrm{Der}(\widehat{\mathbb{Q}\mathcal{G}})$$

$\Delta : \widehat{\mathbb{Q}\mathcal{G}} \to \widehat{\mathbb{Q}\mathcal{G}} \hat{\otimes} \widehat{\mathbb{Q}\mathcal{G}}$ coproduct $\quad \Delta\gamma = \gamma \hat{\otimes} \gamma \ (\forall \gamma \in \mathcal{G}(*_a, *_b))$

<u>in our situation</u> $\mathcal{G} = \Pi S|_E$

$\widehat{\mathbb{Q}(\Pi S|_E)}$ "completed groupoid ring"

$\mathrm{Der}_\partial(\widehat{\mathbb{Q}(\Pi S|_E)}) :=$ the stabilizer of boundary loops $\{\xi_j\}_{j=0}^n$ in $\mathrm{Der}(\widehat{\mathbb{Q}(\Pi S|_E)})$

<u>completion of</u> $\mathbb{Q}\hat{\pi}(S)$

$|\ | : \mathbb{Q}\pi_1(S,*) \to \mathbb{Q}\hat{\pi}(S)$ forgetful map of a basepoint $* \in S$

$\mathbb{Q}\hat{\pi}(S)(m) := |\mathbb{Q}1 + (I\pi_1(S,*))^m|$, ($1 \in \pi_1(S,*)$ identity) $m \geq 0$ indep. of $* \in S$

$[\mathbb{Q}\hat{\pi}(S)(m_1), \mathbb{Q}\hat{\pi}(S)(m_2)] \subset \mathbb{Q}\hat{\pi}(S)(m_1+m_2-2)$ $\quad \forall m_1, \forall m_2 \geq 1$

$$\widehat{\mathbb{Q}\hat{\pi}}(S) := \varprojlim_{m\to\infty} \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\hat{\pi}(S)(m)$$

the completed Goldman (Turaev) Lie (bi)algebra

$\widehat{\mathbb{Q}\hat{\pi}}(S)(m) := \mathrm{Ker}(\widehat{\mathbb{Q}\hat{\pi}}(S) \to \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\hat{\pi}(S)(m))$ $\quad m \geq 0$

Lie subalgebra of $\widehat{\mathbb{Q}\hat{\pi}}(S)$, ($m \geq 3 \Rightarrow$ pro-nilpotent)

<u>Theorem 1</u> (Kuno-K.) (Infinitesimal Dehn-Nielsen Theorem)

σ extends to

$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} \mathrm{Der}_\partial(\widehat{\mathbb{Q}(\Pi S|_E)})$ isomorphism

a corollary of a tensorial description of $\widehat{\mathbb{Q}\hat{\pi}}(S)$ (later)

<u>mapping class group</u>

$\varphi \in \mathcal{M}(S) \longmapsto DN(\varphi) \in \mathrm{Aut}(\mathbb{Q}\widehat{\pi S|_E})$

<u>Assume</u> $\exists \log DN(\varphi) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\widehat{DN}(\varphi) - 1)^k$ —

Then $\log DN(\varphi) \in \mathrm{Der}_\partial(\mathbb{Q}(\widehat{\pi S|_E}))$

$\tau(\varphi) \overset{\mathrm{def}}{=} \sigma^{-1}(\log DN(\varphi)) \in \mathbb{Q}\hat{\hat{\pi}}(S)$ geometric Johnson homomorphism

<u>example</u> $C \subset S \setminus \partial S$ simple closed curve. $C = |x|$, x: based loop

$t_C \in \mathcal{M}(S)$ right-handed Dehn twist along C

$\tau(t_C) = |\frac{1}{2}(\log x)^2| \in \mathbb{Q}\hat{\hat{\pi}}(S)$ --- ($\to$ the reason why we take the completion)

(original $\Sigma_{g,1}$: Kuno-K.
general S ; Massuyeau-Turaev, Kuno-K., independently)

<u>the largest Torelli group</u> (in the sense of Putman)

$C_j := [\xi_j] \in H_1(S:\mathbb{Q})$

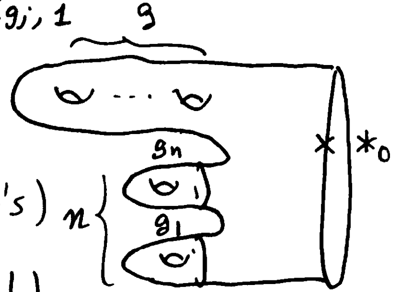
$\mathcal{I}^L(S) := \mathrm{Ker}(\mathcal{M}(S) \to \mathrm{Aut}(H_1(S:\mathbb{Q}) / \sum_{j=0}^{n} \mathbb{Q}C_j))$

the largest Torelli group in the sense of Putman

(which includes all other kinds of Putman's Torelli groups
$\mathcal{I}^L(\Sigma_{g,1}) = \mathcal{I}_{g,1}$ the classical Torelli group)

$g_j \geq 1$, $1 \leq j \leq n$, $\widetilde{S} := S \cup_{\partial S \setminus \partial_0 S} \left(\coprod_{j=1}^{n} \Sigma_{g_j,1} \right) \cong \Sigma_{g+\sum g_j, 1}$

$\iota : S \hookrightarrow \widetilde{S}$ inclusion



$\widehat{\mathbb{Q}\hat{\pi}}(S)(2\frac{1}{3}) := \widehat{\mathbb{Q}\hat{\pi}}(S)(2) \cap (\iota_*)^{-1}(\widehat{\mathbb{Q}\hat{\pi}}(\widetilde{S})(3))$ (indep of g_j's)

pro-nilpotent Lie subalgebra of $\widehat{\mathbb{Q}\hat{\pi}}(S)$

$L^{1/3}(S)$:= the stabilizer of the coproduct Δ in $\widehat{\mathbb{Q}\hat{\pi}}(S)(2\frac{1}{3})$.

pro-nilpotent Lie subalgebra of $\widehat{\mathbb{Q}\hat{\pi}}(S)$

$\Longrightarrow$ pro-nilpotent group via exp & log

Theorem 2 (Kuno-K.)

$\tau : \mathcal{I}^L(S) \to L^{1/3}(S)$ well-defined injective group homomorphism

geometric Johnson homomorphism

$\boxed{\Sigma_{g,1}}$ τ is equivalent to Massuyeau's total Johnson map

$\Longrightarrow \mathrm{gr}(\tau) : \mathrm{gr}(\mathcal{I}_{g,1}) \to \mathrm{gr}(L^{1/3}(\Sigma_{g,1})) = \mathfrak{h}_{g,1}^+$ (Morita's Lie algebra)

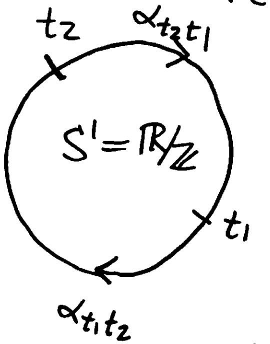
the classical Johnson homomorphism

Turaev cobracket $\quad \hat{\pi} = \hat{\pi}(S)$

$\mathbb{Z}\hat{\pi}' := \mathbb{Z}\hat{\pi}/\mathbb{Z}1.$ Lie algebra $(\because\ 1 \in \mathrm{Center}(\mathbb{Z}\hat{\pi}))$

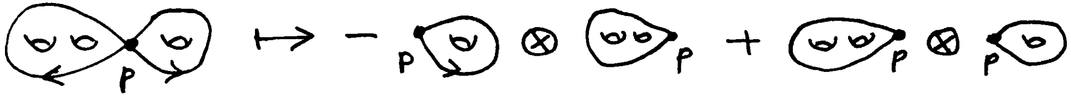
$\delta : \mathbb{Z}\hat{\pi}' \to \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$ Turaev cobracket $\qquad | \ |' : \mathbb{Z}\pi_1(S) \xrightarrow{|\ |} \mathbb{Z}\hat{\pi} \xrightarrow{\text{quotient}} \mathbb{Z}\hat{\pi}'$

$\alpha \in \hat{\pi}(S)$ in general position



$$D_\alpha := \{(t_1, t_2) \in S^1 = \mathbb{R}/\mathbb{Z} \ ;\ t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$$

$$\delta(\alpha) := \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1t_2}|' \otimes |\alpha_{t_2t_1}|' \in \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$$



[Turaev (1) δ : well-defined
(2) $(\mathbb{Z}\hat{\pi}', [\,,\,], \delta)$: Lie bialgebra

[Chas $(\mathbb{Z}\hat{\pi}', [\,,\,], \delta)$: involutive. — $\Rightarrow$ $(\widehat{\mathbb{Q}\hat{\pi}}(S), [\,,\,], \delta)$: the completed Goldman–Turaev Lie bialgebra

δ extends to

$$\delta : \widehat{\mathbb{Q}\hat{\pi}}(S) \to \widehat{\mathbb{Q}\hat{\pi}}(S) \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}(S)$$

(**Theorem 3** (Kuno–K.)

$$\delta \circ \tau = 0 : \mathcal{G}^L(S) \to \widehat{\mathbb{Q}\hat{\pi}}(S) \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}(S) \ —$$

$\Leftarrow$ $\forall$ diffeo. preserves the self-intersection of any curve on S

$\boxed{\Sigma_{g,1}}$

δ includes the Morita traces on $\mathfrak{h}^+_{g,1}$ (Kuno–K.)

$\Uparrow$

tensorial description

$\boxed{\mathrm{II}}$

<u>classical fact due to Magnus et.al.</u>

$G \cong F_n$: free group of rank n

$\Longrightarrow \widehat{\mathbb{Q}G} \underset{\substack{\Uparrow \\ \text{Magnus expansion}}}{\cong} \widehat{T}(G^{abel} \otimes_{\mathbb{Z}} \mathbb{Q}) \cong \mathbb{Q}\langle\langle X_1, X_2, \cdots, X_n \rangle\rangle$

non-commutative formal power series in n indeterminates $X_1, X_2, \cdots, X_n$

where

$\widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$ the completed tensor algebra over a $\mathbb{Q}$-vector space H

$p \geq 1$, $\widehat{T}_p = \widehat{T}(H)_p := \prod_{m \geq p} H^{\otimes m}$ two-sided ideal of $\widehat{T}(H)$

<u>Magnus expansion</u> G as above $\widehat{T} = \widehat{T}(G^{abel} \otimes_{\mathbb{Z}} \mathbb{Q}) \cong \widehat{T}(\mathbb{Q}^n)$

$= \mathbb{Q}\langle\langle X_1, X_2, \cdots, X_n \rangle\rangle$

$\theta : G \to 1 + \widehat{T}_1 \subset \widehat{T}$ Magnus expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\forall x, \forall y \in G$ $\theta(xy) = \theta(x)\theta(y)$

2) $\forall x \in G$, $\theta(x) \equiv 1 + [x] \mod \widehat{T}_2$

$\underset{\text{induces}}{\Longrightarrow}$ $\theta : \widehat{\mathbb{Q}G} \xrightarrow{\cong} \widehat{T}$ isomorphism of filtered algebras for general H,

$$N : \hat{T}(H) \to \hat{T}(H)$$ cyclic symmetrizer (or cyclicizer)

$$N|_{H^{\otimes 0}} := 0$$

$$N(X_1 X_2 \cdots X_m) := \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1} \quad (X_j \in H)$$

S $(\cong \Sigma_{g,n+1},\ g,n \geq 0)$ as above. $\pi_1(S) \cong F_{2g+n}$ free group of rank $2g+n$

<u>Kuno-K.</u> $\forall \theta : \pi_1(S) \to \hat{T} = \hat{T}(H_1(S;\mathbb{Q}))$ Magnus expansion

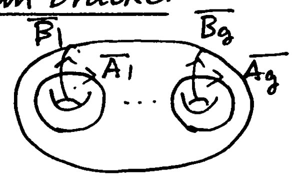
$$N\theta : \mathbb{Q}\hat{\pi}(S) \xrightarrow{\cong} N(\hat{T}_1), \quad |x| \mapsto N(\theta(x))$$

isomorphism of filtered $\mathbb{Q}$-vector spaces.

Goldman bracket ?

<u>An additional datum for describing the Goldman bracket</u> .... $s \in \mathrm{Sect}\, i_*$

$$\overline{S} := S \cup_{\partial S} ((n+1) \text{ disks}) \cong \Sigma_g =$$



$i : S \hookrightarrow \overline{S}$ inclusion map

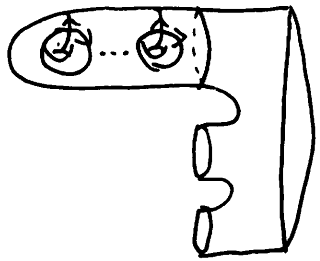
$$H_2(\overline{S}, S) \xrightarrow{\partial_*} H_1(S) \xrightarrow{i_*} H_1(\overline{S}) \to 0 \quad \text{(exact)}$$

$$\mathrm{Im}\, \partial_* = \sum_{j=0}^{n} \mathbb{Q} C_j = \bigoplus_{j=1}^{n} \mathbb{Q} C_j$$

$$\mathrm{Sect}\, i_* := \{\text{sections of } i_* : H_1(S) \to H_1(\overline{S})\}$$

<u>examples</u>

(1) $/\mathbb{Z}$



$\Rightarrow$ a section$/_{\mathbb{Z}}$ $s_0 \in \mathrm{Sect}\, z_*$

(2) $/\mathbb{R}$ Regard $\mathrm{Int}\, S$ as a punctured Riemann surface.

i.e., S a real oriented blow-up of a compact Riemann surface C

at $(n+1)$ distinct points $P_0, P_1, \cdots, P_n$

<u>normalized Abelian differential of the third kind</u>

C : a compact Riemann surface, $P \neq Q \in C$

$\exists!\ \omega = \omega(C : P, Q)$: a meromorphic 1-form on C

s.t. $\begin{cases} \text{holomorphic on } C \setminus \{P, Q\} \\ \mathrm{ord}_P\, \omega = \mathrm{ord}_Q\, \omega = -1 \\ \mathrm{Res}_P\, \omega = -\mathrm{Res}_Q\, \omega = \dfrac{1}{2\pi\sqrt{-1}} \\ \forall \varphi : \text{holo. 1-form on } C \quad \displaystyle\int_C \omega \wedge \overline{\varphi} = 0 \end{cases}$

$n \geqq 1.$ $\quad \mathrm{Int}\, S = C \setminus \{P_0, P_1, \cdots, P_n\}$

$$\bigcap_{i=1}^{n} \mathrm{Ker} \left(\int \mathrm{Re}\, \omega(C : P_0, P_i) : H_1(S) = H_1(\mathrm{Int}\, S) \to \mathbb{R} \right)$$

$\underset{\text{induces}}{\Longrightarrow}$ canonical section $/\mathbb{R}$ $\quad s_{(C : P_0, P_1, \cdots, P_n)} \in \mathrm{Sect}\, i_*$

<u>Remark</u> $n = 1$

$\mathbb{C}_g \to \mathbb{M}_g$ universal Riemann surface over the moduli of compact Riemann surfaces of genus g

$k_0 := \left(\text{the } 1^{st} \text{ variation of } s_{(C : P_0, P_1)}\right)$ extends to $\mathbb{C}_g \underset{\mathbb{M}_g}{\times} \mathbb{C}_g$

$\int_{\text{fiber}} {k_0}^3 =$ the 1^{st} variation of pointed harmonic volumes

(a twisted 1-form on $\mathbb{C}_g$)

$\Longrightarrow$ differential forms representing the Morita-Mumford classes on $\mathbb{M}_g$.

$\underset{\text{related to}}{\Longrightarrow}$ de Jong's study of real analytic functions on $\mathbb{M}_g$

Lie bracket on $N(\hat{T}_1) = N(\hat{T}(H_1(S;\mathbb{Q}))_1)$, the cyclic invariants of $\hat{T}_1$

Fix a section $s \in \mathrm{Sect}\, i_*$ $\quad \{\overline{A_i}, \overline{B_i}\}_{i=1}^{g} \subset H_1(\overline{S})$ symplectic basis

$A_i^s := s(\overline{A_i}),\ B_i^s := s(\overline{B_i}) \in H_1(S)$

$u = \sum_{i=1}^{g} A_i^s u_i' + \sum_{i=1}^{g} B_i^s u_i'' + \sum_{j=1}^{n} C_j u_j^0$, $\quad v = \sum_{i=1}^{g} A_i^s v_i' + \sum_{i=1}^{g} B_i^s v_i'' + \sum_{j=1}^{n} C_j v_j^0 \in N(\hat{T}_1) < H\otimes\hat{T}$

$[u,v]_s \overset{\text{def}}{=} N\left(\sum_{i=1}^{g} u_i' v_i'' - u_i'' v_i' + \sum_{j=1}^{n} C_j (u_j^0 v_j^0 - v_j^0 u_j^0)\right) \in N(\hat{T}_1)$

$\Rightarrow$ Lie algebra $N(\hat{T}_1)_s := (N(\hat{T}_1), [\ ,\ ]_s)$

Action on $\hat{T}_E$

$\hat{T}_E$: $\mathbb{Q}$-linear small category, $\mathrm{Ob}(\hat{T}_E) = E$, $\hat{T}_E(*_a, *_b) := \hat{T}$, $0 \le a, b \le n$

$\mathrm{Der}_\partial(\hat{T}_E)_s := \left\{ D : \text{continuous derivation of } \hat{T}_E.\ \begin{array}{l} D(C_j) = 0 \text{ for } C_j \in \hat{T}_E(*_j, *_j),\ \underset{\sim}{1 \le} \overset{\forall}{j} \le n \\ D(C_0 - \sum_{i=1}^{g}(A_i^s B_i^s - B_i^s A_i^s)) = 0 \quad \text{at } *_0 \end{array} \right\}$

$\sigma_s : N(\hat{T}_1)_s \to \mathrm{Der}_\partial(\hat{T}_E)_s$ Lie algebra homom.

$\sigma_s^0 : N(\hat{T}_1)_s \to \mathrm{Der}(\hat{T}) = \mathrm{Der}(\hat{T}_E(*_0, *_0))$ Lie algebra homom.

$\sigma_s^0(u)(A_i^s) := u_i''$, $\sigma_s^0(u)(B_i^s) := -u_i'$, $\sigma_s^0(u)(C_j) := u_j^0 C_j - C_j u_j^0$

$\sigma_s(u)(v) := -u_a^0 v + \sigma_s^0(u)(v) + v u_b^0 \qquad v \in (\hat{T}_E)(*_a, *_b) \qquad 0 \le a, b \le n$

$(u_0^0 := 0)$

$(1+\hat{T}_1)_E$: groupoid, $\mathrm{Ob}((1+\hat{T}_1)_E) = E$, $(1+\hat{T}_1)_E(*_a, *_b) := 1+\hat{T}_1$, $0 \le a, b \le n$

<u>Magnus expansion.</u>

θ : Magnus expansion of (S, E)

$\overset{\text{def}}{\Longleftrightarrow}$
1) $\theta : \Pi S|_E \to (1+\hat{T}_1)_E$: homomorphism of groupoids over E
2) $0 \le {}^\forall j \le n$, $\theta : \pi_1(S, *_j) \to (1+\hat{T}_1)_E(*_j, *_j) = 1+\hat{T}_1$ Magnus expansion

$s \in \mathrm{Sect}\, i_*$

<u>boundary condition</u> $(\#_s)$

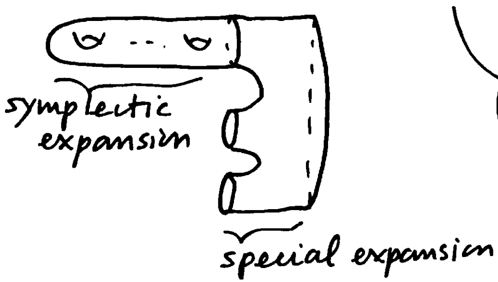
$$\theta(\xi^s_a) = \begin{cases} \exp(C_a) & (1 \le {}^\forall a \le n) \\ \exp\left(C_0 - \sum_{i=1}^{g} (A_i^s B_i^s - B_i^s A_i^s)\right) & (a=0) \end{cases} \qquad \left(C_0 = -\sum_{j=1}^{n} C_j \right)$$

<u>examples for Magnus expansions satisfying</u> $(\#_s)$

(0) $S = \Sigma_{g,1}$, symplectic expansion (Massuyeau)

e.g., K., Massuyeau, Kuno, Bene-K.-Kuno-Penner,

(1) $/\mathbb{Q}$



Magnus expansion satisfying $(\#_{s_0})$

θ_0

(2) $/\mathbb{R}$ $\quad C$: compact Riemann surface of genus g

$P_0, P_1, \cdots, P_n \in C$ $\quad (n+1)$ distinct points

harmonic forms on C, Green operator, iterated integrals

$\Longrightarrow$ Magnus expansion satisfying $(\#_{s(C;P_0,P_1,\cdots,P_n)})$

Theorem 4. (Massuyeau–Turaev, Kuno–K., independently)

$s \in \mathrm{Sect}\, i_*$

θ : Magnus expansion of (S, E) satisfying $(\#_s)$

$\Longrightarrow$ (1) $-N\theta : \mathbb{Q}\hat{\pi}(S) \xrightarrow{\cong} N(\hat{T})_s$ Lie algebra isomorphism

(2)
$$
\begin{tikzcd}
\mathbb{Q}\hat{\pi}(S) \arrow[r, "\sigma"] \arrow[d, "-N\theta"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi_1(S,*_0)}) \arrow[d, "\theta", "\cong"'] \\
N(\hat{T})_s \arrow[r, "\sigma_s^0"'] & \mathrm{Der}_\partial(\hat{T})
\end{tikzcd}
$$

(3)
$$
\begin{tikzcd}
\mathbb{Q}\hat{\pi}(S) \arrow[r, "\sigma"] \arrow[d, "-N\theta"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S|_E}) \arrow[d, "\theta", "\cong"'] \\
N(\hat{T})_s \arrow[r, "\sigma_s"'] & \mathrm{Der}_\partial(\hat{T}_E)
\end{tikzcd}
$$

- original $(\Sigma_{g,1})$ KK
- general S
  (1)(2) MT & KK
  (3) KK
- MT:
  - tensorial description of Papakyriakopoulos–Turaev homotopy intersection form
  - quiver theory

Outline of Proof (KK)

① proof for θ_0 in example (1) ($\Leftarrow$ twisted homology of $(S, \partial S)$)

$\Downarrow$

② $\sigma_{s_0} : N(\hat{T})_{s_0} \xrightarrow{\cong} \mathrm{Der}_\partial(\hat{T}_E)_{s_0}$ isom. ($\Leftarrow$ some linear algebra)

$\Downarrow$

③ $\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi S}_E)$ isom (infinitesimal Dehn-Nielsen theorem)

$\Downarrow$

④ proof for arbitrary $s \in \mathrm{Sect}\, i_*$ //

Remark $\widehat{\mathbb{Q}\hat{\pi}}(S)(2\frac{1}{3}) = (N\theta)^{-1}\{u \in N(\hat{T}_1)_s \,;\, \mathrm{wt}\, u \geq 3\}$

where $\mathrm{wt}(A_i^s) = \mathrm{wt}(B_i^s) = 1$, $\mathrm{wt}(C_j) = 2$

<u>tensorial description of the Turaev cobracket for $\Sigma_{g,1}$</u> , $\hat{T} = \hat{T}(H_1(\Sigma_{g,1};\mathbb{Q}))$

$\theta : \pi_1(\Sigma_{g,1}, *_0) \longrightarrow \{\text{grouplike elements}\} \subset 1 + \hat{T}_1$ Magnus expansion with $(\#_s)$

i.e., symplectic expansion (Massuyeau)

$$\delta^\theta := ((-N\theta)\hat{\otimes}(-N\theta)) \circ \delta \circ (-N\theta)^{-1} : \mathrm{Der}_\omega(\hat{T}) \to \mathrm{Der}_\omega(\hat{T}) \hat{\otimes} \mathrm{Der}_\omega(\hat{T})$$

Turaev cobracket

$$\mathrm{Der}_\omega(\hat{T}) = \prod_{m=1}^{\infty} N(H^{\otimes m}), \quad H = H_1(\Sigma_{g,1};\mathbb{Q})$$

$$\delta^\theta = \sum_{k=-\infty}^{+\infty} \delta^\theta_{(k)}, \quad \delta^\theta_{(k)} : N(H^{\otimes m}) \longrightarrow \bigoplus_{p+q=m+k} N(H^{\otimes p}) \hat{\otimes} N(H^{\otimes q})$$

<u>Theorem 6</u> (Massuyeau-Turaev, Kuno-K. independently)

$$\delta^\theta = \delta^\theta_{(-2)} + \delta^\theta_{(1)} + \delta^\theta_{(2)} + \cdots$$

$$\delta^\theta_{(-2)}(N(X_1\cdots X_m)) = \delta^{alg}(N(X_1\cdots X_m)) = \sum_{i<j} (X_i \cdot X_j) \left\{ \begin{array}{l} N(X_{i+1}\cdots X_{j-1}) \hat{\otimes} N(X_{j+1}\cdots X_m X_1 \cdots X_{i-1}) \\ -N(X_{j+1}\cdots X_m X_1\cdots X_{i-1}) \hat{\otimes} N(X_{i+1}\cdots X_{j-1}) \end{array} \right\}$$

Schedler's cobracket $\qquad (X_j \in H)$

(based on Massuyeau-Turaev's tensorial description of Papakyriakopoulos-Turaev homotopy intersection form)

Morita introduced "traces"

$$\mathrm{Tr}_m : \mathfrak{h}_{g,1}^+ \ (= gr(L^{1/3}(\Sigma_{g,1}))) \longrightarrow \mathrm{Sym}^{m-1}(H)$$

and proved

$$\mathrm{Tr}_m \circ gr(\tau) = 0 : gr(\mathcal{I}_{g,1}) \longrightarrow \mathrm{Sym}^{m-1}(H) \quad \text{for } \forall m \geqq 2$$

$$\mathrm{Der}_\omega(\hat{T}) = \prod_{m=1}^{\infty} N(H^{\otimes m}) \overset{i}{\hookrightarrow} \hat{T} = \prod_{m=0}^{\infty} H^{\otimes m} \xrightarrow[\text{projection}]{p_1} H$$

$$\leq : \mathrm{Der}_\omega(\hat{T})^{\hat{\otimes} 2} \xrightarrow[(p_1 \circ i) \hat{\otimes} i]{} H \otimes \hat{T} \hookrightarrow \hat{T} \xrightarrow[\text{projection}]{} \widehat{\mathrm{Sym}}(H) := \prod_{m=0}^{\infty} \mathrm{Sym}^m(H)$$

$$\leq \circ \delta^{alg} \circ gr(\tau) = 0 \ : gr(\mathcal{I}_{g,1}) \to \widehat{\mathrm{Sym}}(H) \quad (\Leftarrow \text{Thm } 3)$$

Theorem 7 (Kuno-K.)

$\forall m \geqq 2.$ $$\leq \circ \delta^{alg}\big|_{N(H^{\otimes(m+2)})} = (-1)^m m \, \mathrm{Tr}_{m+1} : N(H^{\otimes (m+2)}) \longrightarrow \mathrm{Sym}^m(H)$$

All the Morita traces ($m \geqq 1$) are derived from the geometric fact:

Any diffeomorphism preserves the self-intersection of any curve on the surface.