"Mapping class groups and quantum topology"

IRMA, University of Strasbourg, 25-29 June 2012.

mini-course

"Johnson-Morita theory and the Goldman-Turaev Lie bialgebra"

Part I: "Johnson-Morita theory" (an algebraic approach)

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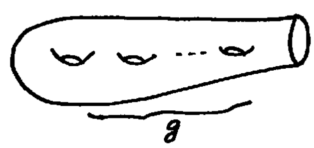
Contents

§ I-1. Johnson homomorphisms

$g \geqq 1$

For simplicity, we confine ourselves to

$\Sigma = \Sigma_{g,1} =$  connected oriented compact surface of genus g with 1 boundary component.

$$\mathcal{M} = \mathcal{M}_{g,1} := \pi_0 \mathrm{Diff}_+(\Sigma \mathrm{\ rel\ } \partial\Sigma)$$

$$= \{\varphi : \Sigma \to \Sigma \,;\ \text{orientation preserving diffeo.},\ \varphi|_{\partial\Sigma} = 1_{\partial\Sigma}\} / \text{isotopy fixing } \partial\Sigma \text{ pointwise}$$

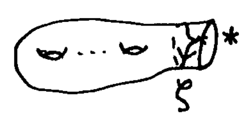
the mapping class group of Σ.

$\mathcal{M} \curvearrowright \pi := \pi_1(\Sigma, *)$, $* \in \partial\Sigma$,

$\Rightarrow$ $DN : \mathcal{M} \to \mathrm{Aut}(\pi)$ group homomorphism

Theorem (Dehn-Nielsen)

(i) $DN : \mathcal{M} \to \mathrm{Aut}(\pi)$ is injective.

(ii) $\mathrm{Image}(DN) = \{\varphi \in \mathrm{Aut}(\pi) \,;\, \varphi(\zeta) = \zeta\}$ where 

Remark We regard $\mathcal{M}_{g,1}$ as a subgroup of $\mathrm{Aut}(\pi)$ via the injective homomorphism DN.

Our motivation of research :

A linear approximation of the mapping class group $\mathcal{M}_{g,1}$

(cf) G : Lie group.
The Lie algebra $\mathrm{Lie}\, G = T_1 G$ is a linear approximation of G)

Slogan : What is "the Lie algebra" of the mapping class group $\mathcal{M}_{g,1}$?

Recall : Generalities on Lower central series
(cf). e.g., J-P. Serre, 1964 Harvard Lectures, LNM1500)

G : a discrete group

$\Gamma_k = \Gamma_k(G) \triangleleft G$, lower central series, $k \geq 1$,

$\Gamma_1(G) := G$

$\Gamma_{k+1}(G) := [\Gamma_k(G), G]$ for $k \geq 1$.

Theorem (classical)

(i) $gr_k \Gamma_\cdot(G) := \Gamma_k(G)/\Gamma_{k+1}(G)$ abelian group

(In particular, $gr_1 \Gamma_\cdot(G) = G/[G,G] = G^{abel}$ abelianization)

(ii) The operation

$$[\ ,\ ] : gr_k \Gamma_\cdot(G) \times gr_\ell \Gamma_\cdot(G) \longrightarrow gr_{k+\ell} \Gamma_\cdot(G)$$

$$(x \bmod \Gamma_{k+1},\ y \bmod \Gamma_{\ell+1}) \longmapsto xyx^{-1}y^{-1} \bmod \Gamma_{k+\ell+1}$$

is well-defined.

(iii) $\left(gr\, \Gamma_\cdot(G) := \bigoplus_{k=1}^{\infty} gr_k \Gamma_\cdot(G),\ [\ ,\ ] \right)$: Lie algebra $/\mathbb{Z}$ —

Remark : $\Bbbk$: comm, ring $\ni 1$

Definition $(\mathfrak{g}, [\ ,\ ])$: Lie algebra $/\Bbbk$

$\overset{\text{def}}{\Longleftrightarrow}$ 0) $\mathfrak{g}$: $\Bbbk$-module

$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $\Bbbk$-bilinear map

1) (skew) $\forall X \in \mathfrak{g}$, $[X, X] = 0$

$\left(\underset{\text{if } \frac{1}{2} \in \Bbbk}{\Longleftrightarrow} \quad \forall X, \forall Y \in \mathfrak{g} \quad [X, Y] = -[Y, X] \right)$

2) (Jacobi) $\forall X, \forall Y, \forall Z \in \mathfrak{g}$

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

The most important example : a free group

$n \geqq 1$

$F_n := \langle x_1, x_2, \cdots ; x_n \rangle$: free group of rank n

$\quad = \pi_1 (\overset{n}{\vee} S^1) \qquad (\text{e.g., } \pi_1(\Sigma_{g,1}) \cong F_{2g})$

an explicit description of the Lie algebra $\mathrm{gr}\,\Gamma_\cdot(F_n)$

$\rightsquigarrow$ free Lie algebra.

$H_{\mathbb{Z}} := F_n^{\mathrm{abel}} = \mathrm{gr}_1 \Gamma_\cdot (F_n) \cong \mathbb{Z}^n$

Theorem (Magnus-Witt)

$\mathrm{gr}\,\Gamma_\cdot(F_n) \cong \mathcal{L}(H_{\mathbb{Z}})$: the free Lie algebra over $H_{\mathbb{Z}}$

universal mapping property of $\mathcal{L}(H_{\mathbb{Z}})$

$$\begin{tikzcd} H_{\mathbb{Z}} \arrow[r, "\forall"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \forall \mathfrak{g} \text{ : Lie algebra}/\mathbb{Z} \\ \mathcal{L}(H_{\mathbb{Z}}) \arrow[ur, dashed, "\exists 1"'] & \end{tikzcd}$$

$\exists 1$ Lie algebra homom.

$U\mathcal{L}(H_{\mathbb{Z}}) = T(H_{\mathbb{Z}}) := \bigoplus_{m=0}^{\infty} H_{\mathbb{Z}}^{\otimes m}$ tensor algebra over $H_{\mathbb{Z}}$

$\mathbb{Q}$-case $\quad H := H_{\mathbb{Z}} \otimes \mathbb{Q} = H_1(F_n ; \mathbb{Q})$

$U\mathcal{L}(H) = T(H)$

$\quad \cup$

$\mathcal{L}(H) = \{ u \in T(H) ; \Delta u = u \otimes 1 + 1 \otimes u \}$

where $\Delta : T(H) \to T(H) \otimes T(H)$ algebra homom.

defined by $\Delta X = X \otimes 1 + 1 \otimes X$ for $\forall X \in H$

$m \geqq 1 \quad \mathcal{L}_m(H) := \mathcal{L}(H) \cap H^{\otimes m}$

$\qquad\qquad \cup \qquad\qquad \cup$

$\qquad \mathcal{L}_m(H_{\mathbb{Z}}) := \mathcal{L}(H_{\mathbb{Z}}) \cap H_{\mathbb{Z}}^{\otimes m}$,

$\widehat{\mathcal{L}}(H) := \prod_{m=1}^{\infty} \mathcal{L}_m(H)$ the completed free Lie algebra over H.

more precisely,

Theorem (Magnus-Witt)

$\mathcal{L}_m(H_{\mathbb{Z}}) = \mathrm{gr}_m \Gamma_\cdot(F_n) = \Gamma_m(F_n) / \Gamma_{m+1}(F_n)$

($\mathbb{Z}$-free of finite rank, rk is explicitly given by Witt.)

return to the mapping class group $\mathcal{M}_{g,1}$

$\mathrm{gr}\,\Gamma_\cdot(\mathcal{M}_{g,1}) = ?$ $\quad$ **Answer** $= 0$ if $g \geqq 3$

Theorem (Mumford - Powell - Harer)

$\mathcal{M}_{g,1}^{\mathrm{abel}} = 0$ i.e., $\mathcal{M}_{g,1} = [\mathcal{M}_{g,1}, \mathcal{M}_{g,1}]$ if $g \geqq 3$.

$\left(\begin{array}{l} \mathcal{M}_{1,1}^{\mathrm{abel}} \cong \mathbb{Z} \\ \mathcal{M}_{2,1}^{\mathrm{abel}} \cong \mathbb{Z}/10 \end{array} \right)$

<u>Revise our slogan</u>: What should be $\underset{\sim}{a}$ Lie algebra of the mapping class group $\mathcal{M}_{g,1}$?

<u>classical answer</u>: $\mathfrak{sp}_{2g} = \left\{ X \,;\, {}^tX \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} X = 0 \right\}$

$\mathcal{M}_{g,1} \curvearrowright H_1(\Sigma_{g,1};\mathbb{Z}) = H_{\mathbb{Z}} \ (\cong \mathbb{Z}^{2g})$

preserving the intersection form on $H_{\mathbb{Z}}$ ← (Jacobivariety of a compact Riemann surface)

$\rho_0 : \mathcal{M}_{g,1} \twoheadrightarrow Sp_{2g}(\mathbb{Z})$ <u>surjective</u> homomorphism

$\varphi \longmapsto |\varphi|$ $\quad$ ($\because$ Dehn twist $\mapsto$ transvection)

$\mathrm{Lie}\, Sp_{2g}(\mathbb{R}) = \mathfrak{sp}_{2g}(\mathbb{R})$

But this classical answer forgets

$\mathcal{I}_{g,1} := \mathrm{Ker}\, \rho_0 = \{ \varphi \in \mathcal{M}_{g,1} \,;\, |\varphi| = 1 \text{ on } H_{\mathbb{Z}} \}$

<u>the Torelli group</u> of $\Sigma_{g,1}$.

<u>Johnson's answer</u>: <u>Look at the whole of the lower central series of</u> $\pi_1(\Sigma,*)$!

(Contemp. Math. <u>20</u> (1983) pp 165–179)

$$\begin{tikzcd} & N_k \arrow[d, dotted] \\ & N_2 \arrow[d] \\ \mathcal{M}_{g,1} \arrow[r, phantom, "\curvearrowright"] \arrow[ur, bend left] \arrow[uur, bend left] & N_1 = H_{\mathbb{Z}} \end{tikzcd}$$

where $N_k := \pi / \Gamma_{k+1}(\pi)$, $k \geq 1$

the k-th nilpotent quotient.

$\mathcal{M}(0) := \mathcal{M}_{g,1}$

$\mathcal{M}(k) := \mathrm{Ker}(\mathcal{M}_{g,1} \to \mathrm{Aut}(N_k))$, $k \geq 1$. $\quad$ ($\mathcal{M}(1) = \mathcal{I}_{g,1}$. Torelli group)

$\{\mathcal{M}(k)\}_{k=0}^{\infty}$: the Johnson filtration.

(<u>Remark</u>: For $\mathrm{Aut}(F_n)$, a similar filtration was introduced by Andreadakis (Proc. London Math. Soc. <u>15</u> (1965), 239–268) about 20 years earlier than Johnson)

<u>Description of the quotient</u> $\mathcal{M}(k)/\mathcal{M}(k+1)$

$=$ <u>the</u> k^{th} <u>Johnson homomorphism</u> (Johnson, ibid.)

$\tau_k : \mathcal{M}(k)/\mathcal{M}(k+1) \longrightarrow H_{\mathbb{Z}}^* \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}})$, $\quad k \geq 1$

<u>Construction</u> $\quad \varphi \in \mathcal{M}(k) = \mathrm{Ker}(\mathcal{M}_{g,1} \to \mathrm{Aut}(\pi/\Gamma_{k+1}(\pi)))$

$\forall x \in \pi \qquad \varphi(x) \equiv x \mod \Gamma_{k+1}(\pi)$

$\varphi(x)x^{-1} \in \Gamma_{k+1}(\pi) \longmapsto \tau_k(\varphi)(x) := \varphi(x)x^{-1} \bmod \Gamma_{k+2} \in \Gamma_{k+1}(\pi)/\Gamma_{k+2}(\pi) \ \left(\overset{\mathcal{L}_{k+1}(H_{\mathbb{Z}})}{\parallel}\right)$

<u>Observations</u> (Johnson)

(i) $\forall x, \forall y \in \pi$

$$\varphi(xy)(xy)^{-1} = \varphi(x)\varphi(y)y^{-1}x^{-1} = (\varphi(x)x^{-1})\, x(\varphi(y)y^{-1})x^{-1}$$

$$\equiv (\varphi(x)x^{-1})(\varphi(y)y^{-1}) \mod \Gamma_{k+2}(\pi)$$

$$\Longrightarrow \begin{tikzcd} \tau_k(\varphi) : \pi \arrow[r] \arrow[d] & \mathcal{L}_{k+1}(H_{\mathbb{Z}}) \quad \text{homomorphism} \\ H_{\mathbb{Z}} \arrow[ur, dashed, "\exists!\ \tau_k(\varphi)"'] & \end{tikzcd} \quad \because \text{abelian} \qquad \text{i.e., } \tau_k(\varphi) \in H_{\mathbb{Z}}^* \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}})$$

(ii) $\forall \varphi, \forall \psi \in \mathcal{M}(k) \quad \tau_k(\varphi\psi) = \tau_k(\varphi) + \tau_k(\psi) \in H_{\mathbb{Z}}^* \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}})$

$$\left[\begin{aligned} (\because)\ \tau_k(\varphi\psi)(x) &= \varphi(\psi(x))x^{-1} = \varphi(\psi(x))\psi(x)^{-1}\psi(x)x^{-1} \\ &\equiv \varphi(x)x^{-1}\psi(x)x^{-1} \bmod \Gamma_{k+2}(\pi) \quad ((\because)\ \psi(x) \equiv x \bmod \Gamma_2(\pi) \ (\because k \geq 1)) \end{aligned} \right.$$

$\tau_k : \mathcal{M}(k) \to H_{\mathbb{Z}}^* \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}})$ homomorphism

(iii) $\tau_k(\varphi) = 0 \iff \forall x \in \pi \ \ \varphi(x)x^{-1} \in \Gamma_{k+1}(\pi) \iff \varphi \in \mathcal{M}(k+1)$

$\Longrightarrow \quad \tau_k : \mathcal{M}(k)/\mathcal{M}(k+1) \to H_{\mathbb{Z}}^* \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}})$ <u>injective</u> homomorphism

the k^{th} Johnson homomorphism

<u>Remarks</u> (i) τ_k can be defined over the Andreadakis filtration of $\mathrm{Aut}(F_n)$.

A. Pettet, Takao Satoh, N. Enomoto, ---- have studied τ_k's on $\mathrm{Aut}(F_n)$ in details.

(ii) The monoid of homology cobordisms $\mathcal{C}_{g,1}$ does not act on π, but <u>does</u> on the nilpotent tower $\{N_k\}_{k\geq 1}$. Hence we can consider the Johnson homomorphisms on $\mathrm{gr}\, \mathcal{C}_{g,1}(\cdot)$. For details, see

Habiro - Massuyeau, arXiv: 1003.2512, and
Sakasai, arXiv: 1005.5501
) both of them are to appear in: Papadopoulos' Handbook.

<u>D. Johnson's results</u>

(I) (Math. Ann. <u>249</u> (1980) 225-242)

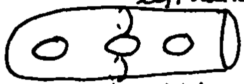
$$\tau_1(\mathcal{I}_{g,1}) = \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}} \subset H_{\mathbb{Z}} \otimes \Lambda^2 H_{\mathbb{Z}} \overset{\text{Poincaré dual}}{=} H_{\mathbb{Z}}^* \otimes \mathcal{L}_2(H_{\mathbb{Z}})$$

where $\Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}} = \langle \sum_{\sigma \in \mathfrak{S}_3} (\mathrm{sgn}\,\sigma) X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes X_{\sigma(3)} \ ;\ X_1, X_2, X_3 \in H_{\mathbb{Z}} \rangle \subset H_{\mathbb{Z}}^{\otimes 3}$

(II) (Ann. of Math., <u>118</u> (1983) 423-442)

$\mathcal{I}_{g,1}$ is generated by BP maps and BSCC maps

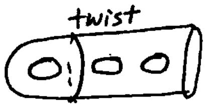
left hand twist



right hand twist

BP map

twist



BSCC map

(III) (Topology <u>24</u> (1985) 113-126)

$\mathcal{M}(2)\ (= \mathrm{Ker}\,\tau_1) = \langle \text{BSCC maps} \rangle$

(IV) (Topology <u>24</u> (1985) 127-144)

$\mathrm{Ker}(\tau_1 : \mathcal{I}_{g,1}^{abel} \twoheadrightarrow \Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}})$ is described by the Birman-Craggs homom's

In particular, it is 2-torsion

$(\Longrightarrow \ \tau_1 : \mathcal{I}_{g,1}^{abel} \otimes \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\cong} (\Lambda^3_{\mathbb{Z}} H_{\mathbb{Z}}) \otimes \mathbb{Z}[\tfrac{1}{2}])$

Remark We can consider the Lie algebra $\mathrm{gr}\,\Gamma_\bullet(\mathcal{I}_{g,1})$.
Hain (J. Amer. Math. Soc. 10 (1997) 591–651) gave an explicit presentation of the Lie algebra $\mathrm{gr}\,\Gamma_\bullet(\mathcal{I}_{g,1}) \otimes \mathbb{Q}$.

§ I-2. Extensions of the Johnson homomorphisms

$$\begin{array}{ccccl} \tau_1 : \mathcal{I}_{g,1} = \mathcal{M}(1) & \twoheadrightarrow & \Lambda^3 H_{\mathbb{Z}} & & \text{the } 1^{st} \text{ Johnson homomorphism} \\ \cap \,\text{+} & \circlearrowleft & \nearrow \exists ?\, \tilde{\tau}_1 & & \\ \mathcal{M}_{g,1} = \mathcal{M}(0) & & & & \end{array}$$

Observations (Morita)

(i) $\mathcal{M}_{g,1}^{abel} = 0$ $\quad \therefore\ \tilde{\tau}_1$ is not a homomorphism

(ii) $\Lambda^3 H_{\mathbb{Z}}$: $\mathcal{M}_{g,1}$-module via $\rho_0 : \mathcal{M}_{g,1} \to Sp_{2g}(\mathbb{Z})$, $\varphi \mapsto |\varphi|$

$\Rightarrow$ $\tilde{\tau}_1$ should be a 1-cocycle, i.e., $\forall \varphi, \forall \psi \in \mathcal{M}_{g,1}$

$$\tilde{\tau}_1(\varphi\psi) = \tilde{\tau}_1(\varphi) + |\varphi|\tilde{\tau}_1(\psi) \qquad (\Leftrightarrow\ d\tilde{\tau}_1 = 0\ )$$

$\Leftrightarrow$ $\tilde{\tau}_1 \in Z^1(C^*(\mathcal{M}_{g,1}; \Lambda^3 H_{\mathbb{Z}}))$

where $C^*(G;M) = \{C^q(G;M), d\}_{q \geq 0}$ normalized standard cochain complex

$$C^q(G;M) = \{c : \overbrace{G \times \cdots \times G}^{q} \to M ;\ \text{map},\ c(\cdots, \overset{\exists}{1}, \cdots) = 0\} \quad (G : \text{group},\ M : G\text{-module})$$

$$d : C^q(G;M) \to C^{q+1}(G;M)$$

$$(dc)(x_1, \cdots, x_{q+1}) = x_1 c(x_2, \cdots, x_{q+1}) + \sum_{i=1}^{q} (-1)^i c(x_1, \cdots, x_i x_{i+1}, \cdots, x_{q+1}) + (-1)^{q+1} c(x_1, \cdots, x_q)$$

$H^*(G;M) = H^*(C^*(G;M))$ the cohomology of G with values in M.

Theorem (Morita, Invent. math. $\underline{111}$ (1993), 197–224)

$\exists \tilde{k} \in Z^1(C^*(\mathcal{M}_{g,1}; \frac{1}{2}\Lambda^3 H_{\mathbb{Z}}))$ unique up to 1-coboundary

s.t. $\tilde{k}|_{\mathcal{I}_{g,1}} = \tau_1 : \mathcal{I}_{g,1} \to \Lambda^3 H_{\mathbb{Z}}$

$\Longrightarrow$ $\rho_1 : \mathcal{M}_{g,1} \to \frac{1}{2}\Lambda^3 H_{\mathbb{Z}} \rtimes Sp_{2g}(\mathbb{Z})$ group homomorphism

$\varphi \longmapsto (\tilde{k}(\varphi), |\varphi|)$

$\Longrightarrow$ $\tilde{k}^* : (\Lambda^*(\Lambda^3 H_{\mathbb{Q}}))^{Sp_{2g}(\mathbb{Q})} \to H^*(\frac{1}{2}\Lambda^3 H_{\mathbb{Z}} \rtimes Sp_{2g}(\mathbb{Z}); \mathbb{Q}) \xrightarrow{\rho_1^*} H^*(\mathcal{M}_{g,1}; \mathbb{Q})$

Morita's description by using trivalent graphs

Theorem (Morita)

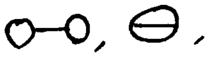
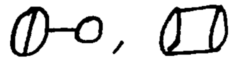
Image $\tilde{k}^* \supset \mathbb{Q}[e_i; i \geq 1]$

where $e_i \in H^{2i}(\mathcal{M}_{g,1}; \mathbb{Z})$ the i^{th} Mumford-Morita-Miller class.

Theorem (Morita–K., Math. Res. Lett. $\underline{3}$ (1996) 629–641)

Image $\tilde{k}^* = \mathbb{Q}[e_i; i \geq 1] \subset H^*(\mathcal{M}_{g,1}; \mathbb{Q})$

(holds also in the unstable range)

(e.g.,  $\mapsto e_1$,  $\mapsto e_2$, ...)

an approximation to the cohomology $H^*(\mathcal{M}_{g,1}; \mathbb{Q})$

Theorem (Madsen-Weiss, Ann. of Math., $\underline{165}$ (2007) 843–941)

$H^*(\mathcal{M}_{g,1}; \mathbb{Q}) = \mathbb{Q}[e_i; i \geq 1]$ for $* < \frac{2}{3}g$ (stable range)

Extensions of the Johnson homomorphisms

τ_1, τ_2 : Morita

$\forall \tau_k$: Hain

other approachs : K., Day, Massuyeau, Day, Massuyeau, ...

→ Magnus expansions

15.

§ I-3. Magnus expansions

$\pi = \pi_1(\Sigma_{g,1}, *)$: a free group of rank $2g$

$H = H_1(\Sigma_{g,1}; \mathbb{Q}) = \pi^{abel} \otimes_{\mathbb{Z}} \mathbb{Q} \ni [x] := (x \bmod [\pi,\pi]) \otimes 1 \quad (x \in \pi)$

$\hat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$ completed tensor algebra.

$\oplus$ $\boxed{\text{Bourbaki Groupes et algèbres de Lie Ch.2}}$

Kitano (Top. Appl., 69 (1996) 165–172)
used the classical Magnus expansion (coming from Fox' derivatives)
to describe the Johnson homomorphisms

$\Downarrow$ (K.) the minimum conditions to define an extension of the Johnson homom's.

Definition

$\theta : \pi \to \hat{T}(H)$ (generalized) Magnus expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 0) $\theta : \pi \to \hat{T}(H)$ map

1) $\forall x, \forall y \in \pi \quad \theta(xy) = \theta(x)\theta(y)$

2) $\forall x \in \pi \quad \theta(x) = 1 + [x] + \text{higher terms} \in \hat{T}(H)$

e.g., $\forall x \in G$
$\log x := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \in \widehat{\mathbb{Q}G}$

$\Longrightarrow \theta : \widehat{\mathbb{Q}\pi} \overset{\cong}{\longrightarrow} \hat{T}(H)$ algebra isomorphism

(where $\widehat{\mathbb{Q}G}$: completed group ring of a group G

$:= \varprojlim_{n\to\infty} \mathbb{Q}G/(IG)^n, \quad IG := \mathrm{Ker}(\varepsilon : \mathbb{Q}G \to \mathbb{Q}), \ \sum_{x\in G} a_x x \mapsto \sum a_x$ augmentation ideal)

16.

$\varphi \in \mathcal{M}_{g,1}$,

$$\begin{array}{ccc} \widehat{\mathbb{Q}\pi} & \overset{\theta}{\cong} & \hat{T}(H) \\ \varphi \downarrow \wr & \circlearrowleft & \wr \downarrow \exists! \, T^\theta(\varphi) \\ \widehat{\mathbb{Q}\pi} & \underset{\theta}{\cong} & \hat{T}(H) \end{array}$$

algebra automorphism preserving the filtration $\{\prod_{m\geq p} H^{\otimes p}\}_{m=1}^{\infty}$

$T^\theta : \mathcal{M}_{g,1} \to \mathrm{Aut}(\hat{T}(H))$ group homomorphism. "the total Johnson map"

$T^\theta(\varphi)|_H \in H^* \otimes \hat{T}(H) = \prod_{k=-1}^{\infty} H^* \otimes H^{\otimes(k+1)}$ (K., arXiv:0505497)

$\quad = (0 + 1_H + \sum_{k=1}^{\infty} \tau_k^\theta(\varphi))|\varphi|, \quad \tau_k^\theta(\varphi) \in H^* \otimes H^{\otimes(k+1)}, \ k \geq 1$

$\tau_k^\theta : \mathcal{M}_{g,1} \to H^* \otimes H^{\otimes(k+1)} \underset{\text{Poincaré dual}}{=} H^{\otimes(k+2)}$ not homomorphism

the k^{th} Johnson map

$\boxed{\text{Classical Result (cf [Bourbaki], [Serre])} \quad (\theta|_\pi)^{-1}(1 + \prod_{m\geq p} H^{\otimes p}) = \Gamma_p(\pi)}$

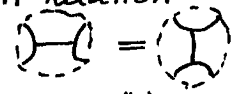
Lemma (K., loc. cit.) $\forall k \geq 1$.
$\tau_k^\theta|_{\mathcal{M}(k)} = \tau_k : \mathcal{M}(k) \to H^* \otimes \mathcal{L}_{k+1}(H) \subset H^* \otimes H^{\otimes(k+1)}$

$\forall \varphi, \forall \psi \in \mathcal{M}_{g,1}, \quad T^\theta(\varphi\psi) = T^\theta(\varphi) T^\theta(\psi)$ (τ_1^θ is a 1-cocycle)

$\Longrightarrow$ coboundary relations

$-d\tau_1^\theta = 0 \in C^*(\mathcal{M}_{g,1}; H^{\otimes 3})$

IH-relation



$-d\tau_2^\theta = (\tau_1^\theta \otimes 1_H + 1_H \otimes \tau_1^\theta) \cup \tau_1^\theta \in C^*(\mathcal{M}_{g,1}; H^{\otimes 4})$

$\vdots$

$\Longrightarrow$ the simplest proof of Theorem (Morita–K.) stated above

The Johnson homomorphism τ as a Lie algebra homomorphism (Morita)

(i) $\mathrm{gr}\,\mathcal{M}(\cdot) = \bigoplus_{k=1}^{\infty} \mathcal{M}(k)/\mathcal{M}(k+1)$: Lie algebra.

$\varphi \in \mathcal{M}(k)$, $\psi \in \mathcal{M}(\ell)$

$[\varphi \bmod \mathcal{M}(k+1), \psi \bmod \mathcal{M}(\ell+1)] := \varphi\psi\varphi^{-1}\psi^{-1} \bmod \mathcal{M}(k+\ell+1)$

(ii) $\mathrm{Der}(\mathcal{L}(H)) \overset{\mathrm{def}}{=} \{D \in \mathrm{End}(\mathcal{L}(H))\ ;\ \forall u, \forall v \in \mathcal{L}(H)\ \ D[u,v] = [Du,v] + [u,Dv]\}$

$\ni D_1, D_2$, $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$ derivation Lie algebra

$\omega := \sum_{i=1}^{g} A_i B_i - B_i A_i \in \Lambda^2 H = \mathcal{L}_2(H)$ symplectic form

independent of the choice of a symplectic basis $\{A_i, B_i\}_{i=1}^{g} \subset H$.

$\mathfrak{h}_{g,1} := \{D \in \mathrm{Der}(\mathcal{L}(H))\ ;\ D\omega = 0\}$

$\mathrm{Der}(\mathcal{L}(H)) \cong H^* \otimes \mathcal{L}(H)$ $\quad(\because\ \mathcal{L}(H)$ free over $H)$

$D \longmapsto D|H$

$\Longrightarrow H^* \otimes \mathcal{L}(H) \overset{\mathrm{P.d.}}{=} H \otimes \mathcal{L}(H)$: Lie algebra

We regard $\mathfrak{h}_{g,1}$ as a Lie subalgebra of $H \otimes \mathcal{L}(H)$

$(\Rightarrow \mathfrak{h}_{g,1} = \mathrm{Ker}([\ ,\ ] : H \otimes \mathcal{L}(H) \to \mathcal{L}(H)))$

$\mathfrak{h}_{g,1}^{+} := \mathfrak{h}_{g,1} \cap H \otimes \left(\bigoplus_{m=2}^{\infty} \mathcal{L}_m(H)\right) \subset \mathfrak{h}_{g,1}$ Lie subalgebra

$(\mathfrak{h}_{g,1} = \mathfrak{h}_{g,1}^{+} \rtimes \mathrm{sp}_{2g}(\mathbb{Q}))$

Theorem (Morita, Duke Math. J., 70 (1993) 699–726)

(i) $\tau : \mathrm{gr}\,\mathcal{M}(\cdot) \to H^* \otimes \mathcal{L}(H)$: Lie algebra homomorphism

(ii) $\tau(\mathrm{gr}\,\mathcal{M}(\cdot)) \subset \mathfrak{h}_{g,1}^{+}$

(iii) $\tau(\mathrm{gr}\,\mathcal{M}(\cdot)) \subsetneqq \mathfrak{h}_{g,1}^{+}$

Remarks (i) $\mathfrak{h}_{g,1}^{+} \cap H^{\otimes 3} = \Lambda^3 H$ (cf. Johnson's result) (Habegger)

(ii) **Theorem** (Garoufalidis–Levine, Proc. Sympos. Pure Math 73 (2005) 173–203)

$\tau(\mathrm{gr}\,\mathcal{C}_{g,1}(\cdot)) = \mathfrak{h}_{g,1}^{+}$

proof of Theorem (Morita)

(i) $\varphi \in \mathcal{M}(k)$, $\psi \in \mathcal{M}(\ell)$, $\mathcal{L}(H) < \hat{T}(H)$, $\forall u \in \hat{T}(H)$

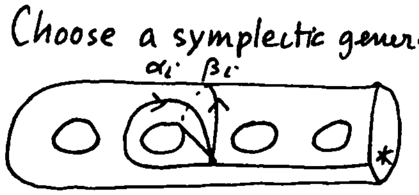
$T^\theta(\varphi)(u) = u + \tau_k(\varphi)(u) +$ higher terms

$T^\theta(\psi)(u) = u + \tau_\ell(\psi)(u) +$ higher terms

(where $\tau_k(\varphi)(u)$ action as a derivation)

$\Longrightarrow T^\theta(\varphi\psi\varphi^{-1}\psi^{-1})(u) = u + [\tau_k(\varphi), \tau_\ell(\psi)](u) +$ higher terms. // (i)

(ii) Choose a symplectic generator $\{\alpha_i, \beta_i\}_{i=1}^{g} \subset \pi$



$\zeta := \prod_{i=1}^{g} [\alpha_i, \beta_i] \in \pi$

boundary loop

$\theta(\zeta) = 1 + \omega +$ higher terms

$^\forall\varphi \in \mathcal{M}(k) \quad \varphi(\zeta) = \zeta$

$0 = T^\theta(\varphi)\,\theta(\zeta) - \theta(\zeta) = \tau_k(\varphi)(\omega) + \text{higher terms}$

$\Rightarrow \quad \tau_k(\varphi)(\omega) = 0$

(iii) (Morita trace)

$$Tr_k : H^* \otimes \mathcal{L}_k(H) \hookrightarrow H^* \otimes H^{\otimes k} \longrightarrow H^{\otimes(k-1)} \xrightarrow{\text{symmetrize}} \mathrm{Sym}^{k-1} H$$

$$f \otimes X_1 \cdots X_k \longmapsto f(X_1)\, X_2 \cdots X_k$$

__Theorem__ (Morita, ibid)

(i) $Tr_k(\mathfrak{h}_{g,1} \cap (H^* \otimes \mathcal{L}_k(H))) = \begin{cases} 0, & \text{if } k : \text{odd} \\ \mathrm{Sym}^{k-1} H, & \text{if } k : \text{even} \end{cases}$

(ii) $Tr_k \circ \tau_{k-1} = 0 \quad (^\forall k \geqq 3)$ — //

__Bad news on__ T^θ

$$T^\theta(\mathcal{I}_{g,1}) \not\subset \widehat{\mathfrak{h}_{g,1}^+} := \underbrace{\prod_{k=1}^{\infty}}_{\text{completion}} ((H^* \otimes \mathcal{L}_{k+1}(H)) \cap \mathfrak{h}_{g,1}^+)$$

$\rightsquigarrow$ Massuyeau's symplectic expansions

__Symplectic expansions__

$\Delta : \hat{T}(H) \to \hat{T}(H) \hat{\otimes} \hat{T}(H)$ coproduct.
algebra homomorphism given by $\Delta(X) = X \hat{\otimes} 1 + 1 \hat{\otimes} X \ (^\forall X \in H)$

__Definition__ (Massuyeau, Bull. Soc. Math. France __140__ (2012) 101–161)

$\theta : \pi \to \hat{T}$ symplectic expansion

$\overset{\text{def}}{\Longleftrightarrow}$
0) $\theta : \pi \to \hat{T}$ Magnus expansion.
1) (group-like) $^\forall x \in \pi \quad \Delta\theta(x) = \theta(x) \hat{\otimes} \theta(x)$
2) (symplectic) $\theta(\zeta) = \exp(\omega) \ \left(= \sum_{n=0}^{\infty} \frac{1}{n!}\omega^n\right)$

(1) $\Leftrightarrow$ $\theta : \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \hat{T}(H)$ isomorphism of complete Hopf algebras)

__Examples__ (1) ($/\mathbb{R}$) (K.) "Harmonic Magnus expansions" parametrized by $\mathcal{T}_{g,1}$ Teichmüller spa[ce]

(2) (Massuyeau, ibid) LMO expansions

(3) (Kuno, Proc. Amer. Math. Soc. __140__ (2012) 1075–1083)
combinatorial symplectic expansions ($\Leftarrow$ free generator of π)

(4) (Bene-K.-Kuno-Penner) $\Leftarrow$ trivalent bordered fatgraph. —

("additional structure" on Σ $\Rightarrow$ symplectic expansion of π)

$\theta : \pi \to \hat{T}(H)$ symplectic expansion

$T^\theta : \mathcal{M}_{g,1} \to \mathrm{Aut}(\hat{T}(H))$ total Johnson map

$\forall \varphi \in \mathcal{I}_{g,1} \quad T^\theta(\varphi)|_H = 1_H + \tau_1^\theta(\varphi) + \tau_2^\theta(\varphi) + \cdots$

$$\tau^\theta(\varphi) := \log T^\theta(\varphi) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (T^\theta(\varphi) - 1)^n \in \mathrm{Der}(\hat{T}(H))$$

converges

$\tau^\theta(\mathcal{I}_{g,1})|_H \subset H^* \otimes \hat{\mathcal{L}}(H)$ ($\because$ group-like condition)

$\tau^\theta(\mathcal{I}_{g,1}) \subset \hat{\mathfrak{h}}^+_{g,1}$ ($\because$ symplectic condition)

$\tau^\theta : \mathcal{I}_{g,1} \to \hat{\mathfrak{h}}^+_{g,1}$ Massuyeau's total Johnson map.

$(\ \forall k \geq 1,\ \tau^\theta|_{\mathcal{M}(k)} = \tau_k + \text{higher terms}\ )$

$\Sigma_{g,r}$
$\parallel$

Questions (i) symplectic expansions are defined only for $\Sigma_{g,1}$, not for $\Sigma_{g,r}$, $(r \geq 2)$

$\rightsquigarrow$ How do we generalize τ^θ to $\Sigma_{g,r}$ $(r \geq 2)$?

(ii) explicit description of $\log$(Dehn twist) $=$?

(iii) geometric meaning of the Morita traces. ?

All these questions will be answered by the completed Goldman-Turaev bialgebra

$\rightsquigarrow$ Part II.